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GENERAL QUASILINEARIZATION METHOD FOR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH A SINGULAR MATRIX

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ABSTRACT. The method of quasilinearization coupled with the method of lower and upper solutions has been very useful in providing an analytical approach to obtaining approximate solutions of non-linear differential equations. In this paper, it is applied to systems of non-linear differential equations with a singular matrix. Sequences of approximate solutions are convergent to the solution and the convergence is quadratic or semiquadratic.

Mathematics Subject Classification: 34A45

Keywords: quasilinearization, monotone iterations, quadratic and semiquadratic convergence

1. INTRODUCTION

Let $y_0, z_0 \in C^1(J, \mathbb{R}^m)$ with $y_0(t) \le z_0(t)$, $y_0'(t) \le z_0'(t)$ on J and define the following set

$$\Omega = \{(t, u, v) : \; y_0(t) \le u \le z_0(t), \quad y_0'(t) \le v \le z_0'(t), \quad t \in J, \; u, v \in \mathbb{R}^m\}.$$

In this paper, the vectorial inequalities mean that the same inequalities hold between their corresponding components.

Assume that A is a singular square matrix of order m and $f \in C(\Omega, \mathbb{R}^m)$. In this paper we shall study the following system of differential equations

$$Ax'(t) = f(t, x(t), x'(t)), \quad t \in J = [0, b] \tag{1.1}$$

with the initial condition

$$x(0) = x_0 \in \mathbb{R}^m. \tag{1.2}$$

The method of quasilinearization offers an approach for obtaining approximate solutions to non-linear differential equations. It has been generalized in recent years by Lakshmikantham and various coauthors to apply to a wide variety of problems, (see, for example, [5–12] and [3, 4]). In this paper, we apply this technique to problems of type (1.1)–(1.2). We show that it is possible to construct monotone sequences that converge to the solution if f is replaced by $f + g$ with $f + \Phi$ convex and $g + \Psi$ concave for some convex function Φ and for some concave function Ψ. This convergence is quadratic or semiquadratic. This paper generalizes the results of [4]. If f does not depend on the third variable with a unit matrix in the place of A, then problem (1.1)–(1.2) is considered in [8].

2. Assumptions

In the place of (1.1)–(1.2), we consider the system of differential equations of the form:

$$\begin{aligned} Ax'(t) &= f(t, x(t), x'(t)) + g(t, x(t), x'(t)) \equiv \mathscr{F}(t, x, x'), \quad t \in J, \\ x(0) &= x_0 \in \mathbb{R}^m, \end{aligned} \tag{2.1}$$

where $J = [0, b]$ and $f, g \in C(J \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$. Note that problem (2.1) is identical with the problem

$$\begin{aligned} x'(t) &= (A + B)^{-1}[\mathscr{F}(t, x, x') + Bx'(t)], \quad t \in J, \\ x(0) &= x_0 \end{aligned}$$

provided that B is an $m \times m$ matrix such that $(A + B)^{-1}$ exists.

A function $v \in C^1(J, \mathbb{R})$ is said to be a lower solution of problem (2.1) if

$$v'(t) \le (A + B)^{-1}[\mathscr{F}(t, v(t), v'(t)) + Bv'(t)], \quad t \in J, \quad v(0) \le x_0,$$

and an upper solution of (2.1) if the inequalities in these relations are reversed.

Let us introduce the following assumptions:

H_1. There exists a square matrix B of order m such that the matrix $A + B$ is non-singular and $(A + B)^{-1}B \ge 0$; moreover, for $f, g \in C(\Omega, \mathbb{R}^m)$, function $\mathscr{F} = f + g$ satisfies the Lipschitz condition with respect to the last variable, so for $u, \alpha, \bar{\alpha} \in \mathbb{R}^m$ such that $y_0(t) \le u \le z_0(t)$, $y_0'(t) \le \alpha, \bar{\alpha} \le z_0'(t)$ on J, the condition

$$|(A + B)^{-1}[\mathscr{F}(t, u, \alpha) - \mathscr{F}(t, u, \bar{\alpha})]| \le (A + B)^{-1}B|\alpha - \bar{\alpha}|$$

holds, where $|\alpha| = (|\alpha_1|, \ldots, |\alpha_m|)^T$ for $\alpha \in \mathbb{R}^m$.

H_2. $f_x, g_x, \Phi, \Phi_x, \Phi_y, \Psi, \Psi_x, \Psi_y \in C(\Omega, \mathbb{R}^m)$; here x and y denote the second and third variable, respectively.

H_3. The matrices $(A + B)^{-1}F_x$, $(A + B)^{-1}\Phi_x$ are non-decreasing with respect to the second variable, and $(A + B)^{-1}G_x$, $(A + B)^{-1}\Psi_x$ are non-increasing with respect to the second variable on Ω with $F = f + \Phi$, $G = g + \Psi$.

H_4. $(A + B)^{-1}F_x$, $(A + B)^{-1}\Phi_x$ are non-decreasing in the third variable, and $(A + B)^{-1}G_x$, $(A + B)^{-1}\Psi_x$ are non-increasing in the third variable on Ω.

H_5. $(A + B)^{-1}V(t, y_0, z_0) \ge 0$, $t \in J$ for some function V defined later [$(A + B)^{-1}V(t, y_0, z_0) \ge 0$ means that the entries of the matrix $(A + B)^{-1}V(t, y_0, z_0)$ are non-negative].

