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RICCI COEFFICIENTS OF ROTATION OF GENERALIZED FINSLER SPACES

SVETISLAV M. MINČIĆ, MIĆA S. STANKOVIĆ, AND MILAN LJ. ZLATANOVIĆ

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Abstract. Generalized Finsler space GF_n is a differentiable N-dimensional manifold with non-symmetric basic tensor $g_{ij}(x,\dot{x})$ defined by condition (1.2). Using the basic tensor, by (1.4) a non-symmetric connection P^* is defined, and also four kinds of covariant derivative in the Rund's sense and five curvature tensors are obtained (Section 1).

In Section 2 two kinds of Ricci coefficients of rotation are defined and their properties are exposed. Also, integrability conditions of the equation expressing covariant derivatives of the congruence vector by means of coefficients of rotation, are obtained.

In Section 3 a geodesic mapping of two spaces GF_n and $G\overline{F}_n$ is defined and some it's properties are proved.

In Section 4 some invariants of such mappings in relation with the coefficients of rotation are studied.

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1. INTRODUCTION

The generalized Finsler space (GF_n) is a differentiable manifold with non-symmetric basic tensor field $g_{ij}(x^1,\dots,x^N,\dot{x}^1,\dots,\dot{x}^N) \equiv g_{ij}(x,\dot{x})$, where

$$g_{ij}(x,\dot{x}) \neq g_{ji}(x,\dot{x}), \quad (g = det(g_{ij}) \neq 0,\ \dot{x} = dx/dt). \tag{1.1}$$

Based on (1.1), the symmetric and anti-symmetric part of g_{ij} are defined

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}), \qquad g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}).$$

Following [18], in our notation, hold

$$a)\ g_{\underline{ij}}(x,\dot{x}) = \frac{1}{2}\frac{\partial^2 F^2(x,\dot{x})}{\partial\dot{x}^i\partial\dot{x}^j}, \qquad b)\ \frac{\partial g_{\underset{\vee}{ij}}}{\partial\dot{x}^k} = 0, \tag{1.2}$$

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where $F(x,\dot{x})$ is a metric function in GF_n, having the properties known from the theory of usual Finsler spaces (F_n). A lot of research papers (see for example [1–4,6,11,14–19,21,22]) are dedicated to the theory of Finsler spaces and their generalizations. The discussed structure is a particular case of the Eisenhart-Lagrange approach which is studied in [12] (Chap. 8).

The lowering and the raising of indices are defined by the tensors $g_{\underline{ij}}$ and h^{ij} respectively, where h^{ij} satisfy equation

$$g_{\underline{ij}}\,h^{jk}=\delta_i^k,\quad (\underline{g}=det(g_{\underline{ij}})\neq 0).$$

Generalized Cristoffel symbols of the 1^{st} and the 2^{nd} kind are defined:

$$\gamma_{i.jk}=\frac{1}{2}(g_{ji,k}-g_{jk,i}+g_{ik,j})\neq\gamma_{i.kj},$$
$$\gamma^i_{jk}=h^{ip}\gamma_{p.jk}=\frac{1}{2}h^{ip}(g_{jp,k}-g_{jk,p}+g_{pk,j})\neq\gamma^i_{kj},$$

where, e.g., $g_{ji,k}=\partial g_{ji}/\partial x^k$.

Then we have

$$\gamma^p_{jk}g_{\underline{ip}}=\gamma_{s.jk}h^{ps}g_{\underline{ip}}=\gamma_{s.jk}\delta^{*s}_i=\gamma_{i.jk}.$$

Introducing a tensor C_{ijk} like as at space F_n, we have

$$C_{ijk}(x,\dot{x})\overset{def}{=}\frac{1}{2}g_{ij,\dot{x}^k}=\frac{1}{2}g_{\underline{ij},\dot{x}^k}=\frac{1}{4}F^2_{\dot{x}^i\dot{x}^j\dot{x}^k}.$$

We see that C_{ijk} is symmetric in relation to each pair of indices. Also, we have

$$C^i_{jk}\overset{def}{=}h^{ip}C_{pjk}=h^{ip}C_{jpk}=h^{ip}C_{jkp}. \tag{1.3}$$

With help of coefficients

$$P^i_{jk}=\gamma^i_{jk}-C^i_{jp}\gamma^p_{sk}\dot{x}^{*s}\neq P^i_{kj}$$

one obtains coefficients of a non-symmetric affine connections in the Rund's sence (see [17,18]):

$$P^{*i}_{jk}=\gamma^i_{jk}-h^{iq}(C_{jqp}P^p_{ks}+C_{kqp}P^p_{js}-C_{jkp}P^p_{qs})\dot{x}^{*s}\neq P^{*i}_{kj}, \tag{1.4}$$

$$P^*_{i.jk}=P^{*r}_{jk}g_{\underline{ir}}=\gamma_{i.jk}-(C_{ijp}P^p_{ks}+C_{ikp}P^p_{js}-C_{jkp}P^p_{is})\dot{x}^{*s}\neq P^*_{i.kj}.$$

In GF_n we denote anti-symmetric and symmetric part for a connection P^* respectively:

$$a)\ T^{*i}_{jk}(x,\dot{x})=\frac{1}{2}(P^{*i}_{jk}-P^{*i}_{kj})=\frac{1}{2}(\gamma^i_{jk}-\gamma^i_{kj}),\quad b)\ P^{*i}_{\underline{jk}}=\frac{1}{2}(P^{*i}_{jk}+P^{*i}_{kj}),$$

where T^{*i}_{jk} is the **torsion tensor** of the connection P^{*i}_{jk}.

