LINK BETWEEN HOSOYA INDEX AND FIBONACCI NUMBERS

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Abstract. Let $G$ be a graph and $Z(G)$ be its Hosoya index. We show how the Hosoya index can be a good tool to establish some new identities involving Fibonacci numbers. This permits to extend Hillard and Windfeldt work.

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1. INTRODUCTION

We denote by $G = (V(G), E(G))$ a simple undirected graph, $V(G)$ is the set of its vertices and $E(G)$ is the set of its edges. The order of $G$ is $|V(G)|$ and the size of $G$ is $|E(G)|$. For a vertex $v$ of $G$, $N(v)$ is the set of vertices adjacent to $v$, $\deg(v) := |N(v)|$ is the degree of $v$; $\text{Link}(v)$ is the set of edges incident to $v$. An edge $\{u, v\}$ of $G$ is denoted $uv$. A path $P_n$, from a vertex $v_1$ to a vertex $v_n$, $n \geq 2$, is a sequence of vertices $v_1, \ldots, v_n$ and edges $v_i v_{i+1}$, for $i = 1, \ldots, n - 1$; for simplicity we denote it by $v_1 \cdots v_n$. We extend the definition of $P_n$ to $n = 0$ and $n = 1$ by setting $P_0$ is empty and $P_1$ is a single vertex, we add the convention that $P_n P_0 = P_0 P_n = P_n$ for all $n \geq 1$.

The graph $G - v$ is obtained from $G$ by removing the vertex $v$ and all edges of $G$ which are incident to $v$. For an edge $e$ of $G$, we denote by $G - e$ the graph obtained from $G$ by removing $e$. The contraction of a graph $G$, associated to an edge $e$, is the graph $G/e$ obtained by removing $e$ and identifying the vertices $u$ and $v$ incident to $e$ and replacing them by a single vertex $v'$ where any edges incident to $u$ or $v$ are redirected to $v'$. We then say that we contract in $G$ the adjacent vertices $u$, $v$ into the vertex $v'$.

The well-known Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. The Fibonacci numbers are connected to the element of Pascal’s triangle using the following identity

$$F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}.$$
For some results and properties related to Fibonacci numbers, see for instance [1]. Many fields widely apply this sequence, particularly in physics and chemistry [10].

A matching $M$ of a graph $G$ is a subset of $E(G)$ such that no two edges in $M$ share a vertex in $G$. A matching of $G$ is also called an independent edge set of $G$. A $k$-matching of a graph $G$ is a matching of $G$ of cardinal $k$, it is then an independent edge set of $G$ of cardinal $k$. We denote by $m(G,k)$ the number of $k$-matchings of $G$ with the convention $m(G,0) = 1$. Note that $m(G,1) = |E(G)|$ and when $k > n/2$, $m(G,k) = 0$.

The Hosoya index of a graph $G$, denoted by $Z(G)$, is an index introduced by Hosoya [9] as follows.

$$Z(G) = \sum_{k=0}^{\lfloor n/2 \rfloor} m(G,k),$$

where $n = |V(G)|$, $\lfloor n/2 \rfloor$ stands for the integer part of $n/2$. This index has several applications in molecular chemistry such as boiling point, entropy or heat of vaporization. The literature includes many papers dealing the Hosoya index [2,3,6].

## 2. Preliminary results

Before proving our main results, we first list the following results. From the definition of the Hosoya index, it is not difficult to deduce the following Lemma.

**Lemma 1** ([7]). Let $G$ be a graph, we have

(1) If $uv \in E(G)$, then $Z(G) = Z(G - uv) + Z(G - \{u,v\}).$
(2) If $v \in V(G)$, then $Z(G) = Z(G - v) + \sum_{w \in N_G(v)} Z(G - \{w,v\}).$
(3) If $G_1, G_2, ..., G_t$ are the components of $G$ then $Z(G) = \prod_{k=1}^{t} Z(G_k).$

Lemma 1 allows us to compute $Z(G)$ for any graph recursively.

