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ON SUBSEQUENTIAL CONVERGENCE OF BOUNDED SEQUENCES

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Abstract. In this paper we establish some Tauberian-like conditions in terms of the weighted general control modulo of integer order to retrieve subsequential convergence of a sequence from its boundedness.

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1. Introduction

Let $p=(p_n)$ be a sequence of nonnegative numbers with $p_0>0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n\to\infty.$$

The weighted means of a sequence (u_n) of real numbers are defined by

$$\sigma_{n,p}^{(1)}(u) := \frac{1}{P_n}\sum_{k=0}^{n} p_k u_k$$

for all nonnegative integers n.

We define the weighted mean method as follows:

Definition 1. If $\lim_{n\to\infty}\sigma_{n,p}^{(1)}(u)=s$, then we say that (u_n) is said to be limitable to s by the weighted mean method $(\overline{N},p_n)$ and we write $u_n\to s(\overline{N},p_n)$.

If $p_n=1$ for all n, then the corresponding weighted mean method is the $(C,1)$ method of Cesàro.

The sequence $\Delta u=(\Delta u_n)$ of the backward differences of (u_n) is defined by $\Delta u_n=u_n-u_{n-1}$, and $\Delta u_0=u_0$ for $n=0$.

The identity

$$u_n-\sigma_{n,p}^{(1)}(u)=V_{n,p}^{(0)}(\Delta u), \tag{1.1}$$

where $V^{(0)}_{n,p}(\Delta u) = \frac{1}{P_n}\sum_{k=1}^{n} P_{k-1}\Delta u_k$, is known as the weighted Kronecker identity.

We define $\sigma^{(m)}_{n,p}(u)$ and $V^{(m)}_{n,p}(\Delta u)$ by

$$\sigma^{(m)}_{n,p}(u) = \begin{cases} \dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n} p_k \sigma^{(m-1)}_{k,p}(u) & ,m \geq 1 \\ u_n & ,m = 0 \end{cases}$$

and

$$V^{(m)}_{n,p}(\Delta u) = \begin{cases} \dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n} p_k V^{(m-1)}_{k,p}(\Delta u) & ,m \geq 1 \\ V^{(0)}_{n,p}(\Delta u) & ,m = 0 \end{cases}$$

respectively (see [4]).

The weighted classical control modulo of (u_n) is denoted by $\omega^{(0)}_{n,p}(u) = \dfrac{P_{n-1}}{p_n}\Delta u_n$ and the weighted general control modulo of integer order $m \geq 1$ of (u_n) is defined by $\omega^{(m)}_{n,p}(u) = \omega^{(m-1)}_{n,p}(u) - \sigma^{(1)}_{n,p}(\omega^{m-1}(u))$ (see [2]).

A new kind of convergence was defined by Dik [5] as follows.

Definition 2. A sequence $u = (u_n)$ is said to be subsequentially convergent if there exists a finite interval $I(u)$ such that all accumulation points of (u_n) are in $I(u)$ and every point of $I(u)$ is an accumulation point of (u_n).

It is clear from the definition that subsequential convergence implies boundedness. But the converse is not true in general. For example, $((-1)^n)$ is bounded, but it is not subsequentially convergent. The converse implication is true under some conditions imposed on the sequence.

The following theorem is a more general one stating that under which condition bounded sequences are subsequential convergent.

Theorem 1 ([5])**.** *Let (u_n) be a bounded sequence. If $\Delta u_n = o(1)$, then (u_n) is subsequentially convergent.*

Throughout this paper, we write $u_n = o(1)$ or $u_n = O(1)$ rather than $\lim_n u_n = 0$ or (u_n) is bounded for large enough n.

