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GBS OPERATORS OF DURRMEYER–STANCU TYPE

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Abstract. Considering four given real parameters α_1, β_1, α_2, and β_2, which satisfy the conditions $0 \le \alpha_1 \le \beta_1$ and $0 \le \alpha_2 \le \beta_2$, we construct the bivariate Durrmeyer–Stancu operators $D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}$ defined by relation (3.1). Next, we consider the associated GBS (Generalized Boolean Sum) operators and establish some approximation properties of these operators.

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1. INTRODUCTION

In 1967, J. L. Durrmeyer [3] introduced the operators $D_m\colon L_1([0,1]) \to C([0,1])$ defined by the formula

$$(D_m f)(x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 p_{m,k}(t)f(t)dt, \tag{1.1}$$

for any $f \in L_1([0,1])$, any $x \in [0,1]$, and any positive integer m, where

$$p_{m,k}(x) = \binom{m}{k} x^k(1-x)^{m-k}, \qquad x \in [0,1],$$

are the fundamental Bernstein polynomials. These operators are called the *Durrmeyer operators*.

Let α and β be two given real parameters satisfying the conditions $0 \le \alpha \le \beta$. In the paper [5], D. D. Stancu introduced and studied the linear positive operators $P_m^{(\alpha,\beta)}\colon C([0,1]) \to C([0,1])$ defined for any function $f \in C([0,1])$ and any positive integer m by the formula

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right) \tag{1.2}$$

for all $x \in [0,1]$. These operators are called the *Stancu operators*.

In the paper [4], the authors introduced and studied the Durrmeyer–Stancu type operators $\big(D_m^{(\alpha,\beta)}\big)_{m\geq 1}$ for two given real parameters α and β with $0 \leq \alpha \leq \beta$. The linear operators $D_m^{(\alpha,\beta)}\colon L_1([0,1]) \to C([0,1])$ are defined, for any function $f \in L_1([0,1])$ and any positive integer m, by the relation

$$\big(D_m^{(\alpha,\beta)} f\big)(x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 p_{m,k}(t) f\left(\frac{mt+\alpha}{m+\beta}\right) dt \tag{1.3}$$

for any $x \in [0,1]$.

2. Preliminaries

In the following, we recall some results which we will use in this paper (see [1,2, 4]). Let the real numbers a,b,c,d so that $a<b$, $c<d$,

$$C_b([a,b]\times[c,d]) = \big\{f\colon [a,b]\times[c,d] \to \mathbb{R} : f \ B\text{-continuous on } [a,b]\times[c,d]\big\},$$
$$M([a,b]\times[c,d]) = \big\{f\colon [a,b]\times[c,d] \to \mathbb{R} : f \text{ bounded on } [a,b]\times[c,d]\big\},$$

and $e_{ij}(x,y) = x^i y^j$, $i,j \in \mathbb{N}$, $0 \leq i,\ j \leq 2$ are the test functions.

If $f\colon [a,b]\times[c,d] \to \mathbb{R}$, f is B-bounded function on $[a,b]\times[c,d]$, then the function $\omega_{\text{mixed}}\colon [0,b-a]\times[0,d-c] \to \mathbb{R}$, defined by the formula

$$\omega_{\text{mixed}}(f;\delta_1,\delta_2) = \sup\big\{|(\Delta f)((x,y),(s,t))| : |s-x| \leq \delta_1,\ |t-y| \leq \delta_2\big\}$$

for any $(\delta_1,\delta_2) \in [0,b-a]\times[0,d-c]$, is called the *mixed modulus of smoothness* of function f.

