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FIXED POINT RESULTS FOR GENERALIZED RATIONAL α-GERAGHTY CONTRACTION

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Abstract. In this paper, an effort has been made to improve the notion of α-Geraghty contraction type mappings and establish some common fixed point theorems for a pair of α-admissible mappings under the improved approach of generalized rational α-Geraghty contractive type condition in a complete metric space. An example has been constructed to demonstrate the novelty of these results.

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1. Preliminaries and scope

In 1973, Geraghty [3] studied a generalization of Banach contraction principle. He generalized the Banach contraction principle in a different way than it was done by different investigators. In 2012, Samet et al. [16], introduced a concept of $\alpha-\psi$- contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards, Karapinar [11], refined the notion and obtained various fixed point results. See more results in [9]. Hussain et al. [7], generalized the concept of α-admissible mappings and proved fixed point theorems. Subsequently, Abdeljawad [1] introduced a pair of $\alpha-$admissible mappings satisfying new sufficient contractive conditions different from those in [7], [16] and obtained fixed point and common fixed point theorems. Salimi et al. [15], modified the concept of $\alpha-\psi-$contractive mappings and established fixed point results. Recently, Hussain et al. [8] proved some fixed point results for single and set-valued $\alpha-\eta-\psi$-contractive mappings in the setting of complete metric space. Mohammadi et al. [13], introduced a new notion of $\alpha-\phi-$contractive mappings and showed that it was a real generalization for some old results. Thereafter, many papers have published on geraghty contractions. For more detail see [4–6, 8, 11, 14] and references therein.

Definition 1 ([16])**.** Let $S : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. We say that S is α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Sx, Sy) \geq 1$.

Example 1 ([12]). Consider $X = [0,\infty)$, and define $S : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by $Sx = 2x$, for all $x, y \in X$ and

$$\alpha(x,y) = \begin{cases} e^{\frac{y}{x}}, & \text{if } x \geq y, x \neq 0 \\ 0, & \text{if } x < y. \end{cases}$$

Then S is $\alpha-$admissible.

Definition 2 ([1]). Let $S, T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that the pair (S,T) is α-admissible if $x, y \in X$ such that $\alpha(x,y) \geq 1$, then we have $\alpha(Sx,Ty) \geq 1$ and $\alpha(Tx,Sy) \geq 1$.

Definition 3 ([10]). Let $S : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that S is triangular α-admissible if $x, y \in X$, $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1 \Rightarrow \alpha(x,y) \geq 1$.

Definition 4 ([10]). Let $S : X \to X$ and $\alpha : X \times X \to (-\infty,+\infty)$. We say that S is a triangular α-admissible mapping if

(T1) $\alpha(x,y) \geq 1$ implies $\alpha(Sx,Sy) \geq 1$, $x, y \in X$,
(T2) $\alpha(x,z) \geq 1$, $\alpha(z,y) \geq 1$, implies $\alpha(x,y) \geq 1$, $x, y, z \in X$.

Definition 5 ([1]). Let $S, T : X \to X$ and $\alpha : X \times X \to (-\infty,+\infty)$. We say that a pair (S,T) is triangular α-admissible if

(T1) $\alpha(x,y) \geq 1$, implies $\alpha(Sx,Ty) \geq 1$ and $\alpha(Tx,Sy) \geq 1$, $x, y \in X$.
(T2) $\alpha(x,z) \geq 1$, $\alpha(z,y) \geq 1$, implies $\alpha(x,y) \geq 1$, $x, y, z \in X$.

Definition 6 ([15]). Let $S : X \to X$ and let $\alpha, \eta : X \times X \to [0,+\infty)$ be two functions. We say that T is α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x,y) \geq \eta(x,y) \Rightarrow \alpha(Sx,Sy) \geq \eta(Sx,Sy)$. Note that if we take $\eta(x,y) = 1$, then this definition reduces to definition in [16]. Also if we take $\alpha(x,y) = 1$, then we says that S is an η-subadmissible mapping.

