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REFINEMENTS OF THE LOWER BOUNDS OF JENSEN'S FUNCTIONAL REVISITED

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Abstract. The functional defined as the difference of the right-hand and the left-hand side of Jensen's inequality is called Jensen's functional. The aim of this paper is to study its lower bounds. More precisely, some refinements of its lower bounds are recaptured under more relaxed conditions on the n-tuple $\mathbf{x}$.

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1. Introduction

Jensen's inequality is possibly one of the most famous and most explored inequalities. It has been generalized in numerous directions and it has found its application in various areas; see for example [8] and the references therein. Classical Jensen's inequality states as follows.

Theorem 1 (Jensen, 1905. [8])**.** *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. Let $n \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ be a positive n-tuple, that is, such that $p_i > 0$ for $i = 1, \ldots, n$. Then*

$$f\left(\frac{1}{P_1^n}\sum_{i=1}^n p_i x_i\right) \leq \frac{1}{P_1^n}\sum_{i=1}^n p_i f(x_i),$$

where $P_1^n = \sum_{i=1}^n p_i$. If f is strictly convex, then the inequality is strict, unless $x_1 = x_2 = \ldots = x_n$.

The functional defined as the difference of the right-hand and the left-hand side of Jensen's inequality:

$$J(\mathbf{x}, \mathbf{p}, f) = \frac{1}{P_1^n}\sum_{i=1}^n p_i f(x_i) - f\left(\frac{1}{P_1^n}\sum_{i=1}^n p_i x_i\right), \tag{1.1}$$

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is called Jensen's functional. What will interest us here in particular are its lower bounds. Some related results can be found, for example, in [1], [4], [5], [7], but the papers of special interest at the moment are [3], [2] and [6].

In [2], the author presented some lower bounds for Jensen's functional which were later recaptured in [6] under relaxed conditions on the n-tuple $\mathbf{x}$. The results from [6] are the basis for the results which are to be presented in this paper. Without any loss of generality, we assume $P_1^n = 1$ since for positive n-tuples such that $P_1^n \neq 1$ results follow easily after substituting p_j by p_j / P_1^n. Furthermore, for $1 \leq i < k \leq n$, we introduce notation:

$$J_{ik}(\mathbf{x}, \mathbf{p}, f) = P_1^i f(x_i) + \sum_{m=i+1}^{k-1} p_m f(x_m) + P_k^n f(x_k) - f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right),$$

where

$$P_i^j = \sum_{m=i}^{j} p_m.$$

Note that $J_{ik}(\mathbf{x}, \mathbf{p}, f)$ can be obtained from $J(\mathbf{x}, \mathbf{p}, f)$ upon taking $x_1 = x_2 = \ldots = x_i$ and $x_k = x_{k+1} = \ldots = x_n$ and that $J_{1n}(\mathbf{x}, \mathbf{p}, f) = J(\mathbf{x}, \mathbf{p}, f)$. In what follows, I is an interval in $\mathbb{R}$ and

$$\bar{X}_i^j = \frac{1}{P_i^j} \sum_{m=i}^{j} p_m x_m$$

Theorem 2 (Theorem 2.1, [6]). *Let f be a convex function on I and $\mathbf{p}$ be a positive n-tuple such that $P_1^n = 1$ for some $n \geq 2$. Let $1 \leq i < k \leq n$ and $x_1, x_2, \ldots, x_k \in I$. If x_i is such that*

$$\bar{X}_1^i \leq x_i \leq \frac{1}{P_{i+1}^n}\left(\sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \tag{1.2}$$

$$\text{or} \quad \frac{1}{P_{i+1}^n}\left(\sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \leq x_i \leq \bar{X}_1^i, \tag{1.3}$$

then we have

$$J_{1k}(\boldsymbol{x}, \boldsymbol{p}, f) \geq J_{ik}(\boldsymbol{x}, \boldsymbol{p}, f). \tag{1.4}$$

Theorem 3 (Theorem 2.3, [6]). *Let f be a convex function on I and $\mathbf{p}$ be a positive n-tuple such that $P_1^n = 1$ for some $n \geq 2$. Let $1 \leq i < k \leq n$ and $x_j, x_{j+1}, \ldots, x_n \in I$.*