The following theorem gives a relation between Hosoya index and Fibonacci number (see [5], [7]).

**Theorem 1.** Let $P_n$ be a path on $n$ vertices, then $Z(P_n) = F_{n+1}$.

## 3. Main results

In this section, inspired by [4], we give another proofs of well-known identities (see Lemmas 2 and 3). Our goal is to prove the formula of Lemma 3 via Theorem 1. For this we give a direct proof of Lemma 2. We also establish two new identities, given in Theorems 2 and 3.

The following identity shows the relation between independent edge subsets in $P_{r_1+r_2}, P_{r_1}$ and $P_{r_2}$ for $r_1$ and $r_2$ two non-negative integers.

**Lemma 2.** Let $r_1, r_2$ be two non-negative integers, then

$$Z(P_{r_1+r_2}) = Z(P_{r_1}) Z(P_{r_2}) + Z(P_{r_1-1}) Z(P_{r_2-1}).$$
**Proof.** Let \( v_1 \cdots v_{r_1+r_2} \) be a path \( P_{r_1+r_2} \) partitioned into two paths \( P_{r_1} \) represented by a sequence of vertices \( v_1 \cdots v_{r_1} \) and \( P_{r_2} \) represented by a sequence of vertices \( v_{r_1+1} \cdots v_{r_1+r_2} \). The vertices \( v_{r_1} \) and \( v_{r_1+1} \) share the same edge in \( P_{r_1+r_2} \). The Hosoya number \( Z(P_{r_1+r_2}) \) of the path \( P_{r_1+r_2} \) represents the number of independent edge subsets between the vertices of \( P_{r_1+r_2} \), this index can be written as
\[
Z(P_{r_1+r_2}) = |M| + |M'|, \text{ where:}
\]

- \( M \) is the set of independent edge subsets of \( P_{r_1+r_2} \) such that the edge \( v_{r_1}v_{r_1+1} \) does not belong to any independent edge subset of \( M \), that means all independent edge subsets of \( M \) are in \( P_{r_1} \) and \( P_{r_2} \), so \( |M| = Z(P_{r_1})Z(P_{r_2}) \).
- \( M' \) is the set of independent edge subsets of \( P_{r_1+r_2} \) such that the edge \( v_{r_1}v_{r_1+1} \) belongs to all independent edge subsets of \( M' \), that means the others independent edges of every subset of \( M' \) are in \( P_{r_1-1} \) and \( P_{r_2-1} \), so \( |M'| = Z(P_{r_1-1})Z(P_{r_2-1}) \).

\[ \square \]

**Lemma 3.** Let \( k, n \) be two integers such that \( 1 \leq k \leq n \). Then
\[
F_{n+1} = F_kF_{n-k} + F_{k+1}F_{n-k+1}.
\]

**Proof.** Consider a path \( P_n = v_1 \cdots v_n \) on \( n \) vertices. We set \( r_1 := k, r_2 := n - k \), and use Theorem 1 and Lemma 2. \( \square \)

We introduce a new identity of Fibonacci numbers which generalize identities of Fibonacci numbers given in [8].

For every integer \( s \geq 2 \), let \( \Omega_s \) be the set of \( \varepsilon := (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s) \) with \( \varepsilon_i \in \{-1,0,1\} \) \( (1 \leq i \leq s) \) such that:

1. The number of \( \varepsilon_i = 0 \) is even. Let \( 2h(\varepsilon) \) this number.
2. If \( 2h(\varepsilon) = 0 \), then \( \varepsilon_i = 1 \) for all \( i \).
3. If \( 2h(\varepsilon) \neq 0 \), let \( L_{\varepsilon} := \{s_1, s_2, \ldots, s_{2h(\varepsilon)} : s_1 < s_2 < \ldots < s_{2h(\varepsilon)} \text{ and } \varepsilon_{s_i} = 0 \text{ for all } i \in \{1, 2, \ldots, 2h(\varepsilon)\}\} \).