Lemma 1 ([2]). *Let $S : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Sx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

Lemma 2. *Let $S, T : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define sequence $x_{2i+1} = Sx_{2i}$, and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \ldots$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

We denote by Ω the family of all functions $\beta : [0,+\infty) \to [0,1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$.

Theorem 1 ([3]). *Let (X,d) be a metric space. Let $S : X \to X$ be a self mapping. Suppose that there exists $\beta \in \Omega$ such that for all $x, y \in X$,*

$$d(Sx,Sy) \leq \beta(d(x,y))\, d(x,y).$$

then S has a fixed unique point $p \in X$ and $\{S^n x\}$ converges to p for each $x \in X$.

2. RESULTS

In this section, we prove some fixed point theorems satisfying generalized rational α-Geraghty contraction type mappings in complete metric space. Let (X,d) be a metric space, and let $\alpha : X\times X \to \mathbb{R}$ be a function. Let $S,T : X \to X$ is called a pair of generalized rational α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x,y \in X$,

$$\alpha(x,y)d(Sx,Ty) \le \beta(M(x,y))M(x,y) \tag{2.1}$$

where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Ty)}{1+d(Sx,Ty)}\right\}.$$

If $S = T$ then T is called generalized rational α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x,y \in X$,

$$\alpha(x,y)d(Tx,Ty) \le \beta(N(x,y))N(x,y)$$

where

$$N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\}.$$

Theorem 2. *Let (X,d) be a complete metric space, $\alpha : X\times X \to \mathbb{R}$ be a function. Let $S,T : X \to X$ be two mappings then suppose that the following holds:*
(i) (S,T) is pair of generalized rational α-Geraghty contraction type mapping;
(ii) (S,T) is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \ge 1$;
(iv) S and T are continuous;
Then (S,T) have common fixed point.

Proof. Let x_1 in X be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that,

$$x_{2i+1} = Sx_{2i}, \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0,1,2,\ldots. \tag{2.2}$$

By assumption $\alpha(x_0,x_1) \ge 1$ and pair (S,T) is α -admissible, by Lemma 2, we have

$$\alpha(x_n,x_{n+1}) \ge 1 \text{ for all } n \in \mathbb{N}\cup\{0\}. \tag{2.3}$$

Then

$$\begin{aligned} d(x_{2i+1},x_{2i+2}) &= d(Sx_{2i},Tx_{2i+1}) \le \alpha(x_{2i},x_{2i+1})d(Sx_{2i},Tx_{2i+1}) \\ &\le \beta(M(x_{2i},x_{2i+1}))M(x_{2i},x_{2i+1}), \end{aligned}$$

for all $i \in \mathbb{N}\cup\{0\}$. Now

$$\begin{aligned} M(x_{2i},x_{2i+1}) &= \max\left\{d(x_{2i},x_{2i+1}), \frac{d(x_{2i},Sx_{2i})d(x_{2i+1},Tx_{2i+1})}{1+d(x_{2i},x_{2i+1})}, \frac{d(x_{2i},Sx_{2i})d(x_{2i+1},Tx_{2i+1})}{1+d(Sx_{2i},Tx_{2i+1})}\right\} \\ &= \max\left\{d(x_{2i},x_{2i+1}), \frac{d(x_{2i},x_{2i+1})d(x_{2i+1},x_{2i+2})}{1+d(x_{2i},x_{2i+1})}, \frac{d(x_{2i},x_{2i+1})d(x_{2i+1},x_{2i+2})}{1+d(x_{2i+1},x_{2i+2})}\right\} \end{aligned}$$

$$\leq \max\{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2})\}.$$

Thus

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq \beta(M(x_{2i}, x_{2i+1}))M(x_{2i}, x_{2i+1}) \\ &\leq \beta(d(x_{2i}, x_{2i+1}))d((x_{2i}, x_{2i+1}) < d(x_{2i}, x_{2i+1}). \end{aligned}$$

so that,

$$d((x_{2i+1}, x_{2i+2}) < d(x_{2i}, x_{2i+1}). \tag{2.4}$$

This implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.5}$$

So, sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. Now, we prove that $d(x_n, x_{n+1}) \to 0$. It is clear that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Therefore, there exists some positive number r such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. From (2.4), we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \beta(d(x_n, x_{n+1})) \leq 1.$$

