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A SUZUKI TYPE COMMON TRIPLED FIXED POINT THEOREM FOR A HYBRID PAIR OF MAPPINGS

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Abstract. In this paper, we prove a tripled common fixed point theorem of Suzuki type for a pair of hybrid mappings in metric spaces. Our result generalizes and modifies several comparable results in the literature. We also provide an example to support our theorem.

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1. Introduction and preliminaries

Banach contraction principle plays a very important role in nonlinear analysis and has many generalizations. In 2008, Suzuki [37, 38] proved generalized versions of both Banach's and Edelstein's basic results. Many other works in this direction have been considered, for example [2, 4, 13, 15, 19, 20, 23, 25, 29, 35, 36] and the references therein.

The study of fixed points for multivalued contraction mappings using the Hausdorff-Pompeu metric was initiated by Nadler [28]. Let (X,d) be a metric space. We denote $CB(X)$ the family of all nonempty closed and bounded subsets of X and $CL(X)$ the set of all nonempty closed subsets of X. For $A,B \in CB(X)$ and $x \in X$, we denote $D(x,A) = \inf\{d(x,a) : a \in A\}$. Let H be the Hausdorff-Pompeu metric induced by the metric d on X, that is

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}$$

for every $A,B \in CB(X)$.

It is clear that for $A,B \in CB(X)$ and $a \in A$ we have $d(a,B) \le H(A,B)$.

Definition 1. An element $x \in X$ is said to be a fixed point of a multivalued mapping $T : X \to CB(X)$ if and only if $x \in Tx$.

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In 1969, Nadler [28] extended the famous Banach contraction principle [10] from singlevalued mapping to multivalued mapping and proved the following fixed point theorem for the multivalued contraction.

Theorem 1 (Nadler, [28]). *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0,1)$ such that*

$$H(Tx,Ty) \le c\, d(x,y),$$

for all $x, y \in X$. Then, T has a fixed point.

Lemma 1 (Nadler, [28]). *Let $A, B \in CB(X)$ and $h > 1$. Then for every $a \in A$, there exists $b \in B$ such that $d(a,b) \le h\, H(A,B)$.*

The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions. For details, we refer the reader to [3, 16–18, 21, 24, 26, 33] and the references therein.

In 2011, Samet and Vetro [34] introduced the concept of coupled fixed point for multivalued mapping. Very recently, Berinde and Borcut [11, 12] proved some tripled fixed and coincidence point theorems for contractive type mappings in partially ordered metric spaces. Later, several authors obtained coincidence and common tripled fixed point theorems in various spaces, for example refer to [1, 5–9, 14, 27, 30–32].

The aim of this paper is to combine the ideas of tripled fixed points and Suzuki type fixed point theorems to obtain a tripled common fixed point theorem for a pair of hybrid mappings in a metric space.

First, we give the following theorem of Suzuki [37].

Theorem 2 ([37]). *Let (X,d) be a complete metric space and let T be a mapping on X. Define a nonincreasing function θ from $[0,1)$ into $(\frac{1}{2},1]$ by*

$$\theta(r) = \begin{cases} 1, & 0 \le \mathrm{r} \le \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2}, & \frac{\sqrt{5}-1}{2} \le \mathrm{r} \le \frac{1}{\sqrt{2}} \\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}} \le \mathrm{r} < 1. \end{cases}$$

Assume that $r \in [0,1)$, such that

$$\theta(r)d(x,Tx) \le d(x,y) \quad \text{implies} \quad d(Tx,Ty) \le r d(x,y).$$

for all $x, y \in X$. Then, there exists a unique fixed point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Now, we give some known definitions which are used to prove our main result.

Definition 2 ([34]). An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to CB(X)$ if $x \in F(x,y)$ and $y \in F(y,x)$.

Definition 3 ([22]). An element $(x,y) \in X \times X$ is called
(i) a coupled coincident point of mappings $F : X \times X \to CB(X)$ and $g : X \to X$ if $gx \in F(x,y)$ and $gy \in F(y,x)$;
(ii) a coupled common fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx \in F(x,y)$ and $y = gy \in F(y,x)$.

Definition 4 ([31]). Let X be a nonempty set, $F : X \times X \times X \to CB(X)$ and $g : X \to X$. Then point $(x,y,z) \in X \times X \times X$ is called a tripled
(i) fixed point of F if $x \in F(x,y,z)$, $y \in F(y,z,x)$ and $z \in F(z,x,y)$.
(ii) coincidence point of F and g if $gx \in F(x,y,z)$, $gy \in F(y,z,x)$ and $gz \in F(z,x,y)$.
(iii) common fixed point of F and g if $x = gx \in F(x,y,z)$, $y = gy \in F(y,z,x)$ and $z = gz \in F(z,x,y)$.

Definition 5 ([31]). Let X be a nonempty set, $F : X \times X \times X \to CB(X)$ and $g : X \to X$. The pair (F,g) is called $w-$compatible if $g(F(x,y,z)) \subseteq F(gx,gy,gz)$, whenever (x,y,z) is a coincidence point of F and g.

In this paper, we use the following function introduced in [20].
Let $\phi : [0,1) \to (0,1]$ be a nonincreasing function defined by

$$\phi(\theta) = \begin{cases} 1, & \text{if } 0 \le \theta < \frac{1}{2}, \\ 1-\theta, & \text{if } \frac{1}{2} \le \theta < 1. \end{cases} \tag{1.1}$$

Using the notion of w-compatibility of the pair (F,g), we establish a tripled common fixed point theorem of Suzuki type for a hybrid pair of mappings in metric spaces. We support our result by an example.

2. Main results

Our main result is the following common tripled fixed point theorem.