We define four kinds of covariant derivative of a tensor in the space GF_n. For example, for a tensor $a^i_j(x,\xi)$:

$$\begin{aligned} a^i_{j\,|m}(x,\xi) &= a^i_{j,m} + a^i_{j,\dot p}\xi^p_{,m} + P^{*i}_{pm}a^p_j - P^{*p}_{jm}a^i_p,\\ a^i_{j\,|m}(x,\xi) &= a^i_{j,m} + a^i_{j,\dot p}\xi^p_{,m} + P^{*i}_{mp}a^p_j - P^{*p}_{mj}a^i_p, \end{aligned} \tag{1.5}$$

$$\begin{aligned} a^i_{j\,|m}(x,\xi) &= a^i_{j,m} + a^i_{j,\dot p}\xi^p_{,m} + P^{*i}_{pm}a^p_j - P^{*p}_{mj}a^i_p,\\ a^i_{j\,|m}(x,\xi) &= a^i_{j,m} + a^i_{j,\dot p}\xi^p_{,m} + P^{*i}_{mp}a^p_j - P^{*p}_{jm}a^i_p, \end{aligned} \tag{1.6}$$

where $\xi(x)$ is an arbitrary tangent vector in the tangent space $T_n(x)$, and $a^i_{j,\dot p} = \partial a^i_j/\partial \dot x^p$.

In the work [10] we obtain 10 Ricci type identities in the general case for a tensor $a^{r_1\dots r_u}_{t_1\dots t_v}(x,\xi)$ and three curvature tensors of GF_n:

$$\underset{1}{\widetilde K}{}^i_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm}P^{*i}_{pn} - P^{*p}_{jn}P^{*i}_{pm} + P^{*p}_{jm,\dot s}\xi^{*s}_{,n} - P^{*p}_{jn,\dot s}\xi^{*s}_{,m}, \tag{1.7}$$

$$\underset{2}{\widetilde K}{}^i_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{mp} + P^{*p}_{mj,\dot s}\xi^{*s}_{,n} - P^{*p}_{nj,\dot s}\xi^{*s}_{,m}, \tag{1.8}$$

$$\begin{aligned} \underset{3}{\widetilde K}{}^i_{jmn} &= P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{nm}(P^{*i}_{pj} - P^{*i}_{jp})\\ &\quad + P^{*p}_{jm,\dot s}\xi^{*s}_{,n} - P^{*p}_{nj,\dot s}\xi^{*s}_{,m}. \end{aligned} \tag{1.9}$$

The magnitudes $\underset{t}{\widetilde K}{}^i_{jmn}$, $t = 1,2,3$ are tensors and we call them curvature tensors of the first, the second and the third kind respectively.

In the work [9] we use the third and the fourth kind of covariant derivative (1.4), and in that manner one gets 10 new Ricci type identities. In these identities appear the same quantities $\underset{t}{\widetilde K}{}^i_{jmn}$, $t = 1,2,3$; $\underset{t}{\widetilde A}{}^i_{jmn}$, $t = 1,\dots,15$, but in different distribution. Only in the last case appears a new curvature tensor (of the fourth kind) $\underset{4}{\widetilde K}$:

$$a^i_{j\,|m|n} - a^i_{j\,|n|m} = \underset{4}{\widetilde K}{}^i_{pmn}a^p_j + \underset{3}{\widetilde K}{}^p_{jnm}a^i_p\ ,$$

where

$$\begin{aligned} \underset{4}{\widetilde K}{}^i_{jmn} &= P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{jm,\dot s}\xi^{*s}_{,n} - P^{*p}_{nj,\dot s}\xi^{*s}_{,m}\\ &\quad + P^{*p}_{mn}(P^{*i}_{pj} - P^{*i}_{jp}). \end{aligned} \tag{1.10}$$

Denoting by semicolon (;) the covariant derivative with respect to the symmetric connection $\underset{jk}{P^{*i}}$, then using (1.3) we have

$$\underset{1}{\widetilde K}{}^i_{jmn} = \widetilde K^i_{jmn} + T^{*i}_{jm;n} - T^{*i}_{jn;m} + T^{*p}_{jm}T^{*i}_{pn} - T^{*p}_{jn}T^{*i}_{pm}, \tag{1.11}$$

$$\underset{2}{\widetilde{K}}{}^{i}_{jmn} = \widetilde{K}^{i}_{jmn} + T^{*i}_{mj;n} - T^{*i}_{nj;m} + T^{*p}_{mj}T^{*i}_{np} - T^{*p}_{nj}T^{i}_{mp}, \tag{1.12}$$

$$\underset{3}{\widetilde{K}}{}^{i}_{jmn} = \widetilde{K}^{i}_{jmn} + T^{*i}_{jm;n} - T^{*i}_{nj;m} + T^{*p}_{jm}T^{*i}_{np} - T^{*p}_{nj}T^{i}_{pm} + T^{*p}_{mn}T^{*i}_{jp}, \tag{1.13}$$

$$\underset{4}{\widetilde{K}}{}^{i}_{jmn} = \widetilde{K}^{i}_{jmn} + T^{*i}_{jm;n} - T^{*i}_{nj;m} + T^{*p}_{jm}T^{*i}_{np} - T^{*p}_{nj}T^{*i}_{pm} - T^{*p}_{mn}T^{*i}_{jp}, \tag{1.14}$$

where $\widetilde{K}^{i}_{jmn}$ is the curvature tensor formed by symmetric connection $P^{*i}_{\underline{jk}}$

$$\widetilde{K}^{i}_{jmn} = P^{*i}_{\underline{jm},n} - P^{*i}_{\underline{jn},m} + P^{*p}_{\underline{jm}}P^{*i}_{\underline{pn}} - P^{*p}_{\underline{jn}}P^{*i}_{\underline{pm}} + P^{*i}_{\underline{jm},\dot{s}}\xi^{*s}_{,n} - P^{*i}_{\underline{jn},\dot{s}}\xi^{*s}_{,m}.$$

In the work [7] we find five linearly independent curvature tensors $\underset{1}{\widetilde{K}}, \ldots, \underset{5}{\widetilde{K}}$, where $\underset{1}{\widetilde{K}}, \ldots, \underset{4}{\widetilde{K}}$ given by equations (1.7-1.10), (1.11-1.14), and

$$\underset{5}{\widetilde{K}}{}^{i}_{jmn} = \widetilde{K}^{i}_{jmn} + T^{*p}_{jm}T^{*i}_{pn} + T^{*p}_{jn}T^{*i}_{pm}, \tag{1.15}$$

$$\begin{aligned}\underset{5}{\widetilde{K}}{}^{i}_{jmn} = \frac{1}{2}(&P^{*i}_{\underline{jm},n} - P^{*i}_{\underline{jn},m} + P^{*p}_{jm}P^{*i}_{pn} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{jn}P^{*i}_{mp}\\ &- P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{\underline{jm},\dot{s}}\xi^{s}_{,n} - P^{*p}_{\underline{jn},\dot{s}}\xi^{s}_{,m}),\end{aligned} \tag{1.16}$$

where $\underline{jm}$ denotes symmetrization by indices j and m.