Example 1. Using Theorem 1, one can easily show that the sequence $(u_n) = (\sin\sqrt{n})$ is subsequentially convergent. Indeed, the sequence (u_n) is bounded. From the fact that

$$\begin{aligned} |\Delta u_n| = |\Delta \sin(\sqrt{n})| &= |\sin(\sqrt{n}) - \sin(\sqrt{n-1})| \leq |\sqrt{n} - \sqrt{n-1}| \\ &= \left|\frac{1}{\sqrt{n}+\sqrt{n-1}}\right| = o(1), \end{aligned}$$

it follows that $\Delta u_n = o(1)$. Thus by Theorem 1, the sequence (u_n) is subsequentially convergent.

Definition 3 ([9])**.** A sequence (u_n) is slowly oscillating if

$$\lim_{\lambda\to 1+}\overline{\lim_{n}}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right| = 0.$$

It is clear from Definition 3 that slow oscillation of (u_n) implies $\Delta u_n = o(1)$.

By Littlewood's condition $n\Delta u_n = O(1)$ [8] and slow oscillation of (u_n), Dik et al. [6] obtained some Tauberian-like theorems to recover subsequential convergence of (u_n) from its boundedness. The condition $n\Delta u_n = O(A_n)$, where (A_n) is an unbounded sequence, is used as a Tauberian-like condition for the recovery of subsequential convergence of (u_n) from its boundedness by Dik et al. [6]. Later in [1], Çanak and Dik introduced some Tauberian-like conditions $\omega_{n,1}^{(m)}(u) = O(A_n)$, where (A_n) is an unbounded sequence, and slow oscillation of $(\Delta\sigma_{n,1}(\omega^{(m)}(u)))$ for recovering subsequential convergence of the sequence of (u_n) from its boundedness.

In this paper we prove that subsequential convergence of (u_n) follows from its boundedness under the condition given in terms of the weighted general control modulo of the oscillatory behavior of integer order of the sequence (u_n).

2. Main results

The main theorem of this paper involves the concepts of a regularly varying sequence of index $\alpha > -1$ and slowly oscillating sequence.

Definition 4 ([7])**.** A positive sequence $(R(n))$ is said to be regularly varying of index $\alpha > -1$ if

$$\lim_{n\to\infty}\frac{R([\lambda n])}{R(n)} = \lambda^{\alpha},\ \ \lambda > 0,\ \ \alpha > -1. \tag{2.1}$$

The following theorems generalize some theorems in [1] that are exactly given in terms of the weighted general control modulo of the oscillatory behavior of the sequence (u_n).

Theorem 2. *Let (u_n) be a bounded sequence and*

$$\frac{P_{n-1}}{p_n} = O(n). \tag{2.2}$$

Let (A_n) be a sequence such that

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k|A_k|^{\mathfrak{p}} = O(1), \mathfrak{p} > 1 \tag{2.3}$$

for some regularly varying sequence (P_n) *of index* $\alpha > -1$. *If*

$$\omega_{n,p}^{(m)}(u) = O(A_n) \tag{2.4}$$

then (u_n) *is subsequentially convergent.*

Theorem 3. *Let* (u_n) *be a bounded sequence and let* (P_n) *be regularly varying of index* $\alpha > -1$. *If* $(\Delta\sigma_{n,p}^{(1)}(\omega^{(m)}(u)))$ *is slowly oscillating for some nonnegative integer m, then* (u_n) *is subsequentially convergent.*

3. Identities and a Lemma

In this section, we present some identities and a lemma to be used in the proof of the main theorems.

The identities

$$\frac{P_{n-1}}{p_n}\Delta\sigma_{n,p}^{(m)}(u) = V_{n,p}^{(m-1)}(\Delta u)$$

and

$$\sigma_{n,p}^{(1)}\left(\frac{P_{n-1}}{p_n}\Delta V_{n,p}^{(0)}(\Delta u)\right) = \frac{P_{n-1}}{p_n}\Delta V_{n,p}^{(1)}(\Delta u).$$

are proved by Totur and Çanak [10].

