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LINK BETWEEN HOSOYA INDEX AND FIBONACCI NUMBERS

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Abstract. Let G be a graph and $Z(G)$ be its Hosoya index. We show how the Hosoya index can be a good tool to establish some new identities involving Fibonacci numbers. This permits to extend Hillard and Windfeldt work.

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1. INTRODUCTION

We denote by $G = (V(G),\ E(G))$ a simple undirected graph, $V(G)$ is the set of its vertices and $E(G)$ is the set of its edges. The order of G is $|V(G)|$ and the size of G is $|E(G)|$. For a vertex v of G, $N(v)$ is the set of vertices adjacent to v, $\deg(v) := |N(v)|$ is the degree of v; $Link(v)$ is the set of edges incident to v. An edge $\{u,v\}$ of G is denoted uv. A path P_n, from a vertex v_1 to a vertex v_n, $n \geq 2$, is a sequence of vertices $v_1,\ldots,v_n$ and edges $v_i v_{i+1}$, for $i = 1,\ldots,n-1$; for simplicity we denote it by $v_1 \cdots v_n$. We extend the definition of P_n to $n = 0$ and $n = 1$ by setting P_0 is empty and P_1 is a single vertex, we add the convention that $P_n P_0 = P_0 P_n = P_n$ for all $n \geq 1$.

The graph $G - v$ is obtained from G by removing the vertex v and all edges of G which are incident to v. For an edge e of G, we denote by $G - e$ the graph obtained from G by removing e. The contraction of a graph G, associated to an edge e, is the graph G/e obtained by removing e and identifying the vertices u and v incident to e and replacing them by a single vertex v' where any edges incident to u or v are redirected to v'. We then say that we contract in G the adjacent vertices u, v into the vertex v'.

The well-known Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. The Fibonacci numbers are connected to the element of Pascal's triangle using the following identity

$$F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}.$$

For some results and properties related to Fibonacci numbers, see for instance [1]. Many fields widely applies this sequence, particularly in physics and chemistry [10].

A matching M of a graph G is a subset of $E(G)$ such that no two edges in M share a vertex in G. A matching of G is also called an independent edge set of G. A k-matching of a graph G is a matching of G of cardinal k, it is then an independent edge set of G of cardinal k. We denote by $m(G,k)$ the number of k-matchings of G with the convention $m(G,0) = 1$. Note that $m(G,1) = |E(G)|$ and when $k > n/2$, $m(G,k) = 0$.

The Hosoya index of a graph G, denoted by $Z(G)$, is an index introduced by Hosoya [9] as follows.

$$Z(G) = \sum_{k=0}^{\lfloor n/2 \rfloor} m(G,k),$$

where $n = |V(G)|$, $\lfloor n/2 \rfloor$ stands for the integer part of $n/2$. This index has several applications in molecular chemistry such as boiling point, entropy or heat of vaporization. The literature includes many papers dealing the Hosoya index [2, 3, 6].

2. Preliminary results

Before proving our main results, we first list the following results. From the definition of the Hosoya index, it is not difficult to deduce the following Lemma.

Lemma 1 ([7]). *Let G be a graph, we have*
(1) *If $uv \in E(G)$, then $Z(G) = Z(G - uv) + Z(G - \{u,v\})$.*
(2) *If $v \in V(G)$, then $Z(G) = Z(G - v) + \sum_{w \in N_G(v)} Z(G - \{w,v\})$.*
(3) *If $G_1, G_2, ..., G_t$ are the components of G then $Z(G) = \prod_{k=1}^{t} Z(G_k)$.*

Lemma 1 allows us to compute $Z(G)$ for any graph recursively.

The following theorem gives a relation between Hosoya index and Fibonacci number (see [5], [7]).

Theorem 1. *Let P_n be a path on n vertices, then $Z(P_n) = F_{n+1}$.*

3. Main results

In this section, inspired by [4], we give another proofs of well-known identities (see Lemmas 2 and 3). Our goal is to prove the formula of Lemma 3 via Theorem 1. For this we give a direct proof of Lemma 2. We also establish two new identities, given in Theorems 2 and 3.

The following identity shows the relation between independent edge subsets in $P_{r_1+r_2}$, P_{r_1} and P_{r_2} for r_1 and r_2 two non-negative integers.