Now by taking limit $n \to \infty$, we have

$$1 \leq \beta(d(x_n, x_{n+1})) \leq 1,$$

that is

$$\lim_{n\to\infty} \beta(d(x_n, x_{n+1})) = 1.$$

By the property of β, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

Now, we show that sequence $\{x_n\}$ is Cauchy sequence. Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that, for all positive integers k, we have $m_k > n_k > k$,

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \tag{2.7}$$

and

$$d(x_{m_k}, x_{n_{k-1}}) < \epsilon. \tag{2.8}$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + d(x_{n_{k-1}}, x_{n_k}). \end{aligned}$$

That is,

$$\epsilon < \epsilon + d(x_{n_{k-1}}, x_{n_k}) \tag{2.9}$$

for all $k \in \mathbb{N}$. In the view of (2.9), (2.6), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{2.10}$$

Again using triangle inequality, we have

$$d(x_{m_k}, x_{n_k}) \le d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$$

and

$$d(x_{m_{k+1}}, x_{n_{k+1}}) \le d(x_{m_{k+1}}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}}).$$

Taking limit as $k \to +\infty$ and using (2.6) and (2.10), we obtain

$$\lim_{k\to+\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon. \tag{2.11}$$

By Lemma 2, $\alpha(x_{n_k}, x_{m_{k+1}}) \ge 1$, we have

$$\begin{aligned} d(x_{n_{k+1}}, x_{m_{k+2}}) = d(Sx_{n_k}, Tx_{m_{k+1}}) &\le \alpha(x_{n_k}, x_{m_{k+1}}) d(Sx_{n_k}, Tx_{m_{k+1}}) \\ &\le \beta(M(x_{n_k}, x_{m_{k+1}})) M(x_{n_k}, x_{m_{k+1}}). \end{aligned}$$

Finally, we conclude that

$$\frac{d(x_{n_{k+1}}, x_{m_{k+2}})}{M(x_{n_k}, x_{m_{k+1}})} \le \beta(M(x_{n_k}, x_{m_{k+1}})).$$

By using (2.6), taking limit as $k \to +\infty$ in the above inequality, we obtain

$$\lim_{k\to\infty} \beta(d(x_{n_k}, x_{m_{k+1}})) = 1. \tag{2.12}$$

So, $\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = 0 < \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete so there exists $p \in X$ such that $x_n \to p$ implies that $x_{2i+1} \to p$ and $x_{2i+2} \to p$. As S and T are continuous, so we get $Tx_{2i+1} \to Tp$ and $Sx_{2i+2} \to Sp$. Thus $p = Sp$ similarly, $p = Tp$, we have $Sp = Tp = p$. Then (S,T) have common fixed point. □

In the following Theorem, we dropped continuity.

Theorem 3. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S,T : X \to X$ be two mappings then suppose that the following holds:*

(i) (S,T) is a pair of generalized rational α -Geraghty contraction type mapping;

(ii) (S,T) is triangular α-admissible;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \ge 1$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \ge 1$ for all k.

Then (S,T) have common fixed point.