For all \( \varepsilon \in L_{\varepsilon} \), we have:

- \( l \) is even \( \iff \varepsilon_i = 1 \) for all \( s_l < i < s_{l+1} \);
- \( l \) is odd \( \iff \varepsilon_i = -1 \) for all \( s_l < i < s_{l+1} \);
- \( \varepsilon_i = 1 \) for all \( i < s_1 \) or \( i > s_{2h(\varepsilon)} \).

For example,
\[
\Omega_2 = \{(1,1),(0,0)\}, \quad \Omega_3 = \{(1,1,1),(1,0,0),(0,0,1),(0,0,0)\}, \quad \Omega_4 = \{(1,1,1,1),(0,0,1,1),(1,0,0,1),(1,1,0,0),(0,0,0,0),(0,0,0,0),(0,0,0,0),(0,0,0,0),(0,0,0,0),(0,0,0,0),(0,0,0,0)\}.
\]

- The lines of the following table represent the elements of \( \Omega_5 \),
The lines of the following table represent the elements of $\Omega_6$.

| 1 1 1 1 1 1 | 0 0 0 0 0 1 |
| 0 0 1 1 1 1 | 0 0 1 0 0 0 |
| 1 0 0 1 1 1 | 0 0 1 0 0 0 |
| 1 1 0 0 1 1 | 0 0 1 0 0 0 |
| 1 1 1 0 0 1 | 0 0 1 0 0 0 |
| 0 1 0 1 1 1 | 0 0 0 0 0 0 |
| 1 0 1 0 1 1 | 0 0 0 0 0 0 |
| 1 1 0 1 0 1 | 0 0 0 0 0 0 |
| 1 1 1 0 1 0 | 0 0 0 0 0 0 |
| 0 1 0 1 0 1 | 0 0 1 0 0 0 |
| 1 0 1 0 1 0 | 0 0 1 0 0 0 |
| 1 1 0 1 0 1 | 0 0 1 0 0 0 |
| 0 0 1 0 1 0 | 0 0 1 0 0 0 |
| 1 0 0 1 0 1 | 0 0 1 0 0 0 |
| 1 1 1 1 0 1 | 0 0 1 0 0 0 |
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| 1 1 1 1 0 1 | 0 0 1 0 0 0 |
| 0 1 0 1 0 1 | 0 0 1 0 0 0 |
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| 0 1 0 1 0 1 | 0 0 1 0 0 0 |
| 1 0 1 0 1 0 | 0 0 1 0 0 0 |
| 0 0 1 0 1 0 | 0 0 1 0 0 0 |
| 1 0 0 1 0 1 | 0 0 1 0 0 0 |
| 1 1 1 1 0 1 | 0 0 1 0 0 0 |

Theorem 2. For any positive integers $r_i$ ($1 \leq i \leq s$) and any integer $s \geq 2$, we have

$$F_{r_1+r_2+\cdots+r_s+1} = \sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_s) \in \Omega_s} \prod_{i=1}^{s} F_{r_i+\epsilon_i}. \quad (3.1)$$

Proof. Let $P_{r_1+r_2+\cdots+r_s}$ be a path with $r_1 + r_2 + \cdots + r_s$ vertices. We subdivide $P_{r_1+r_2+\cdots+r_s}$ in consecutive blocs of paths $P_{r_i}$ with $r_i$ vertices ($1 \leq i \leq s$), see Figure 1.