Applying two kinds of covariant differentiation (1.5), we get

$$\begin{aligned}a^{i}_{\underset{1}{j}|\underset{2}{m}}(x,\xi) &= a^{i}_{j,m} + a^{i}_{j,\dot{p}}\xi^{p}_{,m} + \underset{mp}{P^{*i}_{pm}}a^{p}_{j} - \underset{mj}{P^{*p}_{jm}}a^{i}_{p}\\ &= a^{i}_{j,m} + a^{i}_{j,\dot{p}}\xi^{p}_{,m} + (P^{*i}_{\underline{pm}} \pm T^{*i}_{pm})a^{p}_{j} - (P^{*p}_{\underline{jm}} \pm T^{*p}_{jm})a^{i}_{p}\\ &= a^{i}_{j;m} + \underset{mp}{T^{*i}_{pm}}a^{p}_{j} - \underset{mj}{T^{*p}_{jm}}a^{i}_{p}.\end{aligned}$$

2. Ricci coefficients of rotation in a generalized Finsler space

On Ricci coefficients of rotation in R_n the reader can find e.g. in [20], §54. On that matter in GR_n is written in [8]. On Ricci coefficients of rotation in F_n see [13].

2.1. *Congruence of curves and orthogonal ennuple*

Definition 1. A congruence of curves in a GF_n is such a family of curves that though each point of GF_n passes one curve of the family. N mutually orthogonal congruences of curves constitute an **orthogonal ennuple**. Instead of congruences of curves we shall sometimes speak about congruences of the corresponding tangent vectors.

If $\lambda_{(h)}$, $(h = 1,\dots,N)$ are unit tangent vectors of congruences of curves of an orthogonal ennuple, then in virtue of the previous definition

$$g_{\underline{ij}}(x,\dot{x})\lambda^i_{(h)}\lambda^j_{(k)} = e_{(k)}\delta_{hk},\ e_{(k)} = \pm 1, \tag{2.1}$$

or

$$e_{(k)}\lambda^i_{(h)}\lambda_{(k)i} = \delta_{hk}, \tag{2.2}$$

where δ_{hk} are Kronecker symbols. (of course, we do not suppose summation w.r.t. (k) in (2.1), (2.2) and in similar formulas later on.) The next theorem expresses the basic properties of orthogonal ennuples.

Theorem 1. *For the unit tangent vectors* $\lambda_{(h)}$, $(h = 1,\dots,N)$ *of congruences of curves of an orthogonal ennuple the relations*

$$\begin{aligned} &a)\ \sum_{k=1}^{N} e_{(k)}\lambda_{(k)i}\lambda^j_{(k)} = \delta^j_i, \quad b)\ \sum_{k=1}^{N} e_{(k)}\lambda_{(k)i}\lambda_{(k)j} = g_{\underline{ij}} \\ &c)\ \sum_{k=1}^{N} e_{(k)}\lambda^i_{(k)}\lambda^j_{(k)} = g^{\underline{ij}} \end{aligned} \tag{2.3}$$

are valid.

Proof. In the determinant $det(\lambda^i_{(k)})$, whose value is 1, we can regard $e_{(k)}\lambda_{(k)i}$ as cofactor of the element $\lambda^i_{(k)}$. Developing the determinant either by rows or by columns (2.3a) follows.
Further, we have

$$\sum_k e_{(k)}\lambda_{(k)i}\lambda_{(k)j} = g_{\underline{jl}}\sum_k e_{(k)}\lambda_{(k)i}\lambda^l_{(k)} = g_{\underline{jl}}\delta^l_i \Rightarrow (2.3b).$$

The equation (2.3c) can be obtained in the same manner. □

2.2. *Definition and basic properties of the coefficient of rotation*

Using the two kinds of covariant derivative (1.5) of a vector in a GF_n, we can define two kinds of coefficients of rotation, as two systems of invariants (for $\theta = 1,2$).

Definition 2. The invariants

$$\underset{\theta}{Y}_{(hkm)}(x,\dot{x}) \overset{def}{=} \lambda_{(h)i\,\underset{\theta}{|}\,j}\lambda^i_{(k)}\lambda^j_{(m)} = \lambda^i_{(h)\,\underset{\theta}{|}\,j}\lambda_{(k)i}\lambda^j_{(m)}, \quad \theta = 1,2. \tag{2.4}$$

are said to be coefficients of rotation of the given orthogonal ennuple.

From (1.5, 1.6) it is evidently that

$$\lambda^i_{(h)\,\underset{4}{|}\,j} = \lambda^i_{(h)\,\underset{2}{|}\,j}, \quad \lambda_{(h)i\,\underset{3}{|}\,j} = \lambda_{(h)i\,\underset{2}{|}\,j}, \quad \lambda^i_{(h)\,\underset{3}{|}\,j} = \lambda^i_{(h)\,\underset{1}{|}\,j}, \quad \lambda_{(h)i\,\underset{4}{|}\,j} = \lambda_{(h)i\,\underset{1}{|}\,j},$$

and it is easy to prove that there exist two kinds of coefficients of rotation in the non-symmetric case.