Proof. Follows the similar lines of the Theorem 2. Define a sequence $x_{2i+1} = Sx_{2i}$, and $x_{2i+2} = Tx_{2i+1}$, where $i = 0,1,2,\ldots.$ converges to $p \in X$. By the hypotheses of (iv) there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k}, p) \ge 1$ for all k. Now by using (2.1) for all k, we have

$$d(x_{2n_k+1}, Tp) = d(Sx_{2n_k}, Tp) \le \alpha(x_{2n_k}, p) d(Sx_{2n_k}, Tp)$$

$$\leq \beta\left(M(x_{2n_k},p)\right)M(x_{2n_k},p).$$

so that,

$$d(x_{2n_k+1},Tp) \leq \beta\left(M(x_{2n_k},p)\right)M(x_{2n_k},p). \tag{2.13}$$

On the other hand, we obtain

$$M(x_{2n_k},p) = \max\left\{d(x_{2n_k},p), \frac{d(x_{2n_k},Sx_{2n_k}),d(p,Tp)}{1+d(x_{2n_k},p)}, \frac{d(x_{2n_k},Sx_{2n_k}),d(p,Tp)}{1+d(Sx_{2n_k},Tp)}\right\}.$$

Letting $k \to \infty$ then we have

$$\lim_{k\to\infty} M(x_{2n_k},p) = \max\{d(p,Sp),d(p,Tp)\}. \tag{2.14}$$

Case I.
$\lim_{k\to\infty} M(x_{2n_k},p) = d(p,Tp)$. Suppose that $d(p,Tp) > 0$. From (2.14), for a large k , we have $M(x_{2n_k},p) > 0$, which implies that

$$\beta(M(x_{2n_k},p)) < M(x_{2n_k},p).$$

Then, we have

$$d(x_{2n_k},Tp) < M(x_{2n_k},p) \tag{2.15}$$

Letting $k \to \infty$ in (2.15), we obtain that $d(p,Tp) < d(p,Tp)$, which is a contradiction. Thus, we find that $d(p,Tp) = 0$, implies $p = Tp$.
Case II.
$\lim_{k\to\infty} M(x_{2n_k},p) = d(p,Sp)$. Similarly $p = Sp$. Thus $p = Tp = Sp$. □

If $M(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)d(y,Sy)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Sy)}{1+d(Sx,Sy)}\right\}$ and $S = T$ in Theorem 2 and Theorem 3 , we have the following corollaries.

Corollary 1. *Let (X,d) be a complete metric space and let S is $\alpha-$ admissible mappings such that the following holds:*
(i) S is a generalized rational α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0,T(x_0)) \geq 1$;
(iv) S is continuous;
Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Corollary 2. *Let (X,d) be a complete metric space and let S is $\alpha-$ admissible mappings such that the following holds:*
(i) S is a generalized rational α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$;
(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},p) \geq 1$ for all k.

Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

If $M(x,y) = \max\{d(x,y), d(x,Sx), d(y,Sy)\}$ in Theorem 1, Theorem 2, we obtain the following corollaries.

Corollary 3 ([2])**.** *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S : X \to X$ be a mapping then suppose that the following holds:*
(i) S is a generalized α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;
(iv) S is continuous;
Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Corollary 4 ([2])**.** *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S : X \to X$ be a mapping then suppose that the following holds:*
(i) S is a generalized α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;
(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k.
Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Let (X,d) be a metric space, and let $\alpha, \eta : X \times X \to \mathbb{R}$ be a function. A map $S,T : X \to X$ is called a pair of generalized rational α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x, y \in X$,

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow d(Sx,Ty) \leq \beta(M(x,y))\, M(x,y) \tag{2.16}$$

where

$$M(x,y) = \max\left\{ d(x,y), \frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Ty)}{1+d(Sx,Ty)} \right\}.$$

Theorem 4. *Let (X,d) be a complete metric space. Let S is $\alpha-$admissible mappings with respect to η such that the following holds:*
(i) (S,T) is a generalized rational α-Geraghty contraction type mapping;
(ii) (S,T) is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;
(iv) S and T are continuous;
Then (S,T) have common fixed point.