In one hand side, by Theorem 1, we have $Z(P_{r_1+r_2+\cdots+r_s}) = F_{r_1+r_2+\cdots+r_s+1}$. In the other hand $Z(P_{r_1+r_2+\cdots+r_s})$ is the number of independent edge subsets in $P_{r_1+r_2+\cdots+r_s}$. So, $Z(P_{r_1+r_2+\cdots+r_s}) = \sum_{k=0}^{s-1} |M_k|$ where $M_k$ is the set of independent edge subsets $I$ in $P_{r_1+r_2+\cdots+r_s}$ such that it exists exactly $k$ edges between blocs of paths $P_{r_i}$ ($1 \leq i \leq s$) which belong to $I$. 
\( M_0 \) is the set of independent edge subsets \( I \) in \( P_{r_1+r_2+\ldots+r_s} \) such that doesn’t contain any edge between blocks of paths \( P_{r_i} (1 \leq i \leq s) \), so all these independent edge subsets are in blocks \( P_{r_i} (1 \leq i \leq s) \). Hence, \( |M_0| = \prod_{i=1}^{s} F_{r_i+1} \).

\( M_1 \) is the set of independent edge subsets \( I \) in \( P_{r_1+r_2+\ldots+r_s} \) such that it exists only one edge between blocs of paths \( P_{r_i} (1 \leq i \leq s) \) which belong to \( I \). Let \( H \) be a subset of \( M_1 \) containing the edge \( v_{r_1+\ldots+r_k}v_{r_1+\ldots+r_k+1} (1 \leq k \leq s-1) \) in all of its independent edge subsets. We contract the adjacent vertices \( v_{r_1+\ldots+r_k},v_{r_1+\ldots+r_k+1} \) in \( P_{r_1+\ldots+r_s} \) into one vertex \( v' \) and \( P_{r_1+\ldots+r_s-1} \) is a new path after contraction composed of consecutive blocks of paths \( P_{r_1},P_{r_2},\ldots,P_{r_{k-1}},P_{r_k-1},v',P_{r_{k+1}-1},P_{r_{k+2}} \ldots,P_{r_s} \). A path \( P_{r_1+\ldots+r_s-1} \) does not contain any edge between blocks which belong to independent edge subsets of \( H \), so \( |H| = F_{r_1+1} \times F_{r_2+1} \times \cdots \times F_{r_{k-1}+1} \times F_{r_k} \times F_2 \times F_{r_{k+1}} \times F_{r_{k+2}+1} \times \cdots \times F_{r_s+1} \). Thus, \( |M_1| = \sum_{(e_1,e_2,\ldots,e_s)\in \Delta_1} \prod_{i=1}^{s} F_{r_i+e_i} \) where \( \Delta_1 \) is the set of \((e_1,e_2,\ldots,e_s)\) such that for \( 1 \leq i \leq s \), either \( e_i \in \{0,1\} \) and \( e_1e_2\cdots e_s \) forms a sequence such that there is only one pair of zeros and this pair is of the form \((e_1,e_{i+1})\).

\( M_2 \) is the set of independent edge subsets \( I \) in \( P_{r_1+r_2+\ldots+r_s} \) such that it exists exactly two edges between blocs of paths \( P_{r_i} (1 \leq i \leq s) \) which belong to \( I \). As for computing of \( |M_1| \) and using the contraction method for the two edges between blocs of paths \( P_{r_i} (1 \leq i \leq s) \), we have \( |M_2| = \sum_{(e_1,e_2,\ldots,e_s)\in \Delta_2} \prod_{i=1}^{s} F_{r_i+e_i} \) where \( \Delta_2 \) is the set of \((e_1,e_2,\ldots,e_s)\) in \( \Omega_s \) such that for \( 1 \leq i \leq s \), either \( e_i \in \{-1,0,1\} \) and \( e_1e_2\cdots e_s \) forms a sequence such that there is only one pair of zeros \((e_1,e_{i+1})\) and \( e_i + e_{i+1} = -1 \), or only two pairs of zeros \( \{e_i,e_{i+1}\}, \{e_{i+1+k},e_{i+2+k}\} \) \( (1 \leq k \leq i+k+2 \leq s) \) and \( e_i = 1 \) for all \( i \in \{1,2,\ldots,i-1,i+2,i+3,\ldots,i+k,i+k+3,\ldots,s\} \).