Theorem 2. *Both kinds of coefficients of rotation are antisymmetric in their first two indices, i.e.*

$$Y_{\theta}(hkm) = -Y_{\theta}(khm) \Rightarrow Y_{\theta}(hkm) = 0. \tag{2.5}$$

Proof. By covariant differentation we get from (2.2) the relation

$$e_{(k)}(\lambda^{i}_{(h)\underset{\theta}{|}j}\lambda_{(k)i} + \lambda^{i}_{(h)}\lambda_{(k)i\underset{\theta}{|}j}) = 0,$$

from where, transvecting by $\lambda^{j}_{(m)}$,

$$e_{(k)}(\lambda^{i}_{(h)\underset{\theta}{|}j}\lambda_{(k)i}\lambda^{j}_{(m)} + \lambda^{i}_{(h)}\lambda_{(k)i\underset{\theta}{|}j}\lambda^{j}_{(m)}) \underset{(2.4)}{=} e_{(k)}(Y_{\theta}(hkm) + Y_{\theta}(khm)) = 0$$

that is

$$Y_{\theta}(hkm) + Y_{\theta}(khm) = 0 \Rightarrow (2.5).$$

□

Theorem 3. *If*

$$Y_{(hkm)} = \lambda_{(h)i;j}\lambda^{i}_{(k)}\lambda^{j}_{(m)} \tag{2.6}$$

are coefficients of rotation in the associated Finsler space F_n (see [13]), then

$$Y_{\theta}(hkm) = Y_{(hkm)} + (-1)^{\theta}T^{*p}_{ij}\lambda_{(h)p}\lambda^{i}_{(k)}\lambda^{j}_{(m)}, \quad \theta = 1,2, \tag{2.7}$$

$$Y_{(hkm)} = (Y_{1}(hkm) + Y_{2}(hkm))/2,$$

$$Y_{\theta}(hkh) = Y(hkh), \quad Y_{\theta}(hhk) = Y(hhk), \quad Y_{\theta}(hkk) = Y(hkk), \quad \theta = 1,2. \tag{2.8}$$

Proof. In virtue of (2.4) and

$$\lambda_{i\underset{\theta}{|}n} = \lambda_{i;n} + (-1)^{\theta}T^{*p}_{in}\lambda_{p}, \quad \theta = 1,2,$$

we have

$$Y_{\theta}(hkm) = (\lambda_{(h)i;j} + (-1)^{\theta}T^{*p}_{ij}\lambda_{(h)p})\lambda^{i}_{(k)}\lambda^{j}_{(m)} \underset{(2.6)}{\Rightarrow} (2.7) \quad \theta = 1,2.$$

In virtue of (2.7) for two indices there follows (2.8) because for example (for $h = m$) is

$$\begin{aligned} T^{*p}_{ij}\lambda_{(h)p}\lambda^{i}_{(k)}\lambda^{j}_{(m)} = T^{*}_{p.ij}\lambda^{p}_{(h)}\lambda^{i}_{(k)}\lambda^{j}_{(h)} &= T^{*}_{j.ip}\lambda^{j}_{(h)}\lambda^{i}_{(k)}\lambda^{p}_{(h)} \\ &= -T^{*}_{p.ij}\lambda^{p}_{(h)}\lambda^{i}_{(k)}\lambda^{j}_{(h)} = 0, \end{aligned}$$

where $T^{*}_{p.ij} = T^{*s}_{ij}g_{\underline{sp}}$. Here we applied the fact that $T^{*}_{p.ij}$ is antisymmetric in all pairs of indices. □

2.3. *Expression of the derivative of the vectors of a congruence by the coefficients of rotation*

Theorem 4. *In GF_n the relation*

$$\begin{aligned} a)\ \lambda_{(h)i\underset{\theta}{|}j} &= \sum_{k,m=1}^{N} e_{(k)}e_{(m)}Y_{\theta}(hkm)\lambda_{(k)i}\lambda_{(m)j}, \\ b)\ \lambda^{i}_{(h)\underset{\theta}{|}j} &= \sum_{k,m=1}^{N} e_{(k)}e_{(m)}Y_{\theta}(hkm)\lambda^{i}_{(k)}\lambda_{(m)j} \end{aligned} \tag{2.9}$$

are valid.

Proof. *a)* Multiplying the relation (2.4) by $e_{(k)}e_{(m)}\lambda_{(k)p}\lambda_{(m)q}$ and summing with respect to m,k get

$$\begin{aligned} \sum_{k,m} e_{(k)}e_{(m)}Y_{\theta}(hkm)\lambda_{(k)p}\lambda_{(m)q} &= \sum_{k,m}\lambda_{(h)i\underset{\theta}{|}j}\lambda^{i}_{(k)}\lambda^{j}_{(m)}e_{(k)}e_{(m)}\lambda_{(k)p}\lambda_{(m)q} \\ &= \lambda_{(h)i\underset{\theta}{|}j}\{\sum_{k} e_{(k)}\lambda_{(k)p}\lambda^{i}_{(k)}\}\cdot\{\sum_{m} e_{(m)}\lambda_{(m)q}\lambda^{j}_{(m)}\} \\ &\underset{(2.3a)}{=} \lambda_{(h)i\underset{\theta}{|}j}\delta^{i}_{p}\delta^{j}_{q} = \lambda_{(h)p\underset{\theta}{|}q}. \end{aligned}$$

□

Theorem 5. *The covariant derivatives of vectors $\lambda_{(h)i}$ and $\lambda^{i}_{(h)}$ in the direction of the vector $\lambda^{j}_{(p)}$ can be expressed by the coefficients of rotation as linear combination of the vectors of the ennuple as follows:*

$$a)\ \lambda_{(h)i\underset{\theta}{|}j}\lambda^{j}_{(p)} = \sum_{k} e_{(k)}Y_{\theta}(hkp)\lambda_{(k)i} \quad b)\ \lambda^{i}_{(h)\underset{\theta}{|}j}\lambda^{j}_{(p)} = \sum_{k} e_{(k)}Y_{\theta}(hkp)\lambda^{i}_{(k)} \tag{2.10}$$

Proof. Transvecting the equation (2.9) by $\lambda^{j}_{(p)}$, we obtain

$$\begin{aligned} a)\ \lambda_{(h)i\underset{\theta}{|}j}\lambda^{j}_{(p)} &= \sum_{k,m} e_{(k)}e_{(m)}Y_{\theta}(hkp)\lambda_{(k)i}\lambda_{(m)j}\lambda^{j}_{(p)} \\ &= \sum_{k,m} e_{(k)}Y_{\theta}(hkp)\lambda_{(k)i}\delta_{mp} = \sum_{k} e_{(k)}Y_{\theta}(hkp)\lambda_{(k)i} \Rightarrow (2.10). \end{aligned}$$

□

2.4. *Integrability conditions of the equation (2.9)*

The relation (2.9) is a partial differential equation with respect to the unknown functions $\lambda_{(h)i}$. Now we are going to examine its integrability condition.