Proof. Let x_1 in X be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that,

$$x_{2i+1} = Sx_{2i}, \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where} i = 0,1,2,\ldots. \tag{2.17}$$

By assumption $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$ and the pair (S,T) is α-admissible with respect to η, we have, $\alpha(Sx_0,Tx_1) \geq \eta(Sx_0,Tx_1)$ from which we deduce that $\alpha(x_1,x_2) \geq \eta(x_1,x_2)$ which also implies that $\alpha(Tx_1,Sx_2) \geq \eta(Tx_1,Sx_2)$. Continuing in this way we obtain $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$.

$$\begin{aligned} d(x_{2i+1},x_{2i+2}) &= d(Sx_{2i},Tx_{2i+1}) \leq \alpha(x_{2i},x_{2i+1})d(Sx_{2i},Tx_{2i+1}) \\ &\leq \beta\left(M(x_{2i},x_{2i+1})\right)M(x_{2i},x_{2i+1}), \end{aligned}$$

Therefore,

$$d(x_{2i+1},x_{2i+2}) \leq \alpha(x_{2i},x_{2i+1})d(Sx_{2i},Tx_{2i+1}) \tag{2.18}$$

for all $i \in \mathbb{N}\cup\{0\}$. Now

$$\begin{aligned} M(x_{2i},x_{2i+1}) &= \max\left\{d(x_{2i},x_{2i+1}), \frac{d(x_{2i},Sx_{2i})d(x_{2i+1},Tx_{2i+1})}{1+d(x_{2i},x_{2i+1})}, \frac{d(x_{2i},Sx_{2i})d(x_{2i+1},Tx_{2i+1})}{1+d(Sx_{2i},Tx_{2i+1})}\right\} \\ &= \max\left\{d(x_{2i},x_{2i+1}), \frac{d(x_{2i},x_{2i+1})d(x_{2i+1},x_{2i+2})}{1+d(x_{2i},x_{2i+1})}, \frac{d(x_{2i},Sx_{2i})d(x_{2i+1},Tx_{2i+1})}{1+d(x_{2i+1},x_{2i+2})}\right\} \\ &\leq \max\left\{d(x_{2i},x_{2i+1}), d(x_{2i+1},x_{2i+2})\right\}. \end{aligned}$$

From the definition of β, the case $M(x_{2i},x_{2i+1}) = d(x_{2i+1},x_{2i+2})$ is impossible.

$$\begin{aligned} d(x_{2i+1},x_{2i+2}) &\leq \beta\left(M(x_{2i},x_{2i+1})\right)M(x_{2i},x_{2i+1}) \\ &\leq \beta\left(d(x_{2i+1},x_{2i+2})\right)d(x_{2i+1},x_{2i+2}) < d(x_{2i+1},x_{2i+2}). \end{aligned}$$

Which is a contradiction. Otherwise, in other case

$$\begin{aligned} d(x_{2i+1},x_{2i+2}) &\leq \beta\left(M(x_{2i},x_{2i+1})\right)M(x_{2i},x_{2i+1}) \\ &\leq \beta\left(d(x_{2i},x_{2i+1})\right)d((x_{2i},x_{2i+1}) < d(x_{2i},x_{2i+1}). \end{aligned}$$

This, implies that

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}), \text{ for all } n \in \mathbb{N}\cup\{0\}. \tag{2.19}$$

Follows the similar lines of the Theorem 2. Hence p is common fixed point of S and T. □

Theorem 5. *Let (X,d) be a complete metric space and let (S,T) are $\alpha-$admissible mappings with respect to η such that the following holds:*
(i) (S,T) is a generalized rational α-Geraghty contraction type mapping;
(ii) (S,T) is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq \eta(x_0,Sx_0)$;
(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},p) \geq \eta(x_{n_k},p)$ for all k.
Then S and T has common fixed point.