For \( M_k \) \( (3 \leq k \leq s-2) \), using the contraction method for \( k \) edges between blocs of paths \( P_{r_i} (1 \leq i \leq s) \), we have \( |M_k| = \sum_{(e_1,e_2,\ldots,e_s)\in \Delta_k} \prod_{i=1}^{s} F_{r_i+e_i} \) where \( \Delta_k = \{(e_1,e_2,\ldots,e_s) \in \Omega_s : \text{for all } s_l \in L_r (1 \leq i \leq 2h(s)) \text{, } \sum_{l=1}^{h(s)} (s_{2l} - s_{2l-1}) = k \} \) which represents all sequences of \( \Omega_s \) such that the sum of the difference of the position of each pair of \( 0 \) is equal to \( k \).

We finish by \( M_{s-1} \) which is the set of matchings \( I \) in \( P_{r_1+r_2+\ldots+r_s} \) such that it exists exactly \( s-1 \) edges between blocs of paths \( P_{r_i} (1 \leq i \leq s) \) which belong to
In this case, except the paths $P_{r_1}, P_{r_2}$ that lose one vertex after a contraction all others paths $P_{r_i}$ ($2 \leq i \leq s - 1$) lose two vertices after contraction method. Thus, $|M_{s-1}| = F_{r_1} F_{r_2} \prod_{i=2}^{s-1} F_{r_i} - 1$.

Note that $\{\Delta_k : 1 \leq k \leq s - 1\}$ is a partition of $\Omega_s$. Hence, the identity (3.1) holds.

The following corollaries are the main results given by [8].

**Corollary 1.** For any non-negative integers $r$ and $t$, we have

$$F_{r+t} = F_{r+1} F_t + F_r F_{t-1}. \quad (3.2)$$

**Proof.** From Theorem 2 with $s = 2$ and $\Omega_2 = \{(1, 1), (0, 0)\}$, we obtain the following identity $F_{r_1+r_2+1} = F_{r_1+1} F_{r_2+1} + F_{r_1} F_{r_2}$. We put $r = r_1$ and $t = r_2 + 1$ and we conclude.

**Corollary 2.** For any non-negative integers $u$, $v$ and $w$, we have

$$F_{u+v+w} = F_{u+1} F_{v+1} F_{w+1} + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1}.$$  

**Proof.** From Theorem 2 with $s = 3$ and $\Omega_3 = \{(1, 1, 1), (1, 0, 0), (0, 0, 1), (0, -1, 0)\}$, we obtain the following identity $F_{r_1+r_2+r_3+1} = F_{r_1+1} F_{r_2+1} F_{r_3+1} + F_{r_1+1} F_{r_2} F_{r_3} + F_{r_1} F_{r_2} F_{r_3+1} + F_{r_1} F_{r_2+1} F_{r_3}$. We put $u = r_1$, $v = r_2$ and $w = r_3 + 1$ and using $F_t = F_{t+1} - F_{t-1}$, we have:

$$F_{u+v+w} = F_{u+1} F_{v+1} F_w + F_{u+1} F_v F_{w+1} + F_u F_v F_w + F_u F_{v-1} F_{w+1}$$
$$= F_{u+1} F_{v+1} (F_w + 1) + F_{u+1} F_{v+1} F_{w-1} + F_u F_v F_w + F_u F_{v-1} F_{w-1}$$
$$= F_{u+1} F_{v+1} F_w + F_{u+1} F_v F_{w+1} + F_u F_v F_w + F_u F_{v-1} F_{w-1}$$
$$= F_{u+1} F_{v+1} (F_w + 1) + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1} + F_{u+1} F_{v-1} F_{w-1}$$
$$= F_{u+1} F_{v+1} (F_w + 1) + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1}$$
$$= F_{u+1} F_{v+1} (F_w + 1) + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1}$$
$$= F_{u+1} F_{v+1} (F_w + 1) + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1}.$$