Theorem 3. *Let (X,d) be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying the following :*
(i) $F(X \times X \times X) \subseteq g(X)$ and $g(X)$ is complete
(ii) If there exists $\theta \in [0,1)$ such that

$$\phi(\theta) \min \left\{ \begin{array}{c} d(gx, F(x,y,z)), \\ d(gy, F(y,z,x)), \\ d(gz, F(z,x,y)) \end{array} \right\} \le \max \left\{ \begin{array}{c} d(gx,gu), \\ d(gy,gv), \\ d(gz,gw) \end{array} \right\} \tag{2.1}$$

implies

$$H(F(x,y,z),F(u,v,w)) \le \theta \max \left\{ \begin{array}{c} d(gx,gu), d(gy,gv), d(gz,gw), \\ d(gx,F(x,y,z)), d(gy,F(y,z,x)), d(gz,F(z,x,y)), \\ d(gu,F(u,v,w)), d(gv,F(v,w,u)), d(gw,F(w,u,v)), \\ \frac{1}{2}\max\left\{ \begin{array}{c} d(gx,F(u,v,w)), d(gy,F(v,w,u)), d(gz,F(w,u,v)), \\ d(gu,F(x,y,z)), d(gv,F(y,z,x)), d(gw,F(z,x,y)) \end{array} \right\} \end{array} \right\}. \tag{2.2}$$

for all $x,y,z,u,v,w \in X$. Then F and g have a tripled coincidence point in $X \times X \times X$.

Further, F and g have a tripled common fixed point if one of the following conditions holds:
(a) Suppose the pair (F,g) is w-compatible and there exist $u,v,w \in X$ such that $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$, whenever (x,y,z) is a tripled coincidence point of F and g and g is continuous at u,v and w.
(b) Suppose there exist $u,v,w \in X$ such that $\lim_{n\to\infty} g^n u = x$, $\lim_{n\to\infty} g^n v = y$ and $\lim_{n\to\infty} g^n w = z$, whenever (x,y,z) is a tripled coincidence point of F and g and g is continuous at x,y and z.

Proof. Let $x_0,y_0,z_0 \in X$. From (i) there exist $x_1,y_1,z_1 \in X$ such that $gx_1 \in F(x_0,y_0,z_0), gy_1 \in F(y_0,z_0,x_0)$ and $gz_1 \in F(z_0,x_0,y_0)$. By Lemma 1, there exist $x_2,y_2,z_2 \in X$ such that

$$\begin{aligned} &gx_2 \in F(x_1,y_1,z_1) \text{ with } d(gx_1,gx_2) \le \tfrac{1}{\sqrt{\theta}} H(F(x_0,y_0,z_0),F(x_1,y_1,z_1)), \\ &gy_2 \in F(y_1,z_1,x_1) \text{ with } d(gy_1,gy_2) \le \tfrac{1}{\sqrt{\theta}} H(F(y_0,z_0,x_0),F(y_1,z_1,x_1)), \\ &gz_2 \in F(z_1,x_1,y_1) \text{ with } d(gz_1,gz_2) \le \tfrac{1}{\sqrt{\theta}} H(F(z_0,x_0,y_0),F(z_1,x_1,y_1)). \end{aligned}$$

Continuing in this way, we get sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ in X such that $gx_{n+1} \in F(x_n,y_n,z_n)$, $gy_{n+1} \in F(y_n,z_n,x_n)$ and $gz_{n+1} \in F(z_n,x_n,y_n)$ with

$$\begin{aligned} &d(gx_n,gx_{n+1}) \le \tfrac{1}{\sqrt{\theta}} H(F(x_{n-1},y_{n-1},z_{n-1}),F(x_n,y_n,z_n)), \\ &d(gy_n,gy_{n+1}) \le \tfrac{1}{\sqrt{\theta}} H(F(y_{n-1},z_{n-1},x_{n-1}),F(y_n,z_n,x_n)), \\ &d(gz_n,gz_{n+1}) \le \tfrac{1}{\sqrt{\theta}} H(F(z_{n-1},x_{n-1},y_{n-1}),F(z_n,x_n,y_n)). \end{aligned}$$

Case i: Suppose $gx_n = gx_{n+1}$, $gy_n = gy_{n+1}$ and $gz_n = gz_{n+1}$, for some n. Then $gx_n \in F(x_n,y_n,z_n)$, $gy_n \in F(y_n,z_n,x_n)$ and $gz_n \in F(z_n,x_n,y_n)$. Thus (x_n,y_n,z_n) is a tripled coincidence point of F and g.

Case ii: Assume that $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ or $gz_n \neq gz_{n+1}$ for all n. Since

$$\phi(\theta)\min\left\{\begin{array}{c} d(gx_0,F(x_0,y_0,z_0)),\\ d(gy_0,F(y_0,z_0,x_0)),\\ d(gz_0,F(z_0,x_0,y_0)) \end{array}\right\} \leq \max\left\{\begin{array}{c} d(gx_0,gx_1),\\ d(gy_0,gy_1),\\ d(gz_0,gz_1) \end{array}\right\},$$

from (2.2), we have

$$\begin{array}{l} H(F(x_0,y_0,z_0),F(x_1,y_1,z_1)) \\ \leq \theta\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1),d(gz_0,gz_1),\\ d(gx_0,F(x_0,y_0,z_0)),d(gy_0,F(y_0,z_0,x_0)),d(gz_0,F(z_0,x_0,y_0)),\\ d(gx_1,F(x_1,y_1,z_1)),d(gy_1,F(y_1,z_1,x_1)),d(gz_1,F(z_1,x_1,y_1)),\\ \frac{1}{2}\max\left\{\begin{array}{c} d(gx_0,F(x_1,y_1,z_1)),d(gy_0,F(y_1,z_1,x_1)),d(gz_0,F(z_1,x_1,y_1)),\\ d(gx_1,F(x_0,y_0,z_0)),d(gy_1,F(y_0,z_0,x_0)),d(gz_1,F(z_0,x_0,y_0)) \end{array}\right\} \end{array}\right\}. \end{array}$$

Now we have, using triangular inequality

$$d(gx_1,gx_2) \leq \frac{1}{\sqrt{\theta}}H(F(x_0,y_0,z_0),F(x_1,y_1,z_1)) \leq \sqrt{\theta}\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1),\\ d(gz_0,gz_1),d(gx_1,gx_2),\\ d(gy_1,gy_2),d(gz_1,gz_2) \end{array}\right\}.$$