In [10] we have obtained 10 Ricci-type identities in GF_n. In three of these identities appear the curvature tensors $\underset{1}{\widetilde{K}}$, $\underset{2}{\widetilde{K}}$, $\underset{3}{\widetilde{K}}$, and in the others appear the quantities $\underset{1}{\widetilde{A}},\ldots,\underset{15}{\widetilde{A}}$ which have the form and the role of the curvature tensors, but they are not tensors. In [5] we obtained combined Ricci-type identities, in which appear "derived" curvature tensors $\underset{1}{\widetilde{K}}{}^*,\ldots,\underset{8}{\widetilde{K}}{}^*$. In [7] is proved that only five are independent among the mentioned curvature tensors, for example $\underset{1}{\widetilde{K}}$, $\underset{2}{\widetilde{K}}$, $\underset{3}{\widetilde{K}}$, $\underset{4}{\widetilde{K}}$, $\underset{5}{\widetilde{K}}$ while the others are linear combinations of these five tensors and $\widetilde{K}$. We shall use further those of the Ricci-type identities in which appear the above tensors (the tensor $\underset{1}{\widetilde{K}}{}^*$ is linear combination of $\underset{1}{\widetilde{K}}$, $\underset{2}{\widetilde{K}}$, while the tensor $\underset{2}{\widetilde{K}}{}^*$ does not appear in identities which we need.)

Theorem 6. *In GF_n the first two integrability conditions ($\theta = 1,2$) of equations (2.9) are*

$$
\begin{aligned}
&(-\underset{\theta}{\widetilde{K}}{}^s_{ijr}\lambda_{(h)s} + 2(-1)^\theta T^{*s}_{jr}\lambda_{(h)i\,|\,s}\underset{\theta}{)}\lambda^i_{(p)}\lambda^j_{(q)}\lambda^r_{(t)} = (\underset{\theta}{Y}_{(hpq),j} + \underset{\theta}{Y}_{(hpq),\dot{s}}\dot{\xi}^s_j)\lambda^j_{(t)}\\
&-(\underset{\theta}{Y}_{(hpt),j} + \underset{\theta}{Y}_{(hpt),\dot{s}}\dot{\xi}^s_j)\lambda^j_{(q)} + \sum_{k=1}^{N} e_{(k)}\{\underset{\theta}{Y}_{(hkq)}\underset{\theta}{Y}_{(kpt)} - \underset{\theta}{Y}_{(hkt)}\underset{\theta}{Y}_{(kpq)} \qquad (2.11)\\
&+\underset{\theta}{Y}_{(hpk)}[\underset{\theta}{Y}_{(kqt)} - \underset{\theta}{Y}_{(ktq)}]\}, \quad \theta = 1,2
\end{aligned}
$$

where $\underset{1}{\widetilde{K}}$, $\underset{2}{\widetilde{K}}$ are given by (1.7).

Proof. Applying the Ricci-type identities from [10], we have

$$\lambda_{(h)i\,|\,\underset{\theta}{jr}} - \lambda_{(h)i\,|\,\underset{\theta}{rj}} = -\underset{\theta}{\widetilde{K}}{}^s_{ijr}\lambda_{(h)s} + 2(-1)^\theta T^{*s}_{jr}\lambda_{(h)i\,|\,\underset{\theta}{s}}, \quad \theta = 1,2,$$

By repeated differentiation of (2.9) one can form the difference on the left side of this equation, and then (2.11) easily follows. □

Theorem 7. *The third integrability condition of the equation (2.9) in GF_n is*

$$
\begin{aligned}
&-\underset{3}{\widetilde{K}}{}^s_{ijr}\lambda_{(h)s}\lambda^i_{(p)}\lambda^j_{(q)}\lambda^r_{(t)} = (\underset{1}{Y}_{(hpq),j} + \underset{1}{Y}_{(hpq)\dot{s}}\dot{\xi}^s_j)\lambda^j_{(t)}\\
&-(\underset{2}{Y}_{(hpt),j} + \underset{2}{Y}_{(hpt)\dot{s}}\dot{\xi}^s_j)\lambda^j_{(q)} + \sum_{k=1}^{N} e_{(k)}[\underset{1}{Y}_{(hkq)}\underset{2}{Y}_{(kpt)} - \underset{2}{Y}_{(hkt)}\underset{1}{Y}_{(kpq)} \qquad (2.12)\\
&+\underset{1}{Y}_{(hpk)}\underset{2}{Y}_{(kqt)} - \underset{2}{Y}_{(hpk)}\underset{1}{Y}_{(ktq)}].
\end{aligned}
$$

where $\underset{3}{\widetilde{K}}$ is given by (1.9).

Proof. Applying the corresponding identity from [10] we get

$$\lambda_{(h)i\underset{1}{|}j\underset{2}{|}r}-\lambda_{(h)i\underset{2}{|}r\underset{1}{|}j}=-\underset{3}{\widetilde{K}}{}^{s}_{ijr}\lambda_{(h)s}.$$

Further, we use (2.9) to form the difference on the left side. □

Theorem 8. *The fourth integrability condition of the equation* (2.9) *is*

$$\begin{aligned}&\underset{4}{\widetilde{K}}{}^{i}_{sjr}\lambda^{s}_{(h)}\lambda_{(p)i}\lambda^{j}_{(q)}\lambda^{r}_{(t)}=(\underset{1}{Y}_{(hpq),j}+\underset{1}{Y}_{(hpq)s}\dot{\xi}^{s}_{j})\lambda^{j}_{(t)}\\&-(\underset{2}{Y}_{(hpt),j}+\underset{2}{Y}_{(hpt)s}\dot{\xi}^{s}_{j})\lambda^{j}_{(q)}+\sum_{k=1}^{N}e_{(k)}[\underset{1}{Y}_{(hkq)}\underset{2}{Y}_{(kpt)}+\underset{1}{Y}_{(hpk)}\underset{1}{Y}_{(kqt)}\\&-\underset{2}{Y}_{(hkt)}\underset{1}{Y}_{(kpq)}-\underset{2}{Y}_{(hpk)}\underset{2}{Y}_{(ktq)}].\end{aligned}\tag{2.13}$$

where $\underset{4}{\widetilde{K}}$ *is given by* (1.10).