Theorem 2.1. *Let $(L_{m,n})_{m,n\in\mathbb{N}}$ be a sequence of linear positive bivariate operators, $L_{m,n}\colon C_b([a,b]\times[c,d]) \to M([a,b]\times[c,d])$, $m,n \in \mathbb{N}$. If*

(i) $(L_{m,n}e_{00})(x,y) = 1$,
(ii) $(L_{m,n}e_{10})(x,y) = e_{10}(x,y) + u_{m,n}(x,y)$,
(iii) $(L_{m,n}e_{01})(x,y) = e_{01}(x,y) + v_{m,n}(x,y)$,
(iv) $(L_{m,n}e_{22})(x,y) = e_{22}(x,y) + w_{m,n}(x,y)$,
(v) $\lim_{m,n\to\infty} u_{m,n}(x,y) = \lim_{m,n\to\infty} v_{m,n}(x,y) = \lim_{m,n\to\infty} w_{m,n}(x,y) = 0$ *uniformly on the rectangle* $[a,b]\times[c,d]$,

then, for any $f \in C_b([a,b]\times[c,d])$, the sequence $(U_{m,n}f)_{m,n\in\mathbb{N}}$ converges uniformly to f on the rectangle $[a,b]\times[c,d]$, where, for any $m,n \in \mathbb{N}$, the operator $U_{m,n}\colon \mathbb{R}^{[a,b]\times[c,d]} \to \mathbb{R}^{[a,b]\times[c,d]}$ is defined by

$$(U_{m,n}f)(x,y) = \big(L_{m,n}(f(\cdot,y) + f(x,*) - f(\cdot,*))\big)(x,y) \tag{2.1}$$

for any $f \in \mathbb{R}^{[a,b]\times[c,d]}$, any $(x,y) \in [a,b]\times[c,d]$, "$\cdot$" and "$*$" stand for the first and second variable.

Remark 2.1. For the positive integers m and n, the operator $U_{m,n}$ is called GBS operator associated to $L_{m,n}$ operator. The term “GBS operator” (Generalized Boolean Sum operator) was introduced by C. Badea and C. Cottin in the paper [2].

Remark 2.2. Theorem 2.1 is a Korovkin type theorem for B-continuous functions and is due to C. Badea, I. Badea, and H. H. Gonska (see [1]).

Theorem 2.2. *Let $(L_{m,n})_{m,n\in\mathbb{N}}$ be an arbitrary sequence of linear positive operators $L_{m,n}: C_b([a,b]\times[c,d]) \to M([a,b]\times[c,d])$ and $(U_{m,n})_{m,n\in\mathbb{N}}$ the sequence of associated GBS operators. Then, for any $f \in C_b([a,b]\times[c,d])$, any $(x,y) \in [a,b]\times[c,d]$, any $(\delta_1,\delta_2) \in [0,b-a]\times[0,d-c]$ and any positive integers m,n, the following*

$$\begin{aligned} \left|(U_{m,n}f)(x,y) - f(x,y)\right| &\le |f(x,y)|\,|1-(L_{m,n}e_{00})(x,y)| \\ &+ \Big((L_{m,n}e_{00})(x,y) + \delta_1^{-1}\sqrt{(L_{m,n}(\cdot-x)^2)(x,y)} \\ &+ \delta_2^{-1}\sqrt{(L_{m,n}(*-y)^2)(x,y)} \\ &+ \delta_1^{-1}\delta_2^{-1}\sqrt{(L_{m,n}(\cdot-x)^2(*-y)^2)(x,y)}\Big)\,\omega_{\text{mixed}}(f;\delta_1,\delta_2) \end{aligned} \tag{2.2}$$

holds.

Remark 2.3. Theorem 2.2 is a Shisha–Mond type theorem for B-continuous functions; it is due to C. Badea and C. Cottin (see [2]).