For a sequence $u = (u_n)$, we define

$$\left(\frac{P_{n-1}}{p_n}\Delta\right)_m u_n = \left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1}\left(\frac{P_{n-1}}{p_n}\Delta u_n\right) = \frac{P_{n-1}}{p_n}\Delta\left(\left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1} u_n\right),$$

where $\left(\frac{P_{n-1}}{p_n}\Delta\right)_0 u_n = u_n$, and $\left(\frac{P_{n-1}}{p_n}\Delta\right)_1 u_n = \frac{P_{n-1}}{p_n}\Delta u_n$.

A different representation of the weighted general control modulo of integer order $m \geq 1$ of a sequence (u_n) is given by the identity

$$\omega_{n,p}^{(m)}(u) = \left(\frac{P_{n-1}}{p_n}\Delta\right)_m V_{n,p}^{(m-1)}(\Delta u) \tag{3.1}$$

in ([10]).

We note that any (P_n) with $\lim_{n\to\infty}\frac{P_n}{n} = 1$ is regularly varying of index 1.

Lemma 1. *Let* (P_n) *be regularly varying of index* $\alpha > -1$. *If* (u_n) *is limitable to s by the weighted mean method* $(\overline{N}, p_n)$ *and slowly oscillating, then* (u_n) *converges to s.*

Proof. Assume that (P_n) be regularly varying of index $\alpha > -1$. If (u_n) is slowly oscillating, then $(V_{n,p}^{(0)}(\Delta u))$ is slowly oscillating (see [3]). Since $u_n \to s(\overline{N}, p_n)$ and $(V_{n,p}^{(0)}(\Delta u))$ is slowly oscillating, the proof is completed by [2, Theorem 6]. □

4. Proofs

In this section, proofs of theorems and corollaries are given.

Proof of Theorem 2. By (2.3), it follows that the sequence $\left(\sum_{j=1}^{n}\frac{p_jA_j}{P_{j-1}}\right)$ is slowly oscillating. Indeed,

$$
\begin{aligned}
\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\frac{p_jA_j}{P_{j-1}}\right| &\le \max_{n+1\le k\le[\lambda n]}\sum_{j=n+1}^{k}\left|\frac{p_jA_j}{P_{j-1}}\right| \\
&\le \sum_{j=n+1}^{[\lambda n]}\frac{p_j|A_j|}{P_{j-1}} \le \frac{1}{P_n}\sum_{j=n+1}^{[\lambda n]}p_j|A_j| \\
&= \frac{P_{[\lambda n]}-P_n}{P_n}\frac{1}{P_{[\lambda n]}-P_n}\sum_{j=n+1}^{[\lambda n]}p_j|A_j| \\
&\le \frac{P_{[\lambda n]}-P_n}{P_n}\frac{1}{(P_{[\lambda n]}-P_n)^{\frac{1}{\mathfrak{p}}}}\left(\sum_{j=n+1}^{[\lambda n]}p_j|A_j|^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}} \\
&= \frac{(P_{[\lambda n]}-P_n)^{1-\frac{1}{\mathfrak{p}}}}{P_n}\left(\sum_{j=n+1}^{[\lambda n]}p_j|A_j|^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}} \\
&= \frac{(P_{[\lambda n]}-P_n)^{\frac{1}{\mathfrak{q}}}}{P_n}\left(\sum_{j=n+1}^{[\lambda n]}p_j|A_j|^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}}, \\
&= \frac{(P_{[\lambda n]}-P_n)^{\frac{1}{\mathfrak{q}}}}{(P_n)^{\frac{1}{\mathfrak{q}}}}\left(\frac{P_{[\lambda n]}}{P_n}\frac{1}{P_{[\lambda n]}}\sum_{j=0}^{[\lambda n]}p_j|A_j|^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}} \\
&= \frac{(P_{[\lambda n]}-P_n)^{\frac{1}{\mathfrak{q}}}}{(P_n)^{\frac{1}{\mathfrak{q}}}}\frac{(P_{[\lambda n]})^{\frac{1}{\mathfrak{p}}}}{(P_n)^{\frac{1}{\mathfrak{p}}}}\left(\frac{1}{P_{[\lambda n]}}\sum_{j=0}^{[\lambda n]}p_j|A_j|^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}},
\end{aligned}
$$