H_6. There exist $m\times m$ matrices C_1, C_2, C_3, C_4 with non-negative entries such that

$$\left|(A+B)^{-1}\left[f_x(t,u,v)-f_x(t,\bar{u},\bar{v})\right]\right| \le C_1\sum_{i=1}^{m}\left[|u_i-\bar{u}_i|+|v_i-\bar{v}_i|\right],$$

$$\left|(A+B)^{-1}\left[g_x(t,u,v)-g_x(t,\bar{u},\bar{v})\right]\right| \le C_2\sum_{i=1}^{m}\left[|u_i-\bar{u}_i|+|v_i-\bar{v}_i|\right],$$

$$\left|(A+B)^{-1}\left[\Phi_x(t,u,v)-\Phi_x(t,\bar{u},\bar{v})\right]\right| \le C_3\sum_{i=1}^{m}\left[|u_i-\bar{u}_i|+|v_i-\bar{v}_i|\right],$$

$$\left|(A+B)^{-1}\left[\Psi_x(t,u,v)-\Psi_x(t,\bar{u},\bar{v})\right]\right| \le C_4\sum_{i=1}^{m}\left[|u_i-\bar{u}_i|+|v_i-\bar{v}_i|\right]$$

for $y_0(t)\le u\le z_0(t)$, $y_0'(t)\le v,\bar{v}\le z_0'(t)$, $t\in J$ with $u,\bar{u},v,\bar{v}\in\mathbb{R}^m$.

3. Main results

The next lemma is a special case of Theorem 1.1.4 from [8].

Lemma 1. *Assume that $s_{ij}(t)\ge 0$, $t\in J$ for $i\ne j$, where $S=[s_{ij}]$ is a continuous square matrix of order m. Let $p\in C^1(J,\mathbb{R}^m)$ and*

$$p'(t)\le S(t)p(t),\quad t\in J,$$
$$p(0)\le 0=[\underbrace{0,\dots,0}_{m}]^T.$$

Then $p(t)\le 0$ on J.

Lemma 2. *Let assumptions H_1 and H_3 be satisfied. Then, for $u,v,\bar{u},\bar{v}\in\mathbb{R}^m$ such that $y_0(t)\le u\le\bar{u}\le z_0(t)$, $y_0'(t)\le v\le\bar{v}\le z_0'(t)$, $t\in J$, we have*

$$(A+B)^{-1}[\mathscr{F}(t,u,v)-\mathscr{F}(t,\bar{u},\bar{v})]\le (A+B)^{-1}\{[-F_x(t,u,v)-G_x(t,\bar{u},v)$$
$$+\Phi_x(t,\bar{u},v)+\Psi_x(t,u,v)](\bar{u}-u)+B(\bar{v}-v)\}.$$

Proof. The mean value theorem and assumption H_1 yield

$$\begin{aligned}
&(A+B)^{-1}[\mathscr{F}(t,u,v)-\mathscr{F}(t,\bar{u},\bar{v})]\\
&\quad=(A+B)^{-1}[\mathscr{F}(t,u,v)-\mathscr{F}(t,\bar{u},v)+\mathscr{F}(t,\bar{u},v)-\mathscr{F}(t,\bar{u},\bar{v})]\\
&\quad\le(A+B)^{-1}\left\{\left[\int_0^1\mathscr{F}_x(t,su+(1-s)\bar{u},v)ds\right](u-\bar{u})+B(\bar{v}-v)\right\}\\
&\quad=(A+B)^{-1}\left\{\int_0^1[F_x(t,su+(1-s)\bar{u},v)+G_x(t,su+(1-s)\bar{u},v)\right.\\
&\qquad\left.-\Phi_x(t,su+(1-s)\bar{u},v)-\Psi_x(t,su+(1-s)\bar{u},v)]ds(u-\bar{u})+B(\bar{v}-v)\right\}.
\end{aligned}$$

Hence, we have the assertion of Lemma 2, by using assumption H_3. □

Now we are in a position to prove the following result:

Theorem 1. *Assume that* $f, g \in C(\Omega, \mathbb{R}^m)$, *and*

(i) $y_0, z_0 \in C^1(J, \mathbb{R}^m)$ *are lower and upper solutions of problem* (2.1) *and such that* $y_0(t) \le z_0(t)$ *and* $y_0'(t) \le z_0'(t)$ *on* J,
(ii) *Assumptions* H_1–H_6 *hold with*

$$V(t, y, z) = F_x(t, y, y') + G_x(t, z, z') - \Phi_x(t, z, z') - \Psi_x(t, y, y').$$

(iii) *Problem* (2.1) *has at most one solution.*

Then, there exist monotone sequences $\{y_n\}$, $\{z_n\}$ *which converge uniformly on* J *to the unique solution* x *of problem* (2.1). *Moreover, the convergence is quadratic with respect to* u *and it is semiquadratic with respect to* u' *for* $u = y_n$ *and* $u = z_n$.

Proof. Let y_{n+1} and z_{n+1} be the solutions of the linear initial value problems

$$\begin{aligned} &y_{n+1}'(t) = (A + B)^{-1}\{\mathscr{F}(t, y_n, y_n') + By_n'(t) + V(t, y_n, z_n)[y_{n+1}(t) - y_n(t)]\},\\ &y_{n+1}(0) = x_0, \end{aligned}$$

and

$$\begin{aligned} &z_{n+1}'(t) = (A + B)^{-1}\{\mathscr{F}(t, z_n, z_n') + Bz_n'(t) + V(t, y_n, z_n)[z_{n+1}(t) - z_n(t)]\},\\ &z_{n+1}(0) = x_0, \end{aligned}$$

for $n = 0, 1, \ldots$. Note that the sequences $\{y_n\}, \{z_n\}$ are well defined.