Similarly, we have

$$d(gy_1,gy_2) \leq \sqrt{\theta}\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1),\\ d(gz_0,gz_1),d(gx_1,gx_2),\\ d(gy_1,gy_2),d(gz_1,gz_2) \end{array}\right\}$$

and

$$d(gz_1,gz_2) \leq \sqrt{\theta}\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1),\\ d(gz_0,gz_1),d(gx_1,gx_2),\\ d(gy_1,gy_2),d(gz_1,gz_2) \end{array}\right\}.$$

Thus

$$\max\left\{\begin{array}{c} d(gx_1,gx_2),\\ d(gy_1,gy_2),\\ d(gz_1,gz_2) \end{array}\right\} \leq \sqrt{\theta}\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1),\\ d(gz_0,gz_1),d(gx_1,gx_2),\\ d(gy_1,gy_2),d(gz_1,gz_2) \end{array}\right\}. \tag{2.3}$$

If

$$\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1),\\ d(gz_0,gz_1),d(gx_1,gx_2),\\ d(gy_1,gy_2),d(gz_1,gz_2) \end{array}\right\} = d(gx_1,gx_2),$$

then from (2.3) we have $gx_1 = gx_2$.

Analogous other two cases

$$\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1),\\ d(gz_0,gz_1),d(gx_1,gx_2),\\ d(gy_1,gy_2),d(gz_1,gz_2) \end{array}\right\} = d(gy_1,gy_2),$$

and

$$\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1), \\ d(gz_0,gz_1),d(gx_1,gx_2), \\ d(gy_1,gy_2),d(gz_1,gz_2) \end{array}\right\} = d(gz_1,gz_2),$$

imply that $gy_1 = gy_2$ and $gz_1 = gz_2$. It is a contradiction. Hence from (2.3), we have

$$\max\left\{\begin{array}{c} d(gx_1,gx_2), \\ d(gy_1,gy_2), \\ d(gz_1,gz_2) \end{array}\right\} \le \sqrt{\theta}\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1), \\ d(gz_0,gz_1) \end{array}\right\}.$$

Continuing in this way, we get

$$\max\left\{\begin{array}{c} d(gx_n,gx_{n+1}), \\ d(gy_n,gy_{n+1}), \\ d(gz_n,gz_{n+1}) \end{array}\right\} \le (\sqrt{\theta})^n\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1), \\ d(gz_0,gz_1) \end{array}\right\},$$

which in turn yields that

$$\max\left\{\begin{array}{c} \sum d(gx_n,gx_{n+1}), \\ \sum d(gy_n,gy_{n+1}), \\ \sum d(gz_n,gz_{n+1}) \end{array}\right\} \le \sum(\sqrt{\theta})^n\max\left\{\begin{array}{c} d(gx_0,gx_1),d(gy_0,gy_1), \\ d(gz_0,gz_1) \end{array}\right\} < \infty.$$

Hence $\{gx_n\},\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences. Since $g(X)$ is complete, there exist $p,q,r,u,v,w \in X$ such that $gx_n \to p = gu$, $gy_n \to q = gv$ and $gz_n \to r = gw$. Since $gx_n \ne gx_{n+1}$ or $gy_n \ne gy_{n+1}$ or $gz_n \ne gz_{n+1}$ for all n, it follows that $gx_n \ne gu$ or $gy_n \ne gv$ or $gz_n \ne gw$ for infinitely many n.
Hence $\max\{d(gx_n,gu),d(gy_n,gv),d(gz_n,gw)\} > 0$ for infinitely many n.

Claim I: We will prove that

$$\max\left\{\begin{array}{c} d(gu,F(x,y,z)), \\ d(gv,F(y,z,x)), \\ d(gw,F(z,x,y)) \end{array}\right\} \le \theta\max\left\{\begin{array}{c} d(gu,gx),d(gv,gy), \\ d(gw,gz),d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)),d(gz,F(z,x,y)) \end{array}\right\},$$

for all $x,y,z \in X$ with $\max\{d(gu,gx),d(gv,gy),d(gw,gz)\} > 0$.

Let $x,y,z \in X$ be such that $\max\{d(gu,gx),d(gv,gy),d(gw,gz)\} > 0$. Since $gx_n \to gu$, $gy_n \to gv$ and $gz_n \to gw$, there exists a positive integer n_0 such that for $n \ge n_0$, we have

$$\max\left\{\begin{array}{c} d(gx_n,gu), \\ d(gy_n,gv), \\ d(gz_n,gw) \end{array}\right\} \le \frac{1}{3}\max\left\{\begin{array}{c} d(gu,gx), \\ d(gv,gy), \\ d(gw,gz) \end{array}\right\}. \tag{2.4}$$

Now, for $n \geq n_0$, using (2.4) we have

$$\phi(\theta)\min\left\{\begin{array}{c} d(gx_n,F(x_n,y_n,z_n)),\\ d(gy_n,F(y_n,z_n,x_n)),\\ d(gz_n,F(z_n,x_n,y_n)) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(gx_n,gx_{n+1}),\\ d(gy_n,gy_{n+1}),\\ d(gz_n,gz_{n+1}) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(gx_n,gu)+d(gu,gx_{n+1}),\\ d(gy_n,gv)+d(gv,gy_{n+1}),\\ d(gz_n,gw)+d(gw,gz_{n+1}) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(gx_n,gu),d(gy_n,gv),\\ d(gz_n,gw) \end{array}\right\}+\max\left\{\begin{array}{c} d(gu,gx_{n+1}),d(gv,gy_{n+1}),\\ d(gw,gz_{n+1}) \end{array}\right\}$$