Proof. Using by equations (1.10) we see that

$$\lambda^{i}_{(h)\underset{3}{|}j\underset{4}{|}r}-\lambda^{i}_{(h)\underset{4}{|}r\underset{3}{|}j}=\underset{4}{\widetilde{K}}{}^{i}_{sjr}\lambda^{s}_{(h)},$$

and the use of (2.9) yields the integrability condition (2.13). □

Theorem 9. *The fifth integrability condition of the equation* (2.9) *is*

$$\begin{aligned}&\underset{2}{\widetilde{K}}{}^{*i}_{sjr}\lambda^{s}_{(h)}\lambda_{(p)i}\lambda^{j}_{(q)}\lambda^{r}_{(t)}=(\underset{1}{Y}_{(hp[q),j}+\underset{1}{Y}_{(hp[q)s}\dot{\xi}^{s}_{j})\lambda^{j}_{(t)]}\\&-(\underset{2}{Y}_{(hp[q),j}+\underset{2}{Y}_{(hp[q)s}\dot{\xi}^{s}_{j})\lambda^{j}_{(t)]}+\sum_{k}e_{(k)}[\underset{1}{Y}_{(hkq)}\underset{1}{Y}_{(kpt)}+\underset{1}{Y}_{(hpk)}\underset{1}{Y}_{(kqt)}\\&-\underset{1}{Y}_{(hkt)}\underset{2}{Y}_{(kpq)}-\underset{1}{Y}_{(hkt)}\underset{2}{Y}_{(kpq)}+\underset{2}{Y}_{(hkq)}\underset{2}{Y}_{(kpt)}+\underset{2}{Y}_{(hpk)}\underset{2}{Y}_{(kqt)}\\&-\underset{2}{Y}_{(hkt)}\underset{1}{Y}_{(kpq)}-\underset{2}{Y}_{(hkt)}\underset{1}{Y}_{(kpq)}],\end{aligned}$$

where $[qt]$ *we denoted symmetrization without division by indices* q *and* t *and* $\underset{2}{\widetilde{K}}{}^{*i}_{sjr}\equiv\underset{5}{\widetilde{K}}{}^{i}_{sjr}$ *given by* (1.16).

Proof. We use that Ricci-type identity in which the curvature $\underset{2}{\widetilde{K}}{}^{*}$ appears. From corresponding identities in [5] we have

$$\lambda^{i}_{(h)\underset{1}{|}jr}-\lambda^{i}_{(h)\underset{1}{|}r\underset{2}{|}j}+\lambda^{i}_{(h)\underset{2}{|}jr}-\lambda^{i}_{(h)\underset{2}{|}r\underset{1}{|}j}=2\underset{2}{\widetilde{K}}{}^{*i}_{sjr}\lambda^{s}_{(h)},$$

and then use (2.9). □

3. GEODESIC MAPPINGS

Definition 3. Geodesic in GF_n is given by

$$\frac{d^2x^i}{ds^2} + P^{*i}_{jk}(x, \frac{dx}{ds})\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{3.1}$$

Consider two N-dimensional spaces of non-symmetrical affine connection: GF_n and $G\overline{F}_n$. So, if connection coefficients of these spaces are respectively P^{*i}_{jk} and $\overline{P}^{*i}_{jk}$, we suppose that in general the symmetry with respect to indices j,k is not in force.

One says that reciprocal one valued mapping $f : GF_n \to G\overline{F}_n$ is *geodesic*, if geodesics of the space GF_n pass to geodesics of the space $G\overline{F}_n$. We can consider these spaces in the common by this mapping system of local coordinates, i.e. for $f : M \mapsto \overline{M}$ we have $M(x^1,\dots,x^n,\dot{x}^1,\dots,\dot{x}^n) \equiv M(x,\dot{x})$ and $\overline{M}(x^1,\dots,x^n,x^1,\dots,x^n,\dot{x}^1,\dots,\dot{x}^n) \equiv \overline{M}(x,\dot{x})$, where $M \in GF_n$, $\overline{M} \in G\overline{F}_n$. In the corresponding points $M(x,\dot{x})$ and $\overline{M}(x,\dot{x})$ we can put

$$\overline{P}^{*i}_{jk} = P^{*i}_{jk} + D^i_{jk}, \qquad (i,j,k = 1,\dots,N), \tag{3.2}$$

where $D^i_{jk}(x)$ is **the deformation tensor** of the connection P^* of GF_n according to the mapping $f : GF_n \to G\overline{F}_n$.

Definition 4. The curve

$$l : x^i = x^i(t)$$

is **geodesic** of GF_n if and only if is:

$$\ddot{x}^i + P^{*i}_{pq}\dot{x}^p\dot{x}^q = \rho(t)\dot{x}^i(t), \tag{3.3}$$

where $\rho(t)$ is an invariant and t an arbitrary parameter.

If $f : l \to \overline{l}$, then in the common by the mapping f coordinates $\overline{x}^i \equiv x^i$ it is $\dot{\overline{x}}^i = d\overline{x}^i/dt = \dot{x}^i$, and $\overline{l}$ is geodesic in $G\overline{F}_n$ too, from where we get

$$\ddot{x}^i + \overline{P}^{*i}_{pq}\dot{x}^p\dot{x}^q = \overline{\rho}(t)\dot{x}^i(t) \tag{3.4}$$