First of all, we shall prove that

$$\begin{aligned} &y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \quad t \in J,\\ &y_0'(t) \le y_1'(t) \le z_1'(t) \le z_0'(t), \quad t \in J. \end{aligned} \tag{3.1}$$

Let us put $p = y_0 - y_1$, so $p(0) \le 0$. Then we see that

$$\begin{aligned} p'(t) \le (A + B)^{-1}\{&\mathscr{F}(t, y_0, y_0') + By_0'(t) - \mathscr{F}(t, y_0, y_0') - By_0'(t)\\ &- V(t, y_0, z_0)[y_1(t) - y_0(t)]\} = (A + B)^{-1}V(t, y_0, z_0)p(t), \quad t \in J. \end{aligned}$$

Assumption H_5 and Lemma 1 yield $p(t) \le 0$ on J proving that $y_0(t) \le y_1(t)$ on J. Since $(A + B)^{-1}V(t, y_0, z_0) \ge 0$, and $p(t) \le 0$ on J, then $p'(t) \le 0$, so $y_0'(t) \le y_1'(t)$ on J. By the same way we can show that $z_1(t) \le z_0(t)$ and $z_1'(t) \le z_0'(t)$, $t \in J$. Put

$p = y_1 - z_1$. Then, by Lemma 2 and assumption H_4, we have

$$\begin{aligned} p'(t) &= (A+B)^{-1}\{\mathscr{F}(t,y_0,y_0') - \mathscr{F}(t,z_0,z_0') + B[y_0'(t) - z_0'(t)] \\ &\quad + V(t,y_0,z_0)[y_1(t) - y_0(t) - z_1(t) + z_0(t)]\} \\ &\le (A+B)^{-1}\{[-F_x(t,y_0,y_0') - G_x(t,z_0,y_0') + \Phi_x(t,z_0,y_0') \\ &\quad + \Psi_x(t,y_0,y_0')][z_0(t) - y_0(t)] + B[z_0'(t) - y_0'(t)] \\ &\quad + V(t,y_0,z_0)[y_1(t) - y_0(t) - z_1(t) + z_0(t)] + B[y_0'(t) - z_0'(t)]\} \\ &= (A+B)^{-1}\{[G_x(t,z_0,z_0') - G_x(t,z_0,y_0') + \Phi_x(t,z_0,y_0') \\ &\quad - \Phi_x(t,z_0,z_0')][z_0(t) - y_0(t)] + V(t,y_0,z_0)p(t)\} \\ &\le (A+B)^{-1}V(t,y_0,z_0)p(t) \end{aligned}$$

with $p(0) = 0$. Hence, we have $p(t) \le 0$, and then $p'(t) \le 0$ on J which shows that $y_1(t) \le z_1(t)$, $y_1'(t) \le z_1'(t)$, $t \in J$. This means that (3.1) holds.

In the next step we need to show that y_1 and z_1 are lower and upper solutions of problem (2.1), respectively. By Lemma 2 and assumptions H_3 and H_4, we obtain

$$\begin{aligned} y_1'(t) &= (A+B)^{-1}\{\mathscr{F}(t,y_0,y_0') + By_0'(t) + V(t,y_0,z_0)[y_1(t) - y_0(t)]\} \\ &\le (A+B)^{-1}\{\mathscr{F}(t,y_1,y_1') + By_1'(t) + [-F_x(t,y_0,y_0') - G_x(t,y_1,y_0') \\ &\quad + \Phi_x(t,y_1,y_0') + \Psi_x(t,y_0,y_0')][y_1(t) - y_0(t)] + V(t,y_0,z_0)[y_1(t) - y_0(t)]\} \\ &= (A+B)^{-1}\{\mathscr{F}(t,y_1,y_1') + By_1'(t) + [G_x(t,z_0,z_0') - G_x(t,y_1,y_0') \\ &\quad + \Phi_x(t,y_1,y_0') - \Phi_x(t,z_0,z_0')][y_1(t) - y_0(t)]\} \\ &\le (A+B)^{-1}[\mathscr{F}(t,y_1,y_1') + By_1'(t)], \end{aligned}$$

and

$$\begin{aligned} z_1'(t) &= (A+B)^{-1}\{\mathscr{F}(t,z_0,z_0') + Bz_0'(t) + V(t,y_0,z_0)[z_1(t) - z_0(t)]\} \\ &\ge (A+B)^{-1}\{\mathscr{F}(t,z_1,z_1') + Bz_1'(t) + [F_x(t,z_1,z_1') + G_x(t,z_0,z_1') \\ &\quad - \Phi_x(t,z_0,z_1') - \Psi_x(t,z_1,z_1')][z_0(t) - z_1(t)] + V(t,y_0,z_0)[z_1(t) - z_0(t)]\} \\ &= (A+B)^{-1}\{\mathscr{F}(t,z_1,z_1') + Bz_1'(t) + [F_x(t,z_1,z_1') - F_x(t,y_0,y_0') \\ &\quad + G_x(t,z_0,z_1') - G_x(t,z_0,z_0') + \Phi_x(t,z_0,z_0') - \Phi_x(t,z_0,z_1') \\ &\quad + \Psi_x(t,y_0,y_0') - \Psi_x(t,z_1,z_1')][z_0(t) - z_1(t)]\} \\ &\ge (A+B)^{-1}[\mathscr{F}(t,z_1,z_1') + Bz_1'(t)] \end{aligned}$$