If x_k is such that

$$\frac{1}{P_1^{k-1}}\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m\right) \le x_k \le \bar{X}_k^n \tag{1.5}$$

$$\text{or} \quad \bar{X}_k^n \le x_k \le \frac{1}{P_1^{k-1}}\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m\right), \tag{1.6}$$

then we have

$$J_{in}(\boldsymbol{x},\boldsymbol{p},f) \ge J_{ik}(\boldsymbol{x},\boldsymbol{p},f). \tag{1.7}$$

In the same paper, in Theorems 2.2 and 2.4, alternative sets of conditions under which (1.4) and (1.7) hold were given, respectively, but it can be seen from the proofs that these conditions are actually more restricting than those given in Theorem 2 and Theorem 3, respectively. Finally, another related result, providing a somewhat different type of conditions, was given in the same paper in the following theorem.

Theorem 4 (Theorem 2.7, [6]). *Let f be a convex function on I and $\boldsymbol{p}$ be a positive n-tuple such that $P_1^n = 1$ for some $n \ge 2$. Let $\boldsymbol{x} \in I^n$ be a real n-tuple and let $1 \le i < k \le n$. If x_i and x_k are such that*

$$\bar{X}_1^i \le x_i \le \bar{X}_1^n \le x_k \le \bar{X}_k^n \tag{1.8}$$

$$\text{or} \qquad \bar{X}_k^n \le x_k \le \bar{X}_1^n \le x_i \le \bar{X}_1^i, \tag{1.9}$$

then we have

$$J(\boldsymbol{x},\boldsymbol{p},f) \ge J_{ik}(\boldsymbol{x},\boldsymbol{p},f).$$

These results dealt with obtaining the lower bound for Jensen's functional with two fixed variables. In [3], the author went one step further and gave lower bounds for Jensen's functional with three fixed variables.

Theorem 5. *Let f be a convex function on I and let $x_1, x_2, \ldots, x_n \in I$ $(n \ge 4)$ be such that*

$$x_1 \le x_2 \le \ldots \le x_n$$

and let $p_1, p_2, \ldots, p_n$ be positive weights such that $P_1^n = 1$. For fixed x_i, x_j, x_k $(1 \le i < j < k \le n)$, we have

$$J(\boldsymbol{x},\boldsymbol{p},f) \ge F_{ijk}(\boldsymbol{x},\boldsymbol{p},f) \quad \text{for } x_j \le \frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n}$$

$$\text{and} \qquad J(\boldsymbol{x},\boldsymbol{p},f) \ge G_{ijk}(\boldsymbol{x},\boldsymbol{p},f) \quad \text{for } x_j \ge \frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n},$$

where

$$F_{ijk}(\boldsymbol{x},\boldsymbol{p},f) = P_1^i f(x_i) + P_{i+1}^j f(x_j) + P_k^n f(x_k)$$

$$-(P_1^j+P_k^n)\,f\left(\frac{P_1^i x_i+P_{i+1}^j x_j+P_k^n x_k}{P_1^j+P_k^n}\right) \tag{1.10}$$

$$G_{ijk}(\boldsymbol{x},\boldsymbol{p},f)=P_1^i f(x_i)+P_j^{k-1} f(x_j)+P_k^n f(x_k)$$
$$-(P_1^i+P_j^n)\,f\left(\frac{P_1^i x_i+P_j^{k-1} x_j+P_k^n x_k}{P_1^i+P_j^n}\right) \tag{1.11}$$

In addition, we have $J(\boldsymbol{x},\boldsymbol{p},f)=F_{ijk}(\boldsymbol{x},\boldsymbol{p},f)$ *for*

$$\begin{aligned}
&x_1=x_2=\ldots=x_i,\\
&x_{i+1}=x_{i+2}=\ldots=x_j,\\
&x_{j+1}=x_{j+2}=\ldots=x_{k-1}=\frac{P_1^i x_i+P_{i+1}^j x_j+P_k^n x_k}{P_1^j+P_k^n},\\
&x_k=x_{k+1}=\ldots=x_n
\end{aligned}$$

and $J(\boldsymbol{x},\boldsymbol{p},f)=G_{ijk}(\boldsymbol{x},\boldsymbol{p},f)$ *for*

$$\begin{aligned}
&x_1=x_2=\ldots=x_i,\\
&x_{i+1}=x_{i+2}=\ldots=x_{j-1}=\frac{P_1^i x_i+P_j^{k-1} x_j+P_k^n x_k}{P_1^i+P_j^n},\\
&x_j=x_{j+1}=\ldots=x_{k-1},\\
&x_k=x_{k+1}=\ldots=x_n.
\end{aligned}$$

The principal aim of this paper is to recapture results from Theorem 5 under relaxed conditions on the n-tuple **x**. We shall do so by employing Theorems 2, 3 and 4.