Subtracting (3.3) and (3.4), we obtain

$$(\overline{P}^{*i}_{pq} - P^{*i}_{pq})\dot{x}^p\dot{x}^q = (\overline{\rho}(t) - \rho(t))\dot{x}^i(t),$$

and, because of (3.2):

$$D^i_{pq}\dot{x}^p\dot{x}^q = 2\psi(t)\dot{x}^i(t), \tag{3.5}$$

where $\psi(t) = (\overline{\rho}(t) - \rho(t))/2$. Denoting by $D^i_{\underline{jk}}$, $D^i_{\underset{\vee}{jk}}$ the symmetric and antisymmetric part of D^i_{jk} respectively, we get

$$D^i_{jk} = D^i_{\underline{jk}} + D^i_{\underset{\vee}{jk}},$$

and (3.5) reduces to

$$D^i_{\underline{pq}}\dot{x}^p\dot{x}^q = 2\psi(t)\dot{x}^i(t). \tag{3.6}$$

As in the case of a symmetric connection (see for ex. [4]) one concludes that

$$\psi(t) = \psi_p(x^1(t),\dots,x^n(t),\dot{x}^1(t),\dots,\dot{x}^n(t))\dot{x}^p(t),$$

and from (3.6):

$$D^i_{\underline{pq}}\dot{x}^p\dot{x}^q = 2\psi_p\dot{x}^p\dot{x}^i = \psi_p\dot{x}^p\dot{x}^q\delta^i_q + \psi_q\dot{x}^p\dot{x}^q\delta^i_p = (\psi_p\delta^i_q + \psi_q\delta^i_p)\dot{x}^p\dot{x}^q,$$

wherefrom

$$D^i_{\underline{jk}} = \delta^i_j\psi_k + \delta^i_k\psi_j. \tag{3.7}$$

Denoting also

$$D^i_{\underset{\vee}{jk}} = \xi^i_{jk} = -\xi^i_{kj},$$

by substitution in (3.2) we obtain

$$\overline{P}^{*i}_{jk} = P^{*i}_{jk} + \delta^i_j\psi_k + \delta^i_k\psi_j + \xi^i_{jk}, \tag{3.8}$$

and the deformation tensor is

$$D^i_{jk} = \delta^i_j\psi_k + \delta^i_k\psi_j + \xi^i_{jk}. \tag{3.9}$$

So, the condition (3.8) is necessary that the mapping f be geodesic. It is easily to prove that this condition is sufficient too, and we have

Theorem 10. *A necessary and sufficient condition that the mapping $f : GF_n \to G\overline{F}_n$ be geodesic is the deformation tensor D^i_{jk} from (3.2) at the mapping f to has the form (3.9), where $\psi_j(x,\dot{x})$ is a covariant vector, and $\xi^i_{jk}(x,\dot{x})$ an antisymmetric tensor.*

For $k = i$, we obtain from (3.7):

$$D^i_{\underline{ji}} = \delta^i_j\psi_i + \delta^i_i\psi_j = \psi_j + N\psi_j,$$

wherefrom

$$\psi_i = \frac{1}{N+1}D^p_{\underline{ip}}, \tag{3.10}$$

which, by substitution in (3.8), gives

$$\overline{P}^{*i}_{jk} = P^{*i}_{jk} + \frac{1}{N+1}(\delta^i_j D^p_{\underline{kp}}(x,\dot{x}) + \delta^i_k D^p_{\underline{jp}}(x,\dot{x})) + D^i_{\underset{\vee}{jk}}(x,\dot{x}), \tag{3.11}$$

where $D^i_{jk}(x)$ is the deformation tensor.

On the base of above explained, we get

Theorem 11. *Let a space GF_n be given, i.e. on a differentiable manifold M_n let be defined non-symmetric connection coefficients $P^{*i}_{jk}(x,\dot{x})$. If on M_n is given a tensor $D^i_{jk}(x,\dot{x})$ too and we determine $\overline{P}^{*i}_{jk}(x,\dot{x})$ according to (3.11), then on M_n will be defined a space $G\overline{F}_n$, with connection coefficients $\overline{P}^{*i}_{jk}$, and then GF_n and $G\overline{F}_n$ have common geodesics. We obtain the same result (on the base of (3.8) by choice of a vector $\psi_i(x,\dot{x})$ and antisymmetric tensor $\xi^i_{jk}(x,\dot{x}) = D^i_{\underset{\vee}{jk}}(x,\dot{x})$.*

The question forces itself: Is it possible a geodesic mapping of a space GF_n with a non-symmetric affine connection on to a space $\overline{F}_n$ with a symmetric affine connection? It is easy to see that the next theorem is valid:

Theorem 12. *A necessary and sufficient condition that a mapping $f : GF_n \to \overline{F}_n$ of a non-symmetric affine connection space GF_n onto a symmetric affine connection space $\overline{F}_n$ be geodesic, is*

$$D^i_{\underset{\vee}{jk}}(x,\dot{x}) = -P^{*i}_{\underset{\vee}{jk}}(x,\dot{x}),$$

where $D^i_{\underset{\vee}{jk}}(x,\dot{x})$, $P^{*i}_{\underset{\vee}{jk}}(x,\dot{x})$, are antisymmetric parts of the deformation tensor and connection coefficients of the GF_n respectively.

Remark 1. It is easy to check that a set of geodesic mappings of a space GF_n makes a group.

4. Some projective invariants of geodesic mappings

Putting D into (3.11) in accordance with (3.2) we get

$$\overline{P}^{*i}_{jk} - \overline{P}^{*i}_{\underset{\vee}{jk}} - \frac{1}{N+1}(\delta^i_j \overline{P}^{*p}_{\underline{kp}} + \delta^i_k \overline{P}^{*p}_{\underline{jp}}) = P^{*i}_{jk} - P^{*i}_{\underset{\vee}{jk}} - \frac{1}{N+1}(\delta^i_j P^{*p}_{\underline{kp}} + \delta^i_k P^{*p}_{\underline{jp}}),$$

Denoting

$$\mathcal{T}^i_{jk} = P^{*i}_{\underline{jk}} - \frac{1}{N+1}(\delta^i_j P^{*p}_{\underline{kp}} + \delta^i_k P^{*p}_{\underline{jp}}) = \mathcal{T}^i_{kj}, \tag{4.1}$$

we see that

$$\overline{\mathcal{T}}^i_{jk} = \mathcal{T}^i_{jk}. \tag{4.2}$$

The magnitudes $\mathcal{T}^i_{jk}$ we call **generalized Thomas's projective parameters** at the mapping $f : GF_n \to G\overline{F}_n$. Accordingly, these magnitudes are invariant at a geodesic mapping. Starting from (4.1, 4.2), one obtains (3.11), and we conclude that the next theorem is valid:

Theorem 13. *A necessary and sufficient condition that a mapping $f : GF_n \to G\overline{F}_n$ be geodesic is that generalized Thomas's projective parameters (4.1) are invariant, that is (4.2) to be valid.*