which shows that y_1 and z_1, respectively, are lower and upper solutions of problem (2.1). Let us assume that

$$\begin{aligned} &y_{k-1}(t) \le y_k(t) \le z_k(t) \le z_{k-1}(t), \quad t \in J, \\ &y_{k-1}'(t) \le y_k'(t) \le z_k'(t) \le z_{k-1}'(t), \quad t \in J, \end{aligned}$$

and let y_k, z_k be lower and upper solutions of problem (2.1) for some $k \geq 1$. We shall prove that

$$\begin{aligned} y_k(t) &\leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t), \quad t \in J, \\ y'_k(t) &\leq y'_{k+1}(t) \leq z'_{k+1}(t) \leq z'_k(t), \quad t \in J. \end{aligned} \tag{3.2}$$

Put $p = y_k - y_{k+1}$. Then

$$\begin{aligned} p'(t) \leq (A+B)^{-1}\{&\mathscr{F}(t, y_k, y'_k) + By'_k(t) - \mathscr{F}(t, y_k, y'_k) - By'_k(t) \\ &- V(t, y_k, z_k)[y_{k+1}(t) - y_k(t)]\} = (A+B)^{-1}V(t, y_k, z_k)p(t) \end{aligned}$$

with $p(0) = 0$. Note that, by assumptions H_3–H_5,

$$\begin{aligned} (A+B)^{-1}V(t, y_k, z_k) &= (A+B)^{-1}[F_x(t, y_k, y'_k) + G_x(t, z_k, z'_k) - \Phi_x(t, z_k, z'_k) \\ &\quad - \Psi_x(t, y_k, y'_k)] \\ &\geq (A+B)^{-1}[F_x(t, y_0, y'_0) + G_x(t, z_0, z'_0) - \Phi_x(t, z_0, z'_0) \\ &\quad - \Psi_x(t, y_0, y'_0)] \\ &= (A+B)^{-1}V(t, y_0, z_0) \geq 0, \quad t \in J. \end{aligned}$$

Hence, by Lemma 1, $p(t) \leq 0$, $p'(t) \leq 0$, $t \in J$, which shows that $y_k(t) \leq y_{k+1}(t)$ and $y'_k(t) \leq y'_{k+1}(t)$, $t \in J$. Using the same argument we can prove that $z_{k+1}(t) \leq z_k(t)$, $z'_{k+1}(t) \leq z'_k(t)$, $t \in J$.

Let $p = y_{k+1} - z_{k+1}$. Then $p(0) = 0$. Using Lemma 2 and assumption H_4, we get

$$\begin{aligned} p'(t) &= (A+B)^{-1}\{\mathscr{F}(t, y_k, y'_k) - \mathscr{F}(t, z_k, z'_k) + B[y'_k(t) - z'_k(t)] \\ &\quad + V(t, y_k, z_k)[y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)]\} \\ &\leq (A+B)^{-1}\{[-F_x(t, y_k, y'_k) - G_x(t, z_k, y'_k) + \Phi_x(t, z_k, y'_k) \\ &\quad + \Psi_x(t, y_k, y'_k)][z_k(t) - y_k(t)] \\ &\quad + V(t, y_k, z_k)[y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)]\} \\ &= (A+B)^{-1}\{[G_x(t, z_k, z'_k) - G_x(t, z_k, y'_k) + \Phi_x(t, z_k, y'_k) \\ &\quad - \Phi_x(t, z_k, z'_k)][z_k(t) - y_k(t)] + V(t, y_k, z_k)p(t)\} \\ &\leq (A+B)^{-1}V(t, y_k, z_k)p(t), \quad t \in J. \end{aligned}$$

This proves that $y_{k+1}(t) \leq z_{k+1}(t)$, and $y'_{k+1}(t) \leq z'_{k+1}(t)$, $t \in J$, so relation (3.2) holds. Hence, by induction, for all n, we have

$$\begin{aligned} y_0(t) &\leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t), \quad t \in J, \\ y'_0(t) &\leq y'_1(t) \leq \cdots \leq y'_n(t) \leq z'_n(t) \leq \cdots \leq z'_1(t) \leq z'_0(t), \quad t \in J. \end{aligned}$$

Employing standard techniques (using the Arzeli theorem and the Lebesgue theorem), it can be shown that $y_n \to y$, $y'_n \to y'$, $z_n \to z$, $z'_n \to z'$, $y, z \in C^1(J, \mathbb{R}^m)$, where y and z are solutions of problem (2.1). Hence, by assumption (iii), we have $y = z = x$ on J is the unique solution of (2.1).