The key step in proving the results in [3], [2] and [6] was the following lemma which was presented in [2]. It will play a vital role here as well.

Lemma 1. *Let f be a convex function on I and let p_1, p_2 be non-negative real numbers. If $a_1,a_2,b_1,b_2\in I$ are such that $a_1,a_2\in[b_1,b_2]$ and*

$$p_1a_1+p_2a_2=p_1b_1+p_2b_2,$$

then

$$p_1f(a_1)+p_2f(a_2)\le p_1f(b_1)+p_2f(b_2).$$

2. Main results

We present a series of theorems providing several different sets of conditions on the n-tuple **x** under which the inequalities stated in Theorem 5 hold.

Theorem 6. *Let f be a convex function on I, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive n-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. If x_i is such that*

$$\bar{X}_1^i \leq x_i \leq \bar{X}_{i+1}^n \qquad or \qquad \bar{X}_{i+1}^n \leq x_i \leq \bar{X}_1^i, \tag{2.1}$$

x_j is such that

$$\bar{X}_{i+1}^{j-1} \leq x_j \leq \frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n} \tag{2.2}$$

or

$$\frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n} \leq x_j \leq \bar{X}_{i+1}^{j-1} \tag{2.3}$$

and x_k is such that either (1.5) or (1.6) holds, then we have

$$J(\boldsymbol{x}, \boldsymbol{p}, f) \geq F_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f), \tag{2.4}$$

where $J(\boldsymbol{x}, \boldsymbol{p}, f)$ is defined in (1.1) and $F_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f)$ in (1.10).

Proof. We present the proof in three steps.
First step. We have

$$\begin{aligned} J(\mathbf{x}, \mathbf{p}, f) \geq & \sum_{m=1}^{i} p_m f(x_m) + P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + p_j f(x_j) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) \\ & + \sum_{m=k}^{n} p_m f(x_m) - f\left(\bar{X}_1^n\right). \end{aligned} \tag{2.5}$$

Namely, this is equivalent to

$$\sum_{m=i+1}^{j-1} p_m f(x_m) \geq P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) \quad \text{and} \quad \sum_{m=j+1}^{k-1} p_m f(x_m) \geq P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}),$$

which are both immediate consequences of Jensen's inequality. Note that the right-hand side of (2.5) is equal to the value of $J(\mathbf{x}, \mathbf{p}, f)$ for an n-tuple $\mathbf{x}$ such that $x_{i+1} = \ldots = x_{j-1} = \bar{X}_{i+1}^{j-1}$ and $x_{j+1} = \ldots = x_{k-1} = \bar{X}_{j+1}^{k-1}$.
Second step. We claim that

$$\begin{aligned} & \sum_{m=1}^{i} p_m f(x_m) + P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + p_j f(x_j) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) \\ & \qquad + \sum_{m=k}^{n} p_m f(x_m) - f\left(\bar{X}_1^n\right) \\ & \geq P_1^i f(x_i) + P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + p_j f(x_j) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + P_k^n f(x_k) \end{aligned}$$

$$-f\left(P_1^i x_i+\sum_{m=i+1}^{k-1} p_m x_m+P_k^n x_k\right). \tag{2.6}$$

This inequality follows immediately after first applying Theorem 2 for $k=n$ and then Theorem 3.