$$\leq \tfrac{2}{3}\max\left\{\begin{array}{c} d(gu,gx),\\ d(gv,gy),\\ d(gw,gz) \end{array}\right\}=\max\left\{\begin{array}{c} d(gu,gx),\\ d(gv,gy),\\ d(gw,gz) \end{array}\right\}-\tfrac{1}{3}\max\left\{\begin{array}{c} d(gu,gx),\\ d(gv,gy),\\ d(gw,gz) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(gu,gx),\\ d(gv,gy),\\ d(gw,gz) \end{array}\right\}-\max\left\{\begin{array}{c} d(gu,gx_n),\\ d(gv,gy_n),\\ d(gw,gz_n) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(gu,gx)-d(gu,gx_n),\\ d(gv,gy)-d(gv,gy_n),\\ d(gw,gz)-d(gw,gz_n) \end{array}\right\}$$

$$\leq \max\{d(gx_n,gx),d(gy_n,gy),d(gz_n,gz)\}.$$

From (2.2) and using triangular inequality, we have

$$H(F(x_n,y_n,z_n),F(x,y,z))$$

$$\leq \theta\max\left\{\begin{array}{c} d(gx_n,gx),d(gy_n,gy),d(gz_n,gz),\\ d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1}),\\ d(gx,F(x,y,z)),d(gy,F(y,z,x)),d(gz,F(z,x,y)),\\ \tfrac{1}{2}\max\left\{\begin{array}{c} d(gx_n,F(x,y,z)),d(gy_n,F(y,z,x)),d(gz_n,F(z,x,y)),\\ d(gx,gx_{n+1}),d(gy,gy_{n+1}),d(gz,gz_{n+1}) \end{array}\right\} \end{array}\right\}$$

$$\leq \theta\max\left\{\begin{array}{c} d(gx_n,gx),d(gy_n,gy),d(gz_n,gz),\\ d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1}),\\ d(gx,F(x,y,z)),d(gy,F(y,z,x)),d(gz,F(z,x,y)) \end{array}\right\}.$$

Similarly, we can show that

$$H(F(y_n,z_n,x_n),F(y,z,x))$$
$$\leq \theta \max \left\{ \begin{array}{c} d(gx_n,gx),d(gy_n,gy),d(gz_n,gz), \\ d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1}), \\ d(gx,F(x,y,z)),d(gy,F(y,z,x)),d(gz,F(z,x,y)) \end{array} \right\},$$

and

$$H(F(z_n,x_n,y_n),F(z,x,y))$$
$$\leq \theta \max \left\{ \begin{array}{c} d(gx_n,gx),d(gy_n,gy),d(gz_n,gz), \\ d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1}), \\ d(gx,F(x,y,z)),d(gy,F(y,z,x)),d(gz,F(z,x,y)) \end{array} \right\}.$$

Thus, we get

$$\max \left\{ \begin{array}{c} d(gx_{n+1},F(x,y,z)), \\ d(gy_{n+1},F(y,z,x)), \\ d(gz_{n+1},F(z,x,y)) \end{array} \right\}$$
$$\leq \max \left\{ \begin{array}{c} H(F(x_n,y_n,z_n),F(x,y,z)), \\ H(F(y_n,z_n,x_n),F(y,z,x)), \\ H(F(z_n,x_n,y_n),F(z,x,y)) \end{array} \right\}$$
$$\leq \theta \max \left\{ \begin{array}{c} d(gx_n,gx),d(gy_n,gy),d(gz_n,gz), \\ d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1}), \\ d(gx,F(x,y,z)),d(gy,F(y,z,x)),d(gz,F(z,x,y)) \end{array} \right\}.$$

Letting $n \to \infty$, we get

$$\max \left\{ \begin{array}{c} d(gu,F(x,y,z)), \\ d(gv,F(y,z,x)), \\ d(gw,F(z,x,y)) \end{array} \right\} \leq \theta \max \left\{ \begin{array}{c} d(gu,gx),d(gv,gy), \\ d(gw,gz),d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)),d(gz,F(z,x,y)) \end{array} \right\}.$$

Hence the **Claim I** is true. Now we will show that (u,v,w) is a tripled coincidence point of F and g.

Case a: Suppose now that $0 \leq \theta < \frac{1}{2}$.

On the contrary, assume that $gu \notin F(u,v,w)$ or $gv \notin F(v,w,u)$ or $gw \notin F(w,u,v)$. Let $ga \in F(u,v,w)$, $gb \in F(v,w,u)$ and $gc \in F(w,u,v)$ be such that

$$2\theta \max \left\{ \begin{array}{c} d(ga,gu), \\ d(gb,gv), \\ d(gc,gw) \end{array} \right\} < \max \left\{ \begin{array}{c} d(gu,F(u,v,w)), \\ d(gv,F(v,w,u)), \\ d(gw,F(w,u,v)). \end{array} \right\} \tag{2.5}$$

Since $ga \in F(u,v,w)$, $gb \in F(v,w,u)$ and $gc \in F(w,u,v)$, we have $ga \neq gu$ or $gb \neq gv$ or $gc \neq gw$ and hence $a \neq u$ or $b \neq v$ or $c \neq w$. From **Claim I**, we have

$$\max\left\{\begin{array}{c} d(gu,F(a,b,c)), \\ d(gv,F(b,c,a)), \\ d(gw,F(c,a,b)) \end{array}\right\} \leq \theta \max\left\{\begin{array}{c} d(gu,ga), d(gv,gb), \\ d(gw,gc), d(ga,F(a,b,c)), \\ d(gb,F(b,c,a)), d(gc,F(c,a,b)) \end{array}\right\}. \tag{2.6}$$