The order of convergence of sequences $\{y_n\}$, $\{z_n\}$, $\{y'_n\}$, $\{z'_n\}$ is considered in the next part of our considerations. For this purpose, we put

$$p_{n+1} = x - y_{n+1} \geq 0, \qquad q_{n+1} = z_{n+1} - x \geq 0 \quad \text{on} \quad J,$$

and note that $p_{n+1}(0) = q_{n+1}(0) = 0$ for $n \geq 0$. Using the integral mean value theorem and assumptions H_1, H_3, H_6, we get

$$\begin{aligned}
p'_{n+1}(t) &= (A+B)^{-1}\{\mathscr{F}(t,x,x') + Bx'(t) - \mathscr{F}(t,y_n,x') + \mathscr{F}(t,y_n,x') \\
&\quad - \mathscr{F}(t,y_n,y'_n) - V(t,y_n,z_n)[y_{n+1}(t) - x(t) + x(t) - y_n(t)] - By'_n(t)\} \\
&\leq (A+B)^{-1}\left\{\left[\int_0^1 \mathscr{F}_x(t,sx+(1-s)y_n,x')ds\right] p_n(t) + 2B|p'_n(t)| \right. \\
&\quad \left. + V(t,y_n,z_n)[p_{n+1}(t) - p_n(t)]\right\} \\
&= (A+B)^{-1}\left\{\int_0^1 [F_x(t,sx+(1-s)y_n,x') + G_x(t,sx+(1-s)y_n,x') \right. \\
&\quad - \Phi_x(t,sx+(1-s)y_n,x') - \Psi_x(t,sx+(1-s)y_n,x')]ds\ p_n(t) \\
&\quad \left. + 2B|p'_n(t)| + V(t,y_n,z_n)[p_{n+1}(t) - p_n(t)]\right\} \\
&\leq (A+B)^{-1}\{[F_x(t,x,x') - F_x(t,y_n,y'_n) + G_x(t,y_n,x') - G_x(t,z_n,z'_n) \\
&\quad + \Phi_x(t,z_n,z'_n) - \Phi_x(t,y_n,x') + \Psi_x(t,y_n,y'_n) - \Psi_x(t,x,x')]p_n(t) \\
&\quad + V(t,y_n,z_n)p_{n+1}(t) + 2B|p'_n(t)|\} \\
&\leq \left\{(C_1 + C_2 + 2C_3 + 2C_4)\sum_{i=1}^m p_{ni}(t) \right. \\
&\quad + (C_2 + C_3 + C_4)\sum_{i=1}^m [q_{ni}(t) + |q'_{ni}(t)|] \\
&\quad \left. +(C_1 + C_3 + C_4)\sum_{i=1}^m |p'_{ni}(t)|\right\} p_n(t) \\
&\quad + (A+B)^{-1}\left\{2B|p'_n(t)| + V(t,y_n,z_n)p_{n+1}(t)\right\}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{i=1}^m p_{ni}(t)p_n(t) &\leq \frac{m}{2}p_n^2(t) + \frac{1}{2}Wp_n^2(t), \\
\sum_{i=1}^m q_{ni}(t)p_n(t) &\leq \frac{m}{2}p_n^2(t) + \frac{1}{2}Wq_n^2(t),
\end{aligned} \tag{3.3}$$

where $p_n^2 = [p_{1,n}^2, \dots, p_{m,n}^2]^T$, $W = [w_{ij}]$, $w_{ij} = 1$, $i, j = 1, \dots, m$. This and previous calculations give

$$p'_{n+1}(t) \le Kp_{n+1}(t) + A_1 p_n^2(t) + A_2 q_n^2(t) + A_3|p'_n(t)|^2 + A_4|q'_n(t)|^2 + A_5|p'_n(t)| \qquad (3.4)$$

with $(A+B)^{-1} f_x \le K_1$, $(A+B)^{-1} g_x \le K_2$, $K = K_1 + K_2$ on Ω. Here, K_1, K_2 are $m \times m$ non-negative matrices and

$$\begin{aligned} A_1 &= \frac{1}{2}(C_1 + C_2 + 2C_3 + 2C_4)(mI + W) + (C_2 + C_3 + C_4)m \\ &\quad + (C_1 + C_3 + C_4)\frac{m}{2}, \\ A_2 &= \frac{1}{2}(C_2 + C_3 + C_4)W, \\ A_3 &= \frac{1}{2}(C_1 + C_3 + C_4)W, \\ A_4 &= A_2, \\ A_5 &= 2(A+B)^{-1}B. \end{aligned}$$

There is no loss of generality assuming that K^{-1} exists such that $k_{ij} \ge 0$, where k_{ij} represents the components of this matrix. Hence, for $t \in J$, we have

$$p_{n+1}(t) \le \int_0^t e^{K(t-s)} \Big[A_1 p_n^2(s) + A_2 q_n^2(s) + A_3|p'_n(s)|^2 + A_4|q'_n(s)|^2 + A_5|p'_n(s)| \Big] ds.$$

This implies

$$\begin{aligned} \max_{t\in J} \|p_{n+1}(t)\| \le B_1 \max_{t\in J} \|p_n(t)\|^2 + B_2 \max_{t\in J} \|q_n(t)\|^2 + B_3 \max_{t\in J} \|p'_n(t)\|^2 \\ + B_4 \max_{t\in J} \|q'_n(t)\|^2 + B_5 \max_{t\in J} \|p'_n(t)\|, \end{aligned} \qquad (3.5)$$

where $\|v\|^2 = \big[|v_1|^2, \dots, |v_m|^2\big]^T$, $v \in \mathbb{R}^m$, and

$$A_0 = K^{-1} e^{Kb}, \qquad B_i = A_0 A_i,$$

for $i = \overline{1,5}$. Combining (3.4) and (3.5) we obtain

$$\begin{aligned} \max_{t\in J} \|p'_{n+1}(t)\| \le \bar{A}_1 \max_{t\in J} \|p_n(t)\|^2 + \bar{A}_2 \max_{t\in J} \|q_n(t)\|^2 + \bar{A}_3 \max_{t\in J} \|p'_n(t)\|^2 \\ + \bar{A}_4 \max_{t\in J} \|q'_n(t)\|^2 + \bar{A}_5 \max_{t\in J} \|p'_n(t)\| \end{aligned}$$

with $\bar{A}_i = A_i + KB_i$, $i = \overline{1,5}$.