Third step. The final claim is that

$$P_1^i f(x_i)+P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1})+p_j f(x_j)+P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1})+P_k^n f(x_k)$$
$$-f\left(P_1^i x_i+\sum_{m=i+1}^{k-1} p_m x_m+P_k^n x_k\right)\geq F_{ijk}(\mathbf{x},\mathbf{p},f),$$

which is equivalent to

$$P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1})+P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1})+(P_1^j+P_k^n)\, f\left(\frac{P_1^i x_i+P_{i+1}^j x_j+P_k^n x_k}{P_1^j+P_k^n}\right)$$
$$\geq P_{i+1}^{j-1} f(x_j)+f\left(P_1^i x_i+\sum_{m=i+1}^{k-1} p_m x_m+P_k^n x_k\right).$$

Jensen's inequality implies that we have

$$f\left(P_1^i x_i+\sum_{m=i+1}^{k-1} p_m x_m+P_k^n x_k\right)$$
$$\leq P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1})+(P_1^j+P_k^n) f\left(\frac{P_1^i x_i+\sum_{m=i+1}^{j} p_m x_m+P_k^n x_k}{P_1^j+P_k^n}\right),$$

so we now only need to prove that

$$P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1})+(P_1^j+P_k^n)\, f\left(\frac{P_1^i x_i+P_{i+1}^j x_j+P_k^n x_k}{P_1^j+P_k^n}\right)$$
$$\geq P_{i+1}^{j-1} f(x_j)+(P_1^j+P_k^n) f\left(\frac{P_1^i x_i+\sum_{m=i+1}^{j} p_m x_m+P_k^n x_k}{P_1^j+P_k^n}\right),$$

Here we employ Lemma 1 for

$$a_1=x_j,\ a_2=(P_1^i x_i+\sum_{m=i+1}^{j} p_m x_m+P_k^n x_k)/(P_1^j+P_k^n),$$

$b_1=\bar{X}_{i+1}^{j-1}$ and $b_2=(P_1^i x_i+P_{i+1}^j x_j+P_k^n x_k)/(P_1^j+P_k^n)$ (or the obvious rearrangement) to complete the proof. Condition (2.2) (that is, (2.3)) ensures that the conditions of Lemma 1 are satisfied. This is easily checked. □

Theorem 7. *Let f be a convex function on I, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive n-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. If x_i is such that either (1.2) or (1.3) holds, x_j is such that either (2.2) or (2.3) holds and x_k is such that*

$$\bar{X}_1^{k-1} \leq x_k \leq \bar{X}_k^n \qquad or \qquad \bar{X}_k^n \leq x_k \leq \bar{X}_1^{k-1}, \tag{2.7}$$

then the inequality (2.4) is valid.

Proof. Proof is analogous to the proof of Theorem 6. Namely, the first and the third step of the proof are completely the same, but we have a slight difference in the second step. To obtain inequality (2.6), first apply Theorem 3 for $i = 1$ and then Theorem 2. □

Remark 1. Note that since

$$\frac{1}{P_{i+1}^n}\left(\sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \leq \bar{X}_{i+1}^n \quad \text{if} \quad x_k \leq \bar{X}_k^n$$

and similarly

$$\bar{X}_1^{k-1} \leq \frac{1}{P_1^{k-1}}\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m\right) \quad \text{if} \quad \bar{X}_1^i \leq x_i,$$

Theorem 6 imposes a more restricting condition on x_k, while Theorem 7 imposes a more restricting condition on x_i.

Remark 2. An increasing n-tuple $\mathbf{x}$ satisfies the first condition in (2.1), the left-hand side of the condition (2.2) and the condition (1.5), and also the condition (1.2) and the first condition in (2.7). Thus, Theorems 6 and 7 both provide generalizations of Theorem 5.

Theorem 8. *Let f be a convex function on I, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive n-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. Let x_i and x_k be such that either (1.8) or (1.9) holds and x_j be such that either (2.2) or (2.3) holds. Then the inequality (2.4) is valid.*

Proof. In respect to the proof of Theorem 6, there is a difference again only in the second step. To obtain inequality (2.6), apply Theorem 4. □

Remark 3. Note that an increasing n-tuple $\mathbf{x}$ does not necessarily satisfy either (1.8) or (1.9).

Theorem 9. *Let f be a convex function on I, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive n-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. If x_i is such that either one of the conditions in (2.1) hold, x_j is such that*

$$\bar{X}_{j+1}^{k-1} \leq x_j \leq \frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n} \tag{2.8}$$

or

$$\frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n} \le x_j \le \bar{X}_{j+1}^{k-1} \tag{2.9}$$

and x_k is such that either (1.5) or (1.6) holds, then we have

$$J(\boldsymbol{x},\boldsymbol{p},f) \ge G_{ijk}(\boldsymbol{x},\boldsymbol{p},f), \tag{2.10}$$

where $J(\boldsymbol{x},\boldsymbol{p},f)$ is defined in (1.1) and $G_{ijk}(\boldsymbol{x},\boldsymbol{p},f)$ in (1.11).