Lemma 2. *Let r_1, r_2 be two non-negative integers, then*

$$Z(P_{r_1+r_2}) = Z(P_{r_1}) Z(P_{r_2}) + Z(P_{r_1-1}) Z(P_{r_2-1}).$$

Proof. Let $v_1 \cdots v_{r_1+r_2}$be a path $P_{r_1+r_2}$ partitioned into two paths P_{r_1} represented by a sequence of vertices $v_1 \cdots v_{r_1}$ and P_{r_2} represented by a sequence of vertices $v_{r_1+1} \cdots v_{r_1+r_2}$. The vertices v_{r_1} and v_{r_1+1} share the same edge in $P_{r_1+r_2}$. The Hosoya number $Z(P_{r_1+r_2})$ of the path $P_{r_1+r_2}$ represents the number of independent edge subsets between the vertices of $P_{r_1+r_2}$, this index can be written as $Z(P_{r_1+r_2}) = |M| + |M'|$, where:

- M is the set of independent edge subsets of $P_{r_1+r_2}$ such that the edge $v_{r_1}v_{r_1+1}$ does not belong to any independent edge subset of M, that means all independent edge subsets of M are in P_{r_1} and P_{r_2}, so $|M| = Z(P_{r_1})Z(P_{r_2})$.
- M' is the set of independent edge subsets of $P_{r_1+r_2}$ such that the edge $v_{r_1}v_{r_1+1}$ belongs to all independent edge subsets of M', that means the others independent edges of every subset of M' are in P_{r_1-1} and P_{r_2-1}, so $|M| = Z(P_{r_1-1})Z(P_{r_2-1})$.

□

Lemma 3. *Let k,n be two integers such that $1 \leq k \leq n$. Then*

$$F_{n+1} = F_k F_{n-k} + F_{k+1} F_{n-k+1}.$$

Proof. Consider a path $P_n = v_1 \cdots v_n$ on n vertices. We set $r_1 := k$, $r_2 := n-k$, and use Theorem 1 and Lemma 2. □

We introduce a new identity of Fibonacci numbers which generalize identities of Fibonacci numbers given in [8].

For every integer $s \geq 2$, let Ω_s be the set of $\varepsilon := (\varepsilon_1, \varepsilon_2, ..., \varepsilon_s)$ with $\varepsilon_i \in \{-1, 0, 1\}$ $(1 \leq i \leq s)$ such that :

(1) The number of $\varepsilon_i = 0$ is even. Let $2h(\varepsilon)$ this number.
(2) If $2h(\varepsilon) = 0$, then $\varepsilon_i = 1$ for all i.
(3) If $2h(\varepsilon) \neq 0$, let $L_\varepsilon := \{s_1, s_2, \ldots, s_{2h(\varepsilon)} : s_1 < s_2 < \ldots < s_{2h(\varepsilon)}\ and\ \varepsilon_{s_i} = 0\ for\ all\ i \in \{1, 2, \ldots, 2h(\varepsilon)\}\}$.
For all $l \in L_\varepsilon$, we have :
    - l is even $\implies \varepsilon_i = 1$ for all $s_l < i < s_{l+1}$;
    - l is odd $\implies \varepsilon_i = -1$ for all $s_l < i < s_{l+1}$;
    - $\varepsilon_i = 1$ for all $i < s_1$ or $i > s_{2h(\varepsilon)}$.

For example,

$$\Omega_2 = \{(1,1), (0,0)\},$$
$$\Omega_3 = \{(1,1,1), (1,0,0), (0,0,1), (0,-1,0)\},$$
$$\Omega_4 = \{(1,1,1,1), (0,0,1,1), (1,0,0,1), (1,1,0,0), (0,0,0,0), (0,-1,0,1), (1,0,-1,0), (0,-1,-1,0)\}.$$

- The lines of the following table represent the elements of Ω_5,

1	1	1	1	1	0	−1	−1	0	1
0	0	1	1	1	1	0	−1	−1	0
1	0	0	1	1	0	−1	−1	−1	0
1	1	0	0	1	0	0	0	0	1
1	1	1	0	0	0	0	1	0	0
0	−1	0	1	1	1	0	0	0	0
1	0	−1	0	1	0	−1	0	0	0
1	1	0	−1	0	0	0	0	−1	0