where $\frac{1}{\mathfrak{p}}+\frac{1}{\mathfrak{q}}=1$. Taking $\limsup$ of both sides as $n\to\infty$, we get

$$
\limsup_{n}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\frac{p_jA_j}{P_{j-1}}\right|
$$

$$\leq \limsup_n \frac{(P_{[\lambda n]}-P_n)^{\frac{1}{q}}}{(P_n)^{\frac{1}{q}}} \frac{(P_{[\lambda n]})^{\frac{1}{p}}}{(P_n)^{\frac{1}{p}}} \limsup_n \left(\frac{1}{P_{[\lambda n]}} \sum_{j=0}^{[\lambda n]} p_j |A_j|^{p} \right)^{\frac{1}{p}}$$

$$= \lim_n \left(\frac{P_{[\lambda n]}-P_n}{P_n} \right)^{\frac{1}{q}} \lim_n \left(\frac{P_{[\lambda n]}}{P_n} \right)^{\frac{1}{p}} \limsup_n \left(\frac{1}{P_{[\lambda n]}} \sum_{j=0}^{[\lambda n]} p_j |A_j|^{p} \right)^{\frac{1}{p}}$$

Since (P_n) is regularly varying of index $\alpha > -1$, we have

$$\limsup_n \max_{n+1\leq k\leq [\lambda n]} \left| \sum_{j=n+1}^{k} \frac{p_j A_j}{P_{j-1}} \right| \leq C(\lambda^{\alpha}-1)^{\frac{1}{q}}(\lambda^{\alpha})^{\frac{1}{p}}.$$

for some $C > 0$.

Finally, taking lim sup of both sides as $\lambda \to 1+$, we obtain

$$\lim_{\lambda\to 1^+} \limsup_n \max_{n+1\leq k\leq [\lambda n]} \left| \sum_{j=n+1}^{k} \frac{p_j A_j}{P_{j-1}} \right| \leq C \lim_{\lambda\to 1^+} (\lambda^{\alpha}-1)^{\frac{1}{q}}(\lambda^{\alpha})^{\frac{1}{p}} = 0.$$

Slow oscillation of $\left(\sum_{j=1}^{n} \frac{p_j A_j}{P_{j-1}} \right)$ implies that $\frac{p_j A_j}{P_{j-1}} = o(1)$. Since $\omega_{n,p}^{(m)}(u) = O(A_n)$, it follows from

$$\omega_{n,p}^{(m)}(u) = \frac{P_{n-1}}{p_n} \Delta \left((\frac{P_{n-1}}{p_n}\Delta)_{m-1} V_{n,p}^{(m-1)}(\Delta u) \right) = O(A_n)$$

that

$$\Delta\sigma_{n,p}^{(1)}(\omega^{(m-1)}(u)) = \Delta \left((\frac{P_{n-1}}{p_n}\Delta)_{m-1} V_{n,p}^{(m-1)}(\Delta u) \right) = o(1).$$

Since (u_n) is bounded, $(\sigma_{n,p}^{(1)}(\omega^{(m-2)}(u)))$ is bounded for every nonnegative integer m. From the identity

$$\Delta \left((\frac{P_{n-1}}{p_n}\Delta)_{m-2} V_{n,p}^{(m-2)}(\Delta u) \right) = \frac{p_n (\frac{P_{n-1}}{p_n}\Delta)_{m-2} V_{n,p}^{(m-2)}(\Delta u)}{P_{n-1}} + \Delta \left((\frac{P_{n-1}}{p_n}\Delta)_{m-1} V_{n,p}^{(m-1)}(\Delta u) \right),$$

it follows that

$$\sigma_{n,p}^{(1)}(\omega^{(m-2)}(u)) = \Delta \left((\frac{P_{n-1}}{p_n}\Delta)_{m-2} V_{n,p}^{(m-2)}(\Delta u) \right) = o(1), n \to \infty.$$