Since

$$\phi(\theta)\min\left\{\begin{array}{c} d(gu,F(u,v,w)), \\ d(gv,F(v,w,u)), \\ d(gw,F(w,u,v)) \end{array}\right\} \leq \max\left\{\begin{array}{c} d(gu,ga), \\ d(gv,gb), \\ d(gw,gc) \end{array}\right\},$$

from (2.2) and triangular inequality we have

$$\begin{aligned} &H(F(u,v,w),F(a,b,c)) \\ &\leq \theta \max\left\{\begin{array}{c} d(gu,ga), d(gv,gb), d(gw,gc), \\ d(gu,F(u,v,w)), d(gv,F(v,w,u)), d(gw,F(w,u,v)), \\ d(ga,F(a,b,c)), d(gb,F(b,c,a)), d(gc,F(c,a,b)), \\ \frac{1}{2}\max\left\{\begin{array}{c} d(gu,F(a,b,c)), d(gv,F(b,c,a)), d(gw,F(c,a,b)), \\ d(ga,F(u,v,w)), d(gb,F(v,w,u)), d(gc,F(w,u,v)) \end{array}\right\} \end{array}\right\} \\ &\leq \theta \max\left\{\begin{array}{c} d(gu,ga), d(gv,gb), d(gw,gc), \\ d(ga,F(a,b,c)), d(gb,F(b,c,a)), d(gc,F(c,a,b)) \end{array}\right\}. \end{aligned}$$

Similarly, we have

$$H(F(v,w,u),F(b,c,a)) \leq \theta \max\left\{\begin{array}{c} d(gu,ga), d(gv,gb), \\ d(gw,gc), d(ga,F(a,b,c)), \\ d(gb,F(b,c,a)), d(gc,F(c,a,b)) \end{array}\right\},$$

and

$$H(F(w,u,v),F(c,a,b)) \leq \theta \max\left\{\begin{array}{c} d(gu,ga), d(gv,gb), \\ d(gw,gc), d(ga,F(a,b,c)), \\ d(gb,F(b,c,a)), d(gc,F(c,a,b)) \end{array}\right\}.$$

Thus

$$\max\left\{\begin{array}{c} H(F(u,v,w),F(a,b,c)), \\ H(F(v,w,u),F(b,c,a)), \\ H(F(w,u,v),F(c,a,b)) \end{array}\right\} \leq \theta \max\left\{\begin{array}{c} d(gu,ga), d(gv,gb), \\ d(gw,gc), d(ga,F(a,b,c)), \\ d(gb,F(b,c,a)), d(gc,F(c,a,b)) \end{array}\right\}. \tag{2.7}$$

From (2.7) we have

$$\max\left\{\begin{array}{c} d(ga,F(a,b,c)), \\ d(gb,F(b,c,a)), \\ d(gc,F(c,a,b)) \end{array}\right\} \leq \max\left\{\begin{array}{c} H(F(u,v,w),F(a,b,c)), \\ H(F(v,w,u),F(b,c,a)), \\ H(F(w,u,v),F(c,a,b)) \end{array}\right\}$$

$$\leq \theta \max \left\{ \begin{array}{c} d(gu,ga), d(gv,gb), \\ d(gw,gc), d(ga,F(a,b,c)), \\ d(gb,F(b,c,a)), d(gc,F(c,a,b)) \end{array} \right\}$$

If

$$\max \left\{ \begin{array}{c} d(gu,ga), d(gv,gb), \\ d(gw,gc), d(ga,F(a,b,c)), \\ d(gb,F(b,c,a)), d(gc,F(c,a,b)) \end{array} \right\} = d(ga,F(a,b,c))$$

we conclude that $ga \in F(a,b,c)$. Analogous we conclude that $gb \in F(b,c,a)$ and $gc \in F(c,a,b)$ if

$$\max \left\{ \begin{array}{c} d(gu,ga), d(gv,gb), \\ d(gw,gc), d(ga,F(a,b,c)), \\ d(gb,F(b,c,a)), d(gc,F(c,a,b)) \end{array} \right\} = d(gb,F(b,c,a))$$

or

$$\max \left\{ \begin{array}{c} d(gu,ga), d(gv,gb), \\ d(gw,gc), d(ga,F(a,b,c)), \\ d(gb,F(b,c,a)), d(gc,F(c,a,b)) \end{array} \right\} = d(gc,F(c,a,b)).$$

This is contradiction with $a \neq u$, $b \neq v$ and $c \neq w$.

So,

$$\max \left\{ \begin{array}{c} d(ga,F(a,b,c)), \\ d(gb,F(b,c,a)), \\ d(gc,F(c,a,b)) \end{array} \right\} \leq \theta \max \left\{ \begin{array}{c} d(gu,ga), \\ d(gv,gb), \\ d(gw,gc) \end{array} \right\}. \tag{2.8}$$

From (2.6) and (2.8) we have

$$\max \left\{ \begin{array}{c} d(gu,F(a,b,c)), \\ d(gv,F(b,c,a)), \\ d(gw,F(c,a,b)) \end{array} \right\} \leq \theta \max \left\{ \begin{array}{c} d(gu,ga), \\ d(gv,gb), \\ d(gw,gc) \end{array} \right\}, \tag{2.9}$$

and from (2.7) we get

$$\max \left\{ \begin{array}{c} H(F(u,v,w),F(a,b,c)), \\ H(F(v,w,u),F(b,c,a)), \\ H(F(w,u,v),F(c,a,b)) \end{array} \right\} \leq \theta \max \left\{ \begin{array}{c} d(gu,ga), \\ d(gv,gb), \\ d(gw,gc) \end{array} \right\}. \tag{2.10}$$

Now, using (2.5), (2.9) and (2.10) we obtain

$$\max\left\{\begin{array}{c} d(gu,F(u,v,w)), \\ d(gv,F(v,w,u)), \\ d(gw,F(w,u,v)) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(gu,F(a,b,c))+H(F(a,b,c),F(u,v,w)) \\ d(gv,F(b,c,a))+H(F(b,c,a),F(v,w,u)) \\ d(gw,F(c,a,b))+H(F(c,a,b),F(w,u,v)) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(gu,F(a,b,c)), \\ d(gv,F(b,c,a)), \\ d(gw,F(c,a,b)) \end{array}\right\}+\max\left\{\begin{array}{c} H(F(a,b,c),F(u,v,w)), \\ H(F(b,c,a),F(v,w,u)), \\ H(F(c,a,b),F(w,u,v)) \end{array}\right\}$$

$$\leq 2\theta\max\left\{\begin{array}{c} d(gu,ga), \\ d(gv,gb), \\ d(gw,gc) \end{array}\right\}<\max\left\{\begin{array}{c} d(gu,F(u,v,w)), \\ d(gv,F(v,w,u)), \\ d(gw,F(w,u,v)) \end{array}\right\}.$$

It is a contradiction. Hence $gu\in F(u,v,w)$, $gv\in F(v,w,u)$ and $gw\in F(w,u,v)$.