Proof. Follows the similar line of the Theorem 3. □

If $M(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)d(y,Sy)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Sy)}{1+d(Sx,Sy)}\right\}$ and $S = T$ in the Theorem 4, Theorem 5 , we get the following corollaries.

Corollary 5. *Let (X,d) be a complete metric space and let S is $\alpha-$admissible mappings with respect to η such that the following holds:*
(i) S is a generalized rational α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq \eta(x_0,Sx_0)$;
(iv) S is continuous;
Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Corollary 6. *Let (X,d) be a complete metric space and let S is $\alpha-$admissible mappings with respect to η such that the following holds:*
(i) S is a generalized rational α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq \eta(x_0,Sx_0)$;
(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},p) \geq \eta(x_{n_k},p)$ for all k.
Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Example 2. Let $X = \{1,2,3\}$ with metric

$$d(1,3) = d(3,1) = \frac{5}{7}\; d(1,1) = d(2,2) = d(3,3) = 0$$

$$d(1,2) = d(2,1) = 1,\; d(2,3) = d(3,2) = \frac{4}{7}$$

$$\alpha(x,y) = \left\{\begin{array}{ll} 1, & \text{if } x,y \in X, \\ 0, & \text{otherwise} \end{array}\right\}.$$

Define the mappings $S,T : X \to X$ as follows:

$$Sx = 1 \text{ for each } x \in X.$$

$$T(1) = T(3) = 1,\; T(2) = 3.$$

and $\beta : [0,+\infty) \to [0,1]$, then

$$\alpha(x,y)d(Tx,Ty) \leq \beta(M(x,y))M(x,y).$$

Let $x = 2$ and $y = 3$ then condition (2.1) is not satisfied.

$$d\,(T(2),T(3)) = d(3,1) = \frac{5}{7}$$

$$M(x,y) = \max\{d(2,3), d(2,T(2)), d(3,T(3))\}$$
$$= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}\right\} = \frac{5}{7}$$

and

$$\alpha(2,3)d\,(T(2),T(3)) \nleq \beta(M(x,y))M(x,y).$$

If

$$M(x,y) = \max\left\{d(2,3), \frac{d(2,T(2))d(3,T(3))}{1+d(2,3)}, \frac{d(2,T(2))d(3,T(3))}{1+d(T2,T3)}\right\}$$
$$= \max\left\{\frac{4}{7}, \frac{20}{77}, \frac{20}{84}\right\} = \frac{4}{7}$$

Then the contractions does not holds.

$$\alpha(2,3)d\,(T(2),T(3)) \nleq \beta(M(x,y))M(x,y).$$

We prove that Theorem 1 can be applied to S and T. Let $x,y \in X$, clearly (S,T) is $\alpha-$admissible mapping such that $\alpha(x,y) \geq 1$. Let $x,y \in X$ and so that $Sx,Ty \in X$ and $\alpha(Sx,Ty) = 1$. Hence (S,T) is α-admissible. We show that condition (2.1) of Theorem 1 is satisfied. If $x,y \in X$ then $\alpha(x,y) = 1$, we have

$$\alpha(x,y)d(Sx,Ty) \leq \beta(M(x,y))\,(M(x,y)).$$

where

$$M(x,y) = \max\left\{d(2,3), \frac{d(2,S(2))d(3,T(3))}{1+d(2,3)}, \frac{d(2,S(2))d(3,T(3))}{1+d(S2,T3)}\right\}$$
$$= \max\left\{\frac{4}{7}, \frac{20}{77}, \frac{20}{49}\right\} = \frac{4}{7}$$

and

$$d(S2,T3) = d(1,1) = 0.$$
$$\alpha(x,y)d(Sx,Ty) \leq \beta(M(x,y))\,(M(x,y)).$$

Hence all the hypothesis of the Theorem 1 is satisfied, So S,T have a common fixed point.

Remark 1. More detail, applications and examples see in [2] and references there in. Our results are more general than those in [2], [15] and improve several results existing in the literature.

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