Similarly we have

$$
\begin{aligned}
q'_{n+1}(t) &= (A+B)^{-1}\{\mathscr{F}(t,z_n,z'_n) + Bz'_n(t) - \mathscr{F}(t,x,z'_n) \\
&\quad + \mathscr{F}(t,x,z'_n) - \mathscr{F}(t,x,x') + V(t,y_n,z_n)[q_{n+1}(t)-q_n(t)] - Bx'(t)\} \\
&\le (A+B)^{-1}\left\{\left[\int_0^1 \mathscr{F}_x(t,sz_n+(1-s)x,z'_n)ds\right]q_n(t) + 2B|q'_n(t)| \right. \\
&\quad \left. + V(t,y_n,z_n)[q_{n+1}(t)-q_n(t)]\right\} \\
&\le (A+B)^{-1}\{[F_x(t,z_n,z'_n) - F_x(t,y_n,y'_n) + G_x(t,x,z'_n) - G_x(t,z_n,z'_n) \\
&\quad + \Phi_x(t,z_n,z'_n) - \Phi_x(t,x,z'_n) + \Psi_x(t,y_n,y'_n) - \Psi_x(t,z_n,z'_n)]q_n(t) \\
&\quad + V(t,y_n,z_n)q_{n+1}(t) + 2B|q'_n(t)|\} \\
&\le \left\{(C_1+C_2+2C_3+2C_4)\sum_{i=1}^m q_{ni}(t)\right. \\
&\quad \left. +(C_1+C_3+C_4)\sum_{i=1}^m\left[p_{ni}(t)+|p'_{ni}(t)|+|q'_{ni}(t)|\right]\right\}q_n(t) \\
&\quad + Kq_{n+1}(t) + A_5|q'_n(t)| \\
&\le D_1p_n^2(t) + D_2q_n^2(t) + D_1|p'_n(t)|^2 + D_1|q'_n(t)|^2 + Kq_{n+1}(t) + A_5|q'_n(t)|,
\end{aligned}
$$

where

$$
\begin{aligned}
D_1 &= \frac{1}{2}(C_1+C_3+C_4)W, \\
D_2 &= \frac{3}{2}m(C_1+C_3+C_4) + \frac{1}{2}(C_1+C_2+2C_3+2C_4)(mI+W).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\max_{t\in J}\|q_{n+1}(t)\| \le \overline{B}_1\max_{t\in J}\|p_n(t)\|^2 + \overline{B}_2\max_{t\in J}\|q_n(t)\|^2 + \overline{B}_1\max_{t\in J}\|p'_n(t)\|^2 \\
+ \overline{B}_1\max_{t\in J}\|q'_n(t)\|^2 + \overline{B}_3\max_{t\in J}\|q'_n(t)\|,
\end{aligned}
$$

where $\overline{B}_1 = A_0D_1$, $\overline{B}_2 = A_0D_2$, and $\overline{B}_3 = B_5A$.

Combining this with the last relation for q'_{n+1} we get

$$
\begin{aligned}
\max_{t\in J}\|q'_{n+1}(t)\| \le \overline{L}_1\max_{t\in J}\|p_n(t)\|^2 + \overline{L}_2\max_{t\in J}\|q_n(t)\|^2 + \overline{L}_1\max_{t\in J}\|p'_n(t)\|^2 \\
+ \overline{L}_1\max_{t\in J}\|q'_n(t)\|^2 + \overline{L}_3\max_{t\in J}\|q'_n(t)\|,
\end{aligned}
$$

with $\overline{L}_1 = D_1 + K\overline{B}_1$, $\overline{L}_2 = D_2 + K\overline{B}_2$, and $\overline{L}_3 = A_5 + K\overline{B}_3$. This completes the proof. □

Let us introduce the following assumptions:

H_{17} (i) $(A+B)^{-1}F_x$ is non-decreasing in the third variable on Ω and $V_1 = F_x(t,y,y')$, or
(ii) $(A+B)^{-1}F_x$ is non-increasing in the third variable on Ω and $V_1 = F_x(t,y,z')$.

H_{27} (i) $(A+B)^{-1}G_x$ is non-increasing in the third variable on Ω and $V_2 = G_x(t,z,z')$, or
(ii) $(A+B)^{-1}G_x$ is non-decreasing in the third variable on Ω and $V_2 = G_x(t,z,y')$.

H_{37} (i) $(A+B)^{-1}\Phi_x$ is non-decreasing in the third variable on Ω and $V_3 = \Phi_x(t,z,z')$, or
(ii) $(A+B)^{-1}\Phi_x$ is non-increasing in the third variable on Ω and $V_3 = \Phi_x(t,z,y')$.

H_{47} (i) $(A+B)^{-1}\Psi_x$ is non-increasing in the third variable on Ω and $V_4 = \Psi_x(t,y,y')$, or
(ii) $(A+B)^{-1}\Psi_x$ is non-decreasing in the third variable on Ω and $V_4 = \Psi_x(t,y,z')$.