Proof. The proof follows in the footsteps of the proof of Theorem 6. Namely, the first and the second step are the same (note that the conditions on x_i and x_k are the same). The difference is in the third step. Here we need to prove that

$$P_1^i f(x_i) + P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + p_j f(x_j) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + P_k^n f(x_k) - f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \ge G_{ijk}(\mathbf{x},\mathbf{p},f),$$

which is equivalent to

$$P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + (P_1^i + P_j^n)\, f\left(\frac{P_1^i x_i + P_j^{k-1} x_j + P_k^n x_k}{P_1^i + P_j^n}\right) \ge P_{j+1}^{k-1} f(x_j) + f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right).$$

Jensen's inequality implies that

$$f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \le P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + (P_1^i + P_j^n) f\left(\frac{P_1^i x_i + \sum_{m=j}^{k-1} p_m x_m + P_k^n x_k}{P_1^i + P_j^n}\right),$$

so there is only left to prove that

$$P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + (P_1^i + P_j^n)\, f\left(\frac{P_1^i x_i + P_j^{k-1} x_j + P_k^n x_k}{P_1^i + P_j^n}\right) \ge P_{j+1}^{k-1} f(x_j) + (P_1^i + P_j^n) f\left(\frac{P_1^i x_i + \sum_{m=j}^{k-1} p_m x_m + P_k^n x_k}{P_1^i + P_j^n}\right),$$

Now we apply Lemma 1 for $a_1 = x_j$, $a_2 = (P_1^i x_i + \sum_{m=j}^{k-1} p_m x_m + P_k^n x_k)/(P_1^i + P_j^n)$, $b_1 = \bar{X}_{j+1}^{k-1}$ and $b_2 = (P_1^i x_i + P_j^{k-1} x_j + P_k^n x_k)/(P_1^i + P_j^n)$ (or the obvious rearrangement) to complete the proof. This time condition (2.8) (that is, (2.9)) ensures that the conditions of Lemma 1 are satisfied. □

Theorem 10. *Let f be a convex function on I, $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive n-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. If x_i is such that either (1.2) or (1.3) holds, x_j is such that either (2.8) or (2.9) holds and x_k is such that either one of the conditions in (2.7) holds, then the inequality (2.10) is valid.*

Proof. The first and the second step of the proof are the same as in Theorem 7, while the third step is the same as in the proof of Theorem 9. □

Remark 4. An increasing n-tuple $\mathbf{x}$ satisfies the first condition in (2.1), the right-hand side of the condition (2.9) and the condition (1.5), and also the condition (1.2) and the first condition in (2.7). Thus, Theorems 9 and 10 both provide generalizations of Theorem 5.

Theorem 11. *Let f be a convex function on I, $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive n-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. Let x_i and x_k be such that either (1.8) or (1.9) holds and x_j be such that either (2.8) or (2.9) holds. Then the inequality (2.10) is valid.*

Proof. The first and the second step of the proof are the same as in Theorem 8, while the third step is the same as in the proof of Theorem 9. □

Proposition 1. *Let f be a convex function on I, $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive n-tuple such that $P_1^n = 1$ for some $n \geq 3$.*
If $P_{j+1}^{k-1} \geq P_{i+1}^{j-1}$, then we have

$$G_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f) \geq F_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f), \tag{2.11}$$

while if $P_{i+1}^{j-1} \geq P_{j+1}^{k-1}$, then we have

$$F_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f) \geq G_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f), \tag{2.12}$$

where $F_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f)$ is defined as in (1.10) and $G_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f)$ in (1.11).

Proof. Note that

$$\frac{P_1^i x_i + P_j^{k-1} x_j + P_k^n x_k}{P_1^i + P_j^n} = \frac{P_1^j + P_k^n}{P_1^i + P_j^n} \cdot \frac{P_1^i x_i + P_{i+1}^j x_j + P_k^n x_k}{P_1^j + P_k^n} + \frac{P_{j+1}^{k-1} - P_{i+1}^{j-1}}{P_1^i + P_j^n} \cdot x_j .$$

Inequality (2.11) now follows directly from Jensen's inequality, while the obvious rearrangement yields (2.12). □

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