- The lines of the following table represent the elements of Ω_6,

1	1	1	1	1	1	0	0	0	0	1	1
0	0	1	1	1	1	0	0	1	0	0	1
1	0	0	1	1	1	0	0	1	1	0	0
1	1	0	0	1	1	1	0	0	0	0	1
1	1	1	0	0	1	1	0	0	1	0	0
1	1	1	1	0	0	1	1	0	0	0	0
0	−1	0	1	1	1	0	−1	0	0	0	1
1	0	−1	0	1	1	0	−1	0	1	0	0
1	1	0	−1	0	1	1	0	−1	0	0	0
1	1	1	0	−1	0	1	0	0	0	−1	0
0	−1	−1	0	1	1	0	0	1	0	−1	0
1	0	−1	−1	0	1	0	0	0	−1	0	1
1	1	0	−1	−1	0	0	−1	0	0	−1	0
0	−1	−1	−1	0	1	0	0	0	−1	−1	0
1	0	−1	−1	−1	0	0	−1	−1	0	0	0
0	−1	−1	−1	−1	0	0	0	0	0	0	0

Theorem 2. *For any positive integers r_i $(1 \leq i \leq s)$ and any integer $s \geq 2$, we have*

$$F_{r_1+r_2+\cdots+r_s+1} = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Omega_s} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}, \tag{3.1}$$

Proof. Let $P_{r_1+r_2+\cdots+r_s}$ be a path with $r_1+r_2+\cdots+r_s$ vertices. We subdivide $P_{r_1+r_2+\cdots+r_s}$ in consecutive blocs of paths P_{r_i} with r_i vertices $(1 \leq i \leq s)$, see Figure 1.

In one hand side, by Theorem 1, we have $Z(P_{r_1+\cdots+r_s}) = F_{r_1+\cdots+r_s+1}$. In the other hand $Z(P_{r_1+r_2+\cdots+r_s})$ is the number of independent edge subsets in $P_{r_1+r_2+\cdots+r_s}$. So, $Z(P_{r_1+r_2+\cdots+r_s}) = \sum_{i=0}^{s-1} |M_k|$ where M_k is the set of independent edge subsets I in $P_{r_1+r_2+\cdots+r_s}$ such that it exists exactly k edges between blocs of paths P_{r_i} $(1 \leq i \leq s)$ which belong to I.

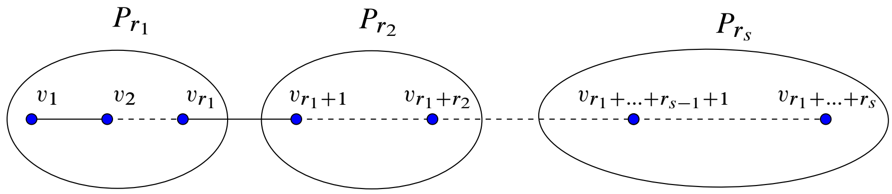

FIGURE 1. Path $P_{r_1+r_2+\ldots+r_s}$ divided in consecutive blocs of paths P_{r_i} with r_i vertices $(1 \le i \le s)$.

M_0 is the set of independent edge subsets I in $P_{r_1+r_2+\cdots+r_s}$ such that doesn't contain any edge between blocs of paths P_{r_i} $(1 \le i \le s)$, so all these independent edge subsets are in blocs P_{r_i} $(1 \le i \le s)$. Hence, $|M_0| = \prod_{i=1}^{s} F_{r_i+1}$.

M_1 is the set of independent edge subsets I in $P_{r_1+r_2+\cdots+r_s}$ such that it exists only one edge between blocs of paths P_{r_i} $(1 \le i \le s)$ which belong to I. Let H be a subset of M_1 containing the edge $v_{r_1+\cdots+r_k} v_{r_1+\cdots+r_k+1}$ $(1 \le k \le s-1)$ in all of its independent edge subsets. We contract the adjacent vertices $v_{r_1+\cdots+r_k}, v_{r_1+\cdots+r_k+1}$ in $P_{r_1+\cdots+r_s}$ into one vertex v' and $P_{r_1+\cdots+r_s-1}$ is a new path after contraction composed of consecutive blocs of paths $P_{r_1}, P_{r_2}, \ldots, P_{r_{k-1}}, P_{r_k-1}, v', P_{r_{k+1}-1}, P_{r_{k+2}}, \ldots, P_{r_s}$. A path $P_{r_1+\cdots+r_s-1}$ does not contain any edge between blocs which belong to independent edge subsets of H, so $|H| = F_{r_1+1} \times F_{r_2+1} \times \cdots \times F_{r_{k-1}+1} \times F_{r_k} \times F_2 \times F_{r_{k+1}} \times F_{r_{k+2}+1} \times \cdots \times F_{r_s+1}$. Thus, $|M_1| = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Delta_1} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}$ where Δ_1 is the set of $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)$ such that for $1 \le i \le s$, $\varepsilon_i \in \{0,1\}$ and $\varepsilon_1\varepsilon_2\cdots\varepsilon_s$ forms a sequence such that there is only one pair of zeros and this pair is of the form $(\varepsilon_l, \varepsilon_{l+1})$.