Continuing in this manner, we obtain that

$$\Delta\sigma^{(1)}_{n,p}(\omega^{(0)}(u)) = \Delta V^{(0)}_{n,p}(\Delta u) = o(1).$$

Using (1.1), we have

$$\Delta u_n = \frac{p_n V^{(0)}_{n,p}(\Delta u)}{P_{n-1}} + \Delta V^{(0)}_n(\Delta u) = o(1).$$

By Theorem 1 we have the proof. □

Corollary 1. *Let (u_n) be a bounded sequence and (A_n) be a sequence such that*

$$\frac{1}{n}\sum_{k=1}^{n} |A_k|^{\mathfrak{p}} = O(1),\ \mathfrak{p} > 1. \tag{4.1}$$

If $\omega^{(m)}_n = O(A_n)$, then (u_n) is subsequentially convergent.

Proof. Take $p_n = 1$ for every nonnegative integer n in Theorem 2. □

Corollary 1 was given by Çanak and Dik [1].

Proof of Theorem 3. Since (u_n) is bounded, then $(\sigma^{(1)}_{n,p}(\omega^{(m)}(u)))$ is bounded for every nonnegative integer m. Thus, we get

$$\sigma^{(1)}_{n,p}(\Delta\sigma^{(1)}(\omega^{(m)}(u))) = \frac{p_n (n\Delta)_m V^{(m)}_{n,p}(\Delta u)}{P_{n-1}} = o(1).$$

Since $(\Delta\sigma^{(1)}_{n,p}(\omega^{(m)}(u)))$ is slowly oscillating, applying Lemma 1 to the sequence $(\Delta\sigma^{(1)}_{n,p}(\omega^{(m)}(u)))$ we obtain that

$$\Delta\sigma^{(1)}_{n,p}(\omega^{(m)}(u)) = \Delta\left((\frac{P_{n-1}}{p_n}\Delta)_m V^{(m)}_{n,p}(\Delta u)\right) = o(1).$$

Since $(\sigma^{(1)}_{n,p}(\omega^{(m-1)}(u)))$ is bounded for every nonnegative integer m, by the identity

$$\Delta\left((\frac{P_{n-1}}{p_n}\Delta)_{m-1} V^{(m-1)}_{n,p}(\Delta u)\right) = \frac{p_n (\frac{P_{n-1}}{p_n}\Delta)_m V^{(m)}_{n,p}(\Delta u)}{P_{n-1}} + \Delta\left((\frac{P_{n-1}}{p_n}\Delta)_m V^{(m)}_{n,p}(\Delta u)\right),$$

it follows that

$$\Delta\sigma^{(1)}_{n,p}(\omega^{(m-1)}(u)) = \Delta\left((\frac{P_{n-1}}{p_n}\Delta)_{m-1} V^{(m-1)}_{n,p}(\Delta u)\right) = o(1).$$

The rest of the proof is as in the proof of Theorem 2. □

Corollary 2. *Let (u_n) be a bounded sequence. If $(\sigma^{(1)}_{n,p}(\omega^{(m)}(u)))$ is slowly oscillating for some nonnegative integer m, then (u_n) is subsequentially convergent.*

Proof. Proof follows from the fact that $\lim_n \Delta u_n = 0$ for every slowly oscillating sequence (u_n). □

Corollary 3. *Let (u_n) be a bounded sequence. If $(\Delta\sigma_n^{(1)}(\omega^{(m)}(u)))$ is slowly oscillating for some nonnegative integer m, then (u_n) is subsequentially convergent.*

Proof. Take $p_n = 1$ for every nonnegative integer n in Theorem 3. □

Corollary 3 was given by Çanak and Dik [1].

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