**Corollary 3.** For any non-negative integers $a$, $b$, $c$ and $d$, we have

$$F_{a+b+c+d+1} = F_{a+1} F_{b+1} F_{c+1} F_{d+1} + F_a F_b F_c F_d + F_{a+1} F_b F_c F_{d+1} + F_{a+1} F_b F_c F_{d+1}$$
$$+ F_b F_c F_d + F_a F_{b-1} F_c F_{d+1} + F_a F_b F_{c+1} F_{d+1}$$
$$+ F_{a+1} F_b F_{c+1} F_{d} + F_a F_{b-1} F_{c+1} F_{d+1}.$$
Proof. From Theorem 2, with \( s = 4 \) and \( \Omega_4 = \{(1, 1, 1, 1), (0, 0, 1, 1), (1, 0, 0, 1), (1, 1, 0, 0), (0, 0, 0, 0), (0, -1, 0, 1), (1, 0, -1, 0), (0, -1, -1, 0)\} \), we obtain the identity.

The following theorem is another identity of Fibonacci number which gives an equivalent of Theorem 2.

**Theorem 3.** Let \( s \geq 2 \) be an integer. For any non-negative integer \( r_i \) \((1 \leq i \leq s)\), we have

\[
F_{\sum_{i=1}^s r_i + 1} = F_{\sum_{i=1}^s r_i} F_{r_s + 1} + \sum_{i=0}^{s-2} \left( \prod_{j=1}^{i} F_{r_{s-j} - 1} \right) F_{\sum_{j=i+1}^{s-2} r_j + 1} F_{r_s - i - 1} F_{r_s} .
\]

Proof. As mentioned in Theorem 2, \( F_{\sum_{i=1}^s r_i + 1} = \sum_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s) \in \Omega_s} \prod_{i=1}^s F_{r_i + \varepsilon_i} \). Then \( F_{\sum_{i=1}^s r_i + 1} = c_1 + c_0 \) where \( c_1 \) corresponds to the case \( \varepsilon_s = 1 \) and \( c_0 \) to the case \( \varepsilon_s = 0 \). That means to count \( F_{\sum_{i=1}^s r_i + 1} \) we have two cases.

Case 1. \( \varepsilon_s = 1 \). Then for all s-uplet \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s)\) we obtain

\[
c_1 = F_{r_s + 1} \left( \sum_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{s-1}) \in \Omega_{s-1}} \prod_{i=1}^{s-1} F_{r_i + \varepsilon_i} \right).
\]

so for \( \varepsilon_s = 1 \) we have \( c_1 = F_{\sum_{i=1}^{s-1} r_i + 1} F_{r_s + 1} \).

Case 2. \( \varepsilon_s = 0 \). Let \( \varepsilon_{s-i-1} = 0 \) with \( i \) the smallest integer \( k \), \( 0 \leq k \leq s-2 \), such that \( \varepsilon_{s-k-1} = 0 \). So for \( 1 \leq j \leq i \) we have \( \varepsilon_{s-j} = -1 \). Hence,

\[
c_0 = \sum_{i=0}^{s-2} \left[ \left( \prod_{j=1}^{i} F_{r_{s-j} - 1} \right) F_{r_{s-i} - 1} F_{r_s} \sum_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{s-i-2}) \in \Omega_{s-i-2}} \prod_{j=1}^{s-i-2} F_{r_j + \varepsilon_j} \right] .
\]

As an immediate consequence of Theorem 3 we have:

**Corollary 4.** For any non-negative integers \( s \) and \( r \), we have

\[
F_{sr + 1} = F_{r + 1} F_{(s-1)r + 1} + \sum_{i=0}^{s-2} F_{r+1} F_{s-i-1} F_{(s-i-2)r + 1} .
\]

Proof. Use Theorem 3 with \( r_1 = r_2 = \cdots = r_s = r \).
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