M_2 is the set of independent edge subsets I in $P_{r_1+r_2+\cdots+r_s}$ such that it exists exactly two edges between blocs of paths P_{r_i} $(1 \le i \le s)$ which belong to I. As for computing of $|M_1|$ and using the contraction method for the two edges between blocs of paths P_{r_i} $(1 \le i \le s)$, we have $|M_2| = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Delta_2} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}$ where Δ_2 is the set of $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s) \in \Omega_s$ such that for $1 \le i \le s$, $\varepsilon_i \in \{-1,0,1\}$ and $\varepsilon_1\varepsilon_2\cdots\varepsilon_s$ forms a sequence such that there is only one pair of zeros $\{\varepsilon_i, \varepsilon_{i+2}\}$ and $\varepsilon_{i+1} = -1$, or only two pairs of zeros $\{\varepsilon_i, \varepsilon_{i+1}\}, \{\varepsilon_{i+1+k}, \varepsilon_{i+2+k}\}$ $(1 \le k$ and $i+k+2 \le s)$ and $\varepsilon_l = 1$ for all $l \in \{1,2,\ldots,i-1,i+2,i+3,\ldots,i+k,i+k+3,\ldots,s\}$.

For M_k $(3 \le k \le s-2)$, using the contraction method for k edges between blocs of paths P_{r_i} $(1 \le i \le s)$, we have $|M_k| = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Delta_k} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}$ where $\Delta_k = \{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s) \in \Omega_s : for\ all\ s_i \in L_\varepsilon(1 \le i \le 2h(\varepsilon)), \sum_{l=1}^{h(\varepsilon)}(s_{2l} - s_{2l-1}) = k\}$ which represents all sequences of Ω_s such that the sum of the difference of the position of each pair of 0 is equal to k.

We finish by M_{s-1} which is the set of matchings I in $P_{r_1+r_2+\cdots+r_s}$ such that it exists exactly $s-1$ edges between blocs of paths P_{r_i} $(1 \le i \le s)$ which belong to

I. In this case, except the paths P_{r_1}, P_{r_s} that lose one vertex after a contraction all others paths P_{r_i} $(2 \le i \le s-1)$ lose two vertices after contraction method. Thus, $|M_{s-1}| = F_{r_1} F_{r_s} \prod_{i=2}^{s-1} F_{r_i - 1}$.

Note that $\{\Delta_k : 1 \le k \le s-1\}$ is a partition of Ω_s. Hence, the identity (3.1) holds. □

The following corollaries are the main results given by [8].

Corollary 1. *For any non-negative integers r and t, we have*

$$F_{r+t} = F_{r+1} F_t + F_r F_{t-1}. \tag{3.2}$$

Proof. From Theorem 2 with $s = 2$ and $\Omega_2 = \{(1,1), (0,0)\}$, we obtain the following identity $F_{r_1+r_2+1} = F_{r_1+1} F_{r_2+1} + F_{r_1} F_{r_2}$. We put $r = r_1$ and $t = r_2 + 1$ and we conclude. □

Corollary 2. *For any non-negative integers u, v and w, we have*

$$F_{u+v+w} = F_{u+1} F_{v+1} F_{w+1} + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1}.$$

Proof. From Theorem 2 with $s = 3$ and $\Omega_3 = \{(1,1,1), (1,0,0), (0,0,1), (0,-1,0)\}$, we obtain the following identity $F_{r_1+r_2+r_3+1} = F_{r_1+1} F_{r_2+1} F_{r_3+1} + F_{r_1+1} F_{r_2} F_{r_3} + F_{r_1} F_{r_2} F_{r_3+1} + F_{r_1} F_{r_2-1} F_{r_3}$. We put $u = r_1$, $v = r_2$ and $w = r_3 + 1$ and using $F_t = F_{t+1} - F_{t-1}$, we have :