Case b: Suppose now that $\frac{1}{2}\leq\theta<1$.

Claim II: We will show that

$$\max\left\{\begin{array}{c} H(F(x,y,z),F(u,v,w)), \\ H(F(y,z,x),F(v,w,u)), \\ H(F(z,x,y),F(w,u,v)) \end{array}\right\}$$

$$\leq\theta\max\left\{\begin{array}{c} d(gx,gu),d(gy,gv),d(gz,gw), \\ d(gx,F(x,y,z)),d(gy,F(y,z,x)),d(gz,F(z,x,y)), \\ d(gu,F(u,v,w)),d(gv,F(v,w,u)),d(gw,F(w,u,v)), \\ \frac{1}{2}\max\left\{\begin{array}{c} d(gx,F(u,v,w)),d(gy,F(v,w,u)),d(gz,F(w,u,v)), \\ d(gu,F(x,y,z)),d(gv,F(y,z,x)),d(gw,F(z,x,y)) \end{array}\right\} \end{array}\right\},$$

for all $x,y,z\in X$ with $\max\left\{\ d(gx,gu),d(gy,gv),d(gz,gw)\ \right\}>0$.

Let $x,y,z\in X$ be such that $\max\left\{\ d(gx,gu),d(gy,gv),d(gz,gw)\ \right\}>0$. Then for every positive integer n, there exist sequences $t_n\in F(x,y,z)$, $t_n^1\in F(y,z,x)$ and $t_n^{11}\in F(z,x,y)$ such that

$$\max\left\{\begin{array}{c} d(gu,t_n), \\ d(gv,t_n^1), \\ d(gw,t_n^{11}) \end{array}\right\}\leq\max\left\{\begin{array}{c} d(gu,F(x,y,z)), \\ d(gv,F(y,z,x)), \\ d(gw,F(z,x,y)) \end{array}\right\}+\frac{1}{n}\max\left\{\begin{array}{c} d(gx,gu), \\ d(gy,gv), \\ d(gz,gw) \end{array}\right\}. \tag{2.11}$$

Now, using (2.11) and triangular inequality we obtain

$$\max\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\} \leq \max\left\{\begin{array}{c} d(gx,gu)+d(gu,t_n), \\ d(gy,gv)+d(gv,t_n^1), \\ d(gz,gw)+d(gw,t_n^{11}) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(gx,gu), \\ d(gy,gv), \\ d(gz,gw) \end{array}\right\} + \max\left\{\begin{array}{c} d(gu,t_n), \\ d(gv,t_n^1), \\ d(gw,t_n^{11}) \end{array}\right\} \tag{2.12}$$

$$\leq \max\left\{\begin{array}{c} d(gx,gu), \\ d(gy,gv), \\ d(gz,gw) \end{array}\right\} + \max\left\{\begin{array}{c} d(gu,F(x,y,z)), \\ d(gv,F(y,z,x)), \\ d(gw,F(z,x,y)) \end{array}\right\} + \tfrac{1}{n}\max\left\{\begin{array}{c} d(gx,gu), \\ d(gy,gv), \\ d(gz,gw) \end{array}\right\}$$

$$\leq (1+\tfrac{1}{n})\max\left\{\begin{array}{c} d(gx,gu), \\ d(gy,gv), \\ d(gz,gw) \end{array}\right\} + \theta\max\left\{\begin{array}{c} d(gu,gx),d(gv,gy), \\ d(gw,gz),d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)),d(gz,F(z,x,y)) \end{array}\right\}.$$

If

$$\max\left\{\begin{array}{c} d(gu,gx), \\ d(gv,gy), \\ d(gw,gz) \end{array}\right\} \geq \max\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\},$$

then from (2.12) we have

$$\max\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\} \leq (1+\theta+\frac{1}{n})\max\left\{\begin{array}{c} d(gu,gx), \\ d(gv,gy), \\ d(gw,gz) \end{array}\right\}.$$

Letting $n\to\infty$, we get

$$\max\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\} \leq (1+\theta)\max\left\{\begin{array}{c} d(gu,gx), \\ d(gv,gy), \\ d(gw,gz) \end{array}\right\}. \tag{2.13}$$

By (1.1), as $\frac{1}{2}\leq\theta<1$ we have $\phi(\theta)=1-\theta$, using (2.13) we have

$$\phi(\theta)\min\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\} \leq (1-\theta)\max\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\} \tag{2.14}$$

$$\leq (1-\theta^2)\max\left\{\begin{array}{c} d(gu,gx), \\ d(gv,gy), \\ d(gw,gz) \end{array}\right\} < \max\left\{\begin{array}{c} d(gu,gx), \\ d(gv,gy), \\ d(gw,gz) \end{array}\right\}.$$

If

$$\max\left\{\begin{array}{c} d(gu,gx), \\ d(gv,gy), \\ d(gw,gz) \end{array}\right\} < \max\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\},$$

then from (2.12), we have

$$(1-\theta)\ \max\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\} \leq (1+\frac{1}{n})\max\left\{\begin{array}{c} d(gu,gx), \\ d(gv,gy), \\ d(gw,gz) \end{array}\right\}.$$

Letting $n \to \infty$, we get

$$\phi(\theta)\min\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\} \leq (1-\theta)\max\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(gu,gx), \\ d(gv,gy), \\ d(gw,gz) \end{array}\right\}.$$