Theorem 14. *If $\lambda^i_{(h)}(x)$ and $\lambda_{(h)i}(x)$ are the contravariant, respectively covariant components of an orthogonal ennuple, then the following geometric entities are invariant under the geodesic mapping:*

$$\underset{\theta}{\mathcal{M}}{}^i_k(x,\dot{x}) = \lambda^i_{(h)|\underset{\theta}{}}(x) - \frac{1}{n+1}\lambda^p_{(h)}(x)\sum_h e_{(h)}\lambda_{(h)q}\delta^q_{(p}\lambda_{(h)|\underset{\theta}{k})},\ \theta = 1,2, \tag{4.3}$$

and

$$\underset{\theta}{\mathcal{M}}{}^*{}_k(x,\dot{x}) = \sum_h e_{(h)}\lambda_{(h)p}\lambda^p_{(h)|\underset{\theta}{k}} - D^q_{\underline{kq}},\ \theta = 1,2, \tag{4.4}$$

where $(p\ldots k)$ denotes symmetrization by respect of indices p and k.

Proof. If we denote by $\lambda^i_{(h)|\underset{\theta}{\overline{k}}}$, $\theta = 1,2$ the covariant derivative wrt the connection $\overline{P}^*$ in sense of (1), we have

$$\lambda^i_{(h)|\underset{1}{\overline{k}}} = \lambda^i_{(h),k} + \lambda^p_{(h)}\overline{P}^{*i}_{\underset{kp}{pk}}. \tag{4.5}$$

Using (3.8, 4.1) we have

$$\begin{aligned}
\lambda^i_{(h)|\underset{1}{\overline{k}}} - \lambda^i_{(h)|\underset{1}{k}} &= \lambda^p_{(h)}(\delta^i_p\psi_k + \delta^i_k\psi_p + \xi^i_{pk}) \\
\lambda^i_{(h)|\underset{2}{\overline{k}}} - \lambda^i_{(a)|\underset{2}{k}} &= \lambda^p_{(h)}(\delta^i_k\psi_p + \delta^i_p\psi_k + \xi^i_{kp}).
\end{aligned} \tag{4.6}$$

Multiplying (4.6) by $\lambda_{(h)i}$ and summing with respect to indices h and using orthogonality condition, we have

$$\begin{aligned}
\sum_h e_{(h)}\lambda_{(h)p}(\lambda^p_{(h)|\underset{1}{\overline{k}}} - \lambda^p_{(h)|\underset{1}{k}}) &= (N+1)\psi_k \\
\sum_h e_{(h)}\lambda_{(h)p}(\lambda^p_{(h)|\underset{2}{\overline{k}}} - \lambda^p_{(h)|\underset{2}{k}}) &= (N+1)\psi_k.
\end{aligned} \tag{4.7}$$

Eliminating the ψ_k from (4.6) and (4.7) we get

$$\lambda^i_{(h)|\underset{1}{\overline{k}}} - \lambda^i_{(h)|\underset{1}{k}} = \frac{1}{n+1}\lambda^p_{(h)}\left(\sum_h e_{(h)}\lambda_{(h)q}\delta^q_{(p}\lambda_{(h)|\underset{1}{\overline{k}})} - \sum_h e_{(h)}\lambda_{(h)q}\delta^q_{(p}\lambda_{(h)|\underset{1}{k})}\right).$$

With help (3.10) we get (4.3). □

Theorem 15. *When GF_n and $G\overline{F}_n$ are in geodesic correspondence, we have the following projective invariant entities:*

$$\underset{\theta}{S}{}_{hkm}(x,\dot{x}) = \underset{\theta}{Y}{}_{(hkm)} - \frac{1}{n+1}(\delta_{hk}\lambda^k_{(m)}P^{*i}_{ki} + \delta_{km}\lambda^j_{(h)}P^{*i}_{ji}),\ \theta = 1,2,$$

where $\underset{\theta}{Y}_{(hkm)}$ $\theta = 1,2$, *are Ricci coefficients of rotation.*

Proof. Using (4.5) we get

$$\lambda^{i}_{(h)\underset{1}{|}\overline{k}\underset{2}{}} - \lambda^{i}_{(h)\underset{1}{|}k\underset{2}{}} = \frac{1}{N+1}\lambda^{j}_{(h)}(\delta^{i}_{j}D^{p}_{\underline{kp}}(x,\dot{x}) + \delta^{i}_{k}D^{p}_{\underline{jp}}(x,\dot{x})). \tag{4.8}$$

Multiplying (4.8) by $\lambda_{(k)i}\lambda^{k}_{(m)}$ and using orthogonality condition, we have

$$\begin{aligned}&\underset{1}{\overset{2}{\overline{Y}}}_{hkm} - \frac{1}{n+1}(\delta_{hk}\lambda^{k}_{(m)}\overline{P}^{*i}_{ki} + \delta_{km}\lambda^{q}_{(h)}\overline{P}^{*p}_{qp})\\ &= \underset{1}{\overset{2}{Y}}_{hkm} - \frac{1}{n+1}(\delta_{hk}\lambda^{k}_{(m)}P^{*p}_{kp} + \delta_{km}\lambda^{j}_{(h)}P^{*p}_{jp}),\end{aligned}$$

where the projectively transformed Ricci coefficients of rotation are given by

$$\underset{\theta}{\overline{Y}}_{hkm} = \lambda_{(h)p\underset{\theta}{|}\overline{q}}\lambda^{p}_{(k)}\lambda^{q}_{(m)} = \lambda^{p}_{(h)\underset{\theta}{|}\overline{q}}\lambda_{(k)p}\lambda^{q}_{(m)}, \qquad \theta = 1,2.$$

□

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Authors' addresses

Svetislav M. Minčić
University of Niš, Faculty of Science and Mathematics, Višegradska 33, 18000 Niš, Serbia
E-mail address: svetislavmincic@yahoo.com

Mića S. Stanković
University of Niš, Faculty of Science and Mathematics, Višegradska 33, 18000 Niš, Serbia
E-mail address: stmica@ptt.rs

Milan Lj. Zlatanović
University of Niš, Faculty of Science and Mathematics, Višegradska 33, 18000 Niš, Serbia
E-mail address: zlatmilan@yahoo.com