The set of all assumptions from H_{17} to H_{47} will be denoted by H_7. Since in any assumptions H_{17}–H_{47} we have two cases (i) or (ii), so we have 16 possibilities for constructing assumption H_7. Note that if we choose case (i) in any assumptions H_{17}–H_{47}, then assumption H_7 is identical with assumption H_4.

Now we can formulate the following

Theorem 2. *Assume that the assumptions of Theorem 1 are satisfied with assumption H_7 instead of H_4 and for*

$$V = V_1 + V_2 - V_3 - V_4.$$

Then the conclusion of Theorem 1 is true.

Proof. Since the proof can be constructed on the basis of the proof of the previous theorem, we shall only indicate the necessary changes. We should create assumption H_7. Let H_7 be produced from assumptions H_{17}(ii), H_{27}(ii), H_{37}(ii), and H_{47}(ii). Note that the sequences $\{y_n\}$, $\{z_n\}$ are constructed as before with

$$V(t,y,z) = F_x(t,y,z') + G_x(t,z,y') - \Phi_x(t,z,y') - \Psi_x(t,y,z').$$

Based on the assumption

$$(A+B)^{-1}V(t,y_0,z_0) \geq 0$$

and Lemma 1, it is quite easy to show that $y_0(t) \leq y_1(t)$, $y_0'(t) \leq y_1'(t)$, $z_1(t) \leq z_0(t)$ and $z_1'(t) \leq z_0'(t)$ on J. If we put $p = y_1 - z_1$, then, by Lemma 2 and assumptions

H_{17}(ii), H_{47}(ii), we have

$$\begin{aligned} p'(t) &\le (A+B)^{-1}\{[-F_x(t,y_0,y_0') - G_x(t,z_0,y_0') + \Phi_x(t,z_0,y_0') \\ &\quad + \Psi_x(t,y_0,y_0')][z_0(t)-y_0(t)] + B[z_0'(t)-y_0'(t)] \\ &\quad + V(t,y_0,z_0)[y_1(t)-y_0(t)-z_1(t)+z_0(t)] + B[y_0'(t)-z_0'(t)]\} \\ &= (A+B)^{-1}\{[F_x(t,y_0,z_0') - F_x(t,y_0,y_0') + \Psi_x(t,y_0,y_0') \\ &\quad - \Psi_x(t,y_0,z_0')][z_0(t)-y_0(t)] \\ &\quad + V(t,y_0,z_0)p(t)\} \le (A+B)^{-1}V(t,y_0,z_0)p(t), \\ p(0) &= 0. \end{aligned}$$

Hence, by Lemma 1, we have $p(t) \le 0$, and therefore $p'(t) \le 0$ on J which shows that $y_1(t) \le z_1(t)$, $y_1'(t) \le z_1'(t)$, $t \in J$. It means that (3.1) holds.

In the next step we need to show that y_1 and z_1 are lower and upper solutions of problem (2.1), respectively. Note that, using Lemma 2 and assumptions H_3 and H_7, we get

$$\begin{aligned} y_1'(t) &\le (A+B)^{-1}\{\mathscr{F}(t,y_1,y_1') + By_1'(t) + [-F_x(t,y_0,y_0') - G_x(t,y_1,y_0') \\ &\quad + \Phi_x(t,y_1,y_0') + \Psi_x(t,y_0,y_0')][y_1(t)-y_0(t)] + V(t,y_0,z_0)[y_1(t)-y_0(t)]\} \\ &= (A+B)^{-1}\{\mathscr{F}(t,y_1,y_1') + By_1'(t) + [F_x(t,y_0,z_0') - F_x(t,y_0,y_0') \\ &\quad + G_x(t,z_0,y_0') - G_x(t,y_1,y_0') + \Phi_x(t,y_1,y_0') - \Phi_x(t,z_0,y_0') + \Psi_x(t,y_0,y_0') \\ &\quad - \Psi_x(t,y_0,z_0')][y_1(t)-y_0(t)]\} \le (A+B)^{-1}[\mathscr{F}(t,y_1,y_1') + By_1'(t)], \end{aligned}$$

and

$$\begin{aligned} z_1'(t) &\ge (A+B)^{-1}\{\mathscr{F}(t,z_1,z_1') + Bz_1'(t) + [F_x(t,z_1,z_1') + G_x(t,z_0,z_1') \\ &\quad - \Phi_x(t,z_0,z_1') - \Psi_x(t,z_1,z_1')][z_0(t)-z_1(t)] + V(t,y_0,z_0)[z_1(t)-z_0(t)]\} \\ &= (A+B)^{-1}\{\mathscr{F}(t,z_1,z_1') + Bz_1'(t) + [F_x(t,z_1,z_1') - F_x(t,y_0,z_0') \\ &\quad + G_x(t,z_0,z_1') - G_x(t,z_0,y_0') + \Phi_x(t,z_0,y_0') - \Phi_x(t,z_0,z_1') \\ &\quad + \Psi_x(t,y_0,z_0') - \Psi_x(t,z_1,z_1')][z_0(t)-z_1(t)]\} \\ &\ge (A+B)^{-1}[\mathscr{F}(t,z_1,z_1') + Bz_1'(t)], \end{aligned}$$

which shows that y_1 and z_1 are lower and upper solutions of problem (2.1), respectively.

By induction in n, we can show that

$$\begin{aligned} &y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t), \quad t \in J, \\ &y_0'(t) \le y_1'(t) \le \cdots \le y_n'(t) \le z_n'(t) \le \cdots \le z_1'(t) \le z_0'(t), \quad t \in J \end{aligned}$$

for all n.