Thus, in both cases we have

$$\phi(\theta)\min\left\{\begin{array}{c} d(gx,F(x,y,z)), \\ d(gy,F(y,z,x)), \\ d(gz,F(z,x,y)) \end{array}\right\} \leq \max\left\{\begin{array}{c} d(gu,gx), \\ d(gv,gy), \\ d(gw,gz) \end{array}\right\}. \tag{2.15}$$

From (2.2) and (2.15), we have

$$H(F(x,y,z),F(u,v,w))$$

$$\leq \theta\max\left\{\begin{array}{c} d(gx,gu),d(gy,gv),d(gz,gw), \\ d(gx,F(x,y,z)),d(gy,F(y,z,x)),d(gz,F(z,x,y)), \\ d(gu,F(u,v,w)),d(gv,F(v,w,u)),d(gw,F(w,u,v)), \\ \frac{1}{2}\max\left\{\begin{array}{c} d(gx,F(u,v,w)),d(gy,F(v,w,u)),d(gz,F(w,u,v)), \\ d(gu,F(x,y,z)),d(gv,F(y,z,x)),d(gw,F(z,x,y)) \end{array}\right\} \end{array}\right\}.$$

Similarly, we can show that

$$H(F(y,z,x),F(v,w,u))$$

$$\leq \theta\max\left\{\begin{array}{c} d(gx,gu),d(gy,gv),d(gz,gw), \\ d(gx,F(x,y,z)),d(gy,F(y,z,x)),d(gz,F(z,x,y)), \\ d(gu,F(u,v,w)),d(gv,F(v,w,u)),d(gw,F(w,u,v)), \\ \frac{1}{2}\max\left\{\begin{array}{c} d(gx,F(u,v,w)),d(gy,F(v,w,u)),d(gz,F(w,u,v)), \\ d(gu,F(x,y,z)),d(gv,F(y,z,x)),d(gw,F(z,x,y)) \end{array}\right\} \end{array}\right\},$$

and

$$H(F(z,x,y),F(w,u,v))$$

$$\leq \theta \max \left\{ \begin{array}{c} d(gx,gu), d(gy,gv), d(gz,gw), \\ d(gx,F(x,y,z)), d(gy,F(y,z,x)), d(gz,F(z,x,y)), \\ d(gu,F(u,v,w)), d(gv,F(v,w,u)), d(gw,F(w,u,v)), \\ \frac{1}{2}\max \left\{ \begin{array}{c} d(gx,F(u,v,w)), d(gy,F(v,w,u)), d(gz,F(w,u,v)), \\ d(gu,F(x,y,z)), d(gv,F(y,z,x)), d(gw,F(z,x,y)) \end{array} \right\} \end{array} \right\}.$$

Thus

$$\max \left\{ \begin{array}{c} H(F(x,y,z),F(u,v,w)), \\ H(F(y,z,x),F(v,w,u)), \\ H(F(z,x,y),F(w,u,v)) \end{array} \right\}$$

$$\leq \theta \max \left\{ \begin{array}{c} d(gx,gu), d(gy,gv), d(gz,gw), \\ d(gx,F(x,y,z)), d(gy,F(y,z,x)), d(gz,F(z,x,y)), \\ d(gu,F(u,v,w)), d(gv,F(v,w,u)), d(gw,F(w,u,v)), \\ \frac{1}{2}\max \left\{ \begin{array}{c} d(gx,F(u,v,w)), d(gy,F(v,w,u)), d(gz,F(w,u,v)), \\ d(gu,F(x,y,z)), d(gv,F(y,z,x)), d(gw,F(z,x,y)) \end{array} \right\} \end{array} \right\}.$$

Hence the **Claim II** is true. Now consider

$$\max \left\{ \begin{array}{c} d(gu,F(u,v,w)), \\ d(gv,F(v,w,u)), \\ d(gw,F(w,u,v)) \end{array} \right\}$$

$$\leq \lim_{n\to\infty} \max \left\{ \begin{array}{c} d(gx_{n+1},F(u,v,w)), \\ d(gy_{n+1},F(v,w,u)), \\ d(gz_{n+1},F(w,u,v)) \end{array} \right\}$$

$$\leq \lim_{n\to\infty} \max \left\{ \begin{array}{c} H(F(x_n,y_n,z_n),F(u,v,w)), \\ H(F(y_n,z_n,x_n),F(v,w,u)), \\ H(F(z_n,x_n,y_n),F(w,u,v)) \end{array} \right\}$$

$$\leq \lim_{n\to\infty} \theta \max \left\{ \begin{array}{c} d(gx_n,gu), d(gy_n,gv), d(gz_n,gw), \\ d(gx_n,F(x_n,y_n,z_n)), d(gy_n,F(y_n,z_n,x_n)), \\ d(gz_n,F(z_n,x_n,y_n)), d(gu,F(u,v,w)), \\ d(gv,F(v,w,u)), d(gw,F(w,u,v)), \\ \max \left\{ \begin{array}{c} d(gx_n,F(u,v,w)), d(gy_n,F(v,w,u)), \\ d(gz_n,F(w,u,v)), d(gu,F(x_n,y_n,z_n)), \\ d(gv,F(y_n,z_n,x_n)), d(gw,F(z_n,x_n,y_n)) \end{array} \right\} \end{array} \right\}$$

$$\leq \theta \lim_{n\to\infty} \max \left\{ \begin{array}{c} d(gx_n, gu), d(gy_n, gv), d(gz_n, gw), \\ d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), \\ d(gu, F(u,v,w)), d(gv, F(v,w,u)), d(gw, F(w,u,v)), \\ \max \left\{ \begin{array}{c} d(gx_n, F(u,v,w)), d(gy_n, F(v,w,u)), d(gz_n, F(w,u,v)), \\ d(gu, gx_{n+1}), d(gv, gy_{n+1}), d(gw, gz_{n+1}) \end{array} \right\} \end{array} \right\}.$$

Letting $n \to \infty$ we get

$$\max \left\{ \begin{array}{l} d(gu, F(u,v,w)), \\ d(gv, F(v,w,u)), \\ d(gw, F(w,u,v)) \end{array} \right\} \leq \theta \max \left\{ \begin{array}{l} d(gu, F(u,v,w)), \\ d(gv, F(v,w,u)), \\ d(gw, F(w,u,v)) \end{array} \right\},$$

which in turn yields that $gu \in F(u,v,w)$, $gv \in F(v,w,u)$ and $gw \in F(w,u,v)$. Thus (u,v,w) is a tripled coincidence point of F and g.