$$\begin{aligned}
F_{u+v+w} &= F_{u+1} F_{v+1} F_w + F_{u+1} F_v F_{w-1} + F_u F_v F_w + F_u F_{v-1} F_{w-1} \\
&= F_{u+1} F_{v+1} (F_{w+1} - F_{w-1}) + F_{u+1} F_v F_{w-1} + F_u F_v F_w \\
&+ (F_{u+1} - F_{u-1}) F_{v-1} F_{w-1} \\
&= F_{u+1} F_{v+1} F_{w+1} - F_{u+1} F_{v+1} F_{w-1} + F_{u+1} F_v F_{w-1} + F_u F_v F_w \\
&+ F_{u+1} F_{v-1} F_{w-1} - F_{u-1} F_{v-1} F_{w-1} \\
&= F_{u+1} F_{v+1} F_{w+1} + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1} + F_{u+1} F_{v-1} F_{w-1} \\
&+ F_{u+1} F_v F_{w-1} - F_{u+1} F_{v+1} F_{w-1} \\
&= F_{u+1} F_{v+1} F_{w+1} + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1} \\
&+ F_{u+1} (F_{v-1} + F_v - F_{v+1}) F_{w-1} \\
&= F_{u+1} F_{v+1} F_{w+1} + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1}.
\end{aligned}$$

□

Corollary 3. *For any non-negative integers a, b, c and d, we have*

$$\begin{aligned}
&F_{a+b+c+d+1} \\
&= F_{a+1} F_{b+1} F_{c+1} F_{d+1} + F_a F_b F_c F_d + F_{a+1} F_b F_c F_{d+1} + F_{a+1} F_{b+1} \\
&\quad + F_c F_d + F_a F_{b-1} F_c F_{d+1} + F_a F_b F_{c+1} F_{d+1} \\
&\quad + F_{a+1} F_b F_{c-1} F_d + F_a F_{b-1} F_{c-1} F_d.
\end{aligned}$$

Proof. From Theorem 2, with $s = 4$ and $\Omega_4 = \{(1,1,1,1),(0,0,1,1),(1,0,0,1),(1,1,0,0),(0,0,0,0),(0,-1,0,1),(1,0,-1,0),(0,-1,-1,0)\}$, we obtain the identity. □

The following theorem is another identity of Fibonacci number which gives an equivalent of Theorem 2.

Theorem 3. *Let $s \geq 2$ be an integer. For any non-negative integer r_i $(1 \leq i \leq s)$, we have*

$$F_{\sum_{i=1}^{s} r_i+1} = F_{\sum_{i=1}^{s-1} r_i+1} F_{r_s+1} + \sum_{i=0}^{s-2} \left[\left(\prod_{j=1}^{i} F_{r_{s-j}-1} \right) F_{\sum_{j=1}^{s-i-2} r_j+1} F_{r_{s-i-1}} F_{r_s} \right].$$

Proof. As mentioned in Theorem 2, $F_{\sum_{i=1}^{s} r_i+1} = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Omega_s} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}$. Then $F_{\sum_{i=1}^{s} r_i+1} = c_1 + c_0$ where c_1 corresponds to the case $\varepsilon_s = 1$ and c_0 to the case $\varepsilon_s = 0$. That means to count $F_{\sum_{i=1}^{s} r_i+1}$ we have two cases.

Case 1. $\varepsilon_s = 1$. Then for all s-uplet $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s)$ we obtain

$$c_1 = F_{r_s+1} \left(\sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{s-1})\in\Omega_{s-1}} \prod_{i=1}^{s-1} F_{r_i+\varepsilon_i} \right),$$

so for $\varepsilon_s = 1$ we have $c_1 = F_{\sum_{i=1}^{s-1} r_i+1} F_{r_s+1}$.

Case 2. $\varepsilon_s = 0$. Let $\varepsilon_{s-i-1} = 0$ with i the smallest integer k, $0 \leq k \leq s-2$, such that $\varepsilon_{s-k-1} = 0$. So for $1 \leq j \leq i$ we have $\varepsilon_{s-j} = -1$. Hence,

$$\begin{aligned} c_0 &= \sum_{i=0}^{s-2} \left[\left(\prod_{j=1}^{i} F_{r_{s-j}-1} \right) F_{r_{s-i-1}} F_{r_s} \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{s-i-2})\in\Omega_{s-i-2}} \prod_{j=1}^{s-i-2} F_{r_j+\varepsilon_j} \right] \\ &= \sum_{i=0}^{s-2} \left[\left(\prod_{j=1}^{i} F_{r_{s-j}-1} \right) F_{\sum_{j=1}^{s-i-2} r_j+1} F_{r_{s-i-1}} F_{r_s} \right]. \end{aligned}$$

□

As an immediate consequence of Theorem 3 we have :

Corollary 4. *For any non-negative integers s and r, we have*

$$F_{sr+1} = F_{r+1} F_{(s-1)r+1} + F_r^2 \sum_{i=0}^{s-2} F_{r-1}^{i} F_{(s-i-2)r+1}.$$

Proof. Use Theorem 3 with $r_1 = r_2 = \cdots = r_s = r$. □

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