Employing standard techniques, it is easy to conclude that $y_n \to y$, $y'_n \to y'$, $z_n \to z$, $z'_n \to z'$, $y, z \in C^1(J, \mathbb{R}^m)$, where y and z are solutions of problem (2.1). Hence, by assumption (iii), we have $y = z = x$ on J is the unique solution of (2.1).

To show the quadratic and semiquadratic convergence, we set

$$p_{n+1} = x - y_{n+1} \geq 0, \qquad q_{n+1} = z_{n+1} - x \geq 0$$

on J. Note that $p_{n+1}(0) = q_{n+1}(0) = 0$ for $n \geq 0$. The beginning for p'_{n+1} is the same as in the proof of Theorem 1, so

$$\begin{aligned} p'_{n+1}(t) \leq (A+B)^{-1}\bigg\{ \int_0^1 [&F_x(t, sx + (1-s)y_n, x') + G_x(t, sx + (1-s)y_n, x') \\ &- \Phi_x(t, sx + (1-s)y_n, x') - \Psi_x(t, sx + (1-s)y_n, x')]ds \; p_n(t) \\ &+ 2B|p'_n(t)| + V(t, y_n, z_n)[p_{n+1}(t) - p_n(t)] \bigg\}. \end{aligned}$$

Now, using the same argument as in the proof of Theorem 1, we can prove that

$$\begin{aligned} \max_{t\in J} \|p_{n+1}(t)\| \leq \alpha_1 \max_{t\in J} \|p_n(t)\|^2 + \alpha_2 \max_{t\in J} \|q_n(t)\|^2 + \alpha_3 \max_{t\in J} \|p'_n(t)\|^2 \\ + \alpha_4 \max_{t\in J} \|q'_n(t)\|^2 + \alpha_5 \max_{t\in J} \|p'_n(t)\| \end{aligned}$$

and

$$\begin{aligned} \max_{t\in J} \|p'_{n+1}(t)\| \leq \bar{\alpha}_1 \max_{t\in J} \|p_n(t)\|^2 + \bar{\alpha}_2 \max_{t\in J} \|q_n(t)\|^2 + \bar{\alpha}_3 \max_{t\in J} \|p'_n(t)\|^2 \\ + \bar{\alpha}_4 \max_{t\in J} \|q'_n(t)\|^2 + \bar{\alpha}_5 \max_{t\in J} \|p'_n(t)\| \end{aligned}$$

where

$$\begin{aligned} \delta_1 &= \frac{1}{2}C_1(2mI + W) + \frac{1}{2}C_2(3mI + W) + \frac{1}{2}(C_3 + C_4)(5mI + 2W), \\ \delta_2 &= \frac{1}{2}(C_2 + C_3 + C_4)W, \\ \delta_3 &= \frac{1}{2}(C_3 + C_4)W, \\ \delta_4 &= \frac{1}{2}(C_1 + C_2 + C_3 + C_4)W, \\ \delta_5 &= A_5, \end{aligned}$$

and $\alpha_i = A_0\delta_i$, $\bar{\alpha}_i = \delta_i + K\alpha_i$, $i = \overline{1, 5}$, with A_0 and K defined as in the proof of Theorem 1.

Similarly, we can show that

$$\max_{t\in J}\|q_{n+1}(t)\| \le \beta_1 \max_{t\in J}\|p_n(t)\|^2 + \beta_2 \max_{t\in J}\|q_n(t)\|^2 + \beta_3 \max_{t\in J}\|p_n'(t)\|^2 + \beta_4 \max_{t\in J}\|q_n'(t)\|^2 + \beta_5 \max_{t\in J}\|q_n'(t)\|$$

and

$$\max_{t\in J}\|q_{n+1}'(t)\| \le \overline{\beta}_1 \max_{t\in J}\|p_n(t)\|^2 + \overline{\beta}_2 \max_{t\in J}\|q_n(t)\|^2 + \overline{\beta}_3 \max_{t\in J}\|p_n'(t)\|^2 + \overline{\beta}_4 \max_{t\in J}\|q_n'(t)\|^2 + \overline{\beta}_5 \max_{t\in J}\|q_n'(t)\|,$$

with

$$\eta_1 = \frac{1}{2}(C_1 + 2C_3 + C_4)W,$$
$$\eta_3 = \eta_4 = \frac{1}{2}(C_2 + C_3 + C_4)W,$$
$$\eta_5 = A_5,$$
$$\eta_2 = \frac{1}{2}C_1(2mI + W) + \frac{1}{2}C_2(3mI + W) + \frac{1}{2}(C_3 + C_4)(5mI + W),$$

and

$$\beta_i = A_0\eta_i, \quad \overline{\beta_i} = \eta_i + K\beta_i$$

for $i = \overline{1,5}$.

It is now easy to construct the proofs of the assertions corresponding to the remaining cases of assumption H_7 following the proof of Theorem 1 and the proof given above. We omit the details. The proof of this theorem is therefore complete. □

Remark 1. Note that if v is a lower solution of problem (2.1) and $(A + B)^{-1} \geq 0$, then v satisfies the relation

$$Av'(t) \le \mathscr{F}(t, v(t), v'(t)), \quad t \in J, \quad v(0) \le x_0.$$

Here, $(A + B)^{-1} \geq 0$ means that some entries of $(A + B)^{-1}$ may be equal to zero.

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