Suppose now that (a) holds. Let (x,y,z) be a tripled coincidence point of F and g. Then there exist $u,v,w \in X$ such that $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$.

Since g is continuous at u,v,w, we have $gu = u$, $gv = v$ and $gw = w$. Based on the assumption that the pair (F,g) is w-compatible and $gx \in F(x,y,z)$, $gy \in F(y,z,x)$ and $gz \in F(z,x,y)$, we have $g^2x \in F(gx,gy,gz)$, $g^2y \in F(gy,gz,gx)$ and $g^2z \in F(gz,gx,gy)$ respectively. Thus (gx,gy,gz) is a tripled coincidence point of F and g. Similarly, we can show that $(g^n x, g^n y, g^n z)$ is a tripled coincidence point of F and g. Also it is clear that $g^n x \in F(g^{n-1}x, g^{n-1}y, g^{n-1}z)$, $g^n y \in F(g^{n-1}y, g^{n-1}z, g^{n-1}x)$ and $g^n z \in F(g^{n-1}z, g^{n-1}x, g^{n-1}y)$. Now we have

$$\phi(\theta) \min \left\{ \begin{array}{l} d(g^n x, F(g^{n-1}x, g^{n-1}y, g^{n-1}z)), \\ d(g^n y, F(g^{n-1}y, g^{n-1}z, g^{n-1}x)), \\ d(g^n z, F(, g^{n-1}z, g^{n-1}x, g^{n-1}y)) \end{array} \right\} = 0 \leq \max \left\{ \begin{array}{l} d(g^n x, gu), \\ d(g^n y, gv), \\ d(g^n z, gw), \end{array} \right\}.$$

Hence from (2.2), we have

$$d(g^n x, F(u,v,w))) \leq H(F(g^{n-1}x, g^{n-1}y, g^{n-1}z), F(u,v,w))$$

$$\leq \theta \max \left\{ \begin{array}{c} d(g^n x, gu), d(g^n y, gv), d(g^n z, gw) \\ d(g^n x, F(g^{n-1}x, g^{n-1}y, g^{n-1}z)), d(g^n y, F(g^{n-1}y, g^{n-1}z, g^{n-1}x)), \\ d(g^n z, F(g^{n-1}z, g^{n-1}x, g^{n-1}y)), d(gu, F(u,v,w)), \\ d(gv, F(v,w,u)), d(gw, F(w,u,v)) \\ \frac{1}{2} \max \left\{ \begin{array}{c} d(g^n x, F(u,v,w)), d(g^n y, F(v,w,u)), \\ d(g^n z, F(w,u,v)), d(gu, F(g^{n-1}x, g^{n-1}y, g^{n-1}z)), \\ d(gv, F(g^{n-1}y, g^{n-1}z, g^{n-1}x)), d(gw, F(g^{n-1}z, g^{n-1}x, g^{n-1}y)) \end{array} \right\} \end{array} \right\}.$$

Letting $n \to \infty$, we get

$$d(u, F(u,v,w)) \leq \theta \max\{d(u, F(u,v,w)), d(v, F(v,w,u)), d(w, F(w,u,v))\}.$$

Similarly, we have

$$d(v, F(v,w,u)) \leq \theta \max\{d(u, F(u,v,w)), d(v, F(v,w,u)), d(w, F(w,u,v))\},$$

and

$$d(w,F(w,u,v)) \leq \theta \max\{d(u,F(u,v,w)), d(v,F(v,w,u)), d(w,F(w,u,v))\}.$$

Thus

$$\max\left\{\begin{array}{l} d(u,F(u,v,w)), \\ d(v,F(v,w,u)), \\ d(w,F(w,u,v)) \end{array}\right\} \leq \theta \max\left\{\begin{array}{l} d(u,F(u,v,w)), \\ d(v,F(v,w,u)), \\ d(w,F(w,u,v)) \end{array}\right\},$$

which in turn yields that $u \in F(u,v,w)$, $v \in F(v,w,u)$ and $w \in F(w,u,v)$. Thus (u,v,w) is a tripled common fixed point of F and g.

Suppose now that (b) holds. Let (x,y,z) be a tripled coincidence point of F and g. Then there exist $u,v,w \in X$ such that $\lim_{n\to\infty} g^n u = x$, $\lim_{n\to\infty} g^n v = y$ and $\lim_{n\to\infty} g^n w = z$. Since g is continuous at x, y and z, we have $gx = x$, $gy = y$ and $gz = z$. Thus (x,y,z) is a tripled common fixed point of F and g. □

The following example illustrates Theorem 3.

Example 1. Let $X = [0,1]$, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be defined by $F(x,y,z) = [0, \frac{x}{8} + \frac{y}{4} + \frac{z}{3}]$ and $gx = \frac{7x}{8}$. Then for all $x,y,z,u,v,w \in X$, consider

$$\begin{aligned} H(F(x,y,z),F(u,v,w)) &= \left|(\frac{x}{8} + \frac{y}{4} + \frac{z}{3}) - (\frac{u}{8} + \frac{v}{4} + \frac{w}{3})\right| \\ &\leq \frac{1}{8}|x-u| + \frac{1}{4}|y-v| + \frac{1}{3}|z-w| \\ &= \frac{1}{7}|gx-gu| + \frac{2}{7}|gy-gv| + \frac{8}{21}|gz-gw| \\ &\leq \frac{17}{21}\max\{d(gx,gu), d(gy,gv), d(gz,gw)\}. \end{aligned}$$

Clearly all conditions of Theorem 3 are satisfied and $(0,0,0)$ is the tripled common fixed point of F and g.

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