Theorem 2.3. *Let two given real parameters α and β with $0 \le \alpha \le \beta$. The oprators $\big(D_m^{(\alpha,\beta)}\big)_{m\ge 1}$ verify the following*

$$\big(D_m^{(\alpha,\beta)}e_0\big)(x) = 1, \tag{2.3}$$

$$\big(D_m^{(\alpha,\beta)}e_1\big)(x) = \frac{m^2}{(m+\beta)(m+2)}x + \frac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)}, \tag{2.4}$$

$$\begin{aligned} \big(D_m^{(\alpha,\beta)}e_2\big)(x) &= \frac{m^3(m-1)}{(m+\beta)^2(m+2)(m+3)}x^2 \\ &+ \frac{4m^3+2\alpha m^2(m+3)}{(m+\beta)^2(m+2)(m+3)}x \\ &+ \frac{2m^2+2\alpha m(m+3)+\alpha^2(m+2)(m+3)}{(m+\beta)^2(m+2)(m+3)}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \big(D_m^{(\alpha,\beta)}\varphi_x^2\big)(x) &= \left(\frac{m}{m+\beta}\right)^2 \frac{2(m-3)x(1-x)+2}{(m+2)(m+3)} \\ &+ \frac{[\beta^2(m+2)+4\beta m]x^2 - 2x[\alpha\beta(m+2)+\beta m+2\alpha m]x + \alpha^2(m+2)+2\alpha m}{(m+\beta)^2(m+2)}, \end{aligned} \tag{2.6}$$

and

$$\delta_m^{(\alpha,\beta)}(x) \leq \delta_m^{(\alpha,\beta)}, \tag{2.7}$$

where

$$\delta_m^{(\alpha,\beta)}(x) = \sqrt{\left(D_m^{(\alpha,\beta)}\varphi_x^2\right)(x)} \tag{2.8}$$

and

$$\delta_m^{(\alpha,\beta)} = \left(\left(\frac{m}{m+\beta}\right)^2 \frac{m+1}{2(m+2)(m+3)} + \max\left\{\frac{\alpha^2(m+2)+2\alpha m}{(m+\beta)^2(m+2)}, \frac{(\alpha-\beta)^2(m+2)-2(\alpha-\beta)m}{(m+\beta)^2(m+2)}\right\}\right)^{\frac{1}{2}} \tag{2.9}$$

for any positive integer m, and any $x \in [0,1]$.

Proof. For the proof, see [4]. □

3. Main results

In this section, let the given real parameters α_1, α_2, β_1, and β_2 satisfying the conditions $0 \leq \alpha_1 \leq \beta_1$ and $0 \leq \alpha_2 \leq \beta_2$.

We construct the bivariate Durrmeyer–Stancu operators

$$D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} : L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1]),$$

defined for any positive integers m,n and any $f \in L_1([0,1]\times[0,1])$ by

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) = (m+1)(n+1)\sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y) \times \int_0^1\int_0^1 p_{m,k}(s)p_{n,j}(t) f\left(\frac{ms+\alpha_1}{m+\beta_1}, \frac{nt+\alpha_2}{n+\beta_2}\right) ds\,dt \tag{3.1}$$

for any $(x,y) \in [0,1]\times[0,1]$.

Let ${}^xD_m^{(\alpha_1,\beta_1)}$, ${}^yD_n^{(\alpha_2,\beta_2)}$ be the parametric extensions of the operator (1.3), where $x,y \in [0,1]$.

If $x \in [0,1]$, then ${}^xD_m^{(\alpha_1,\beta_1)} : L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ is defined for any positive integer m and any $f \in L_1([0,1]\times[0,1])$ by the formula

$$\left({}^xD_m^{(\alpha_1,\beta_1)} f\right)(x,y) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 p_{m,k}(s) f\left(\frac{ms+\alpha_1}{m+\beta_1}, y\right) ds \tag{3.2}$$

for any $y \in [0,1]$ and if $y \in [0,1]$, then ${}^{y}D_n^{(\alpha_2,\beta_2)}: L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ is defined for any positive integer n and any $f \in L_1([0,1]\times[0,1])$ by the formula

$$\left({}^{y}D_n^{(\alpha_1,\beta_2)} f\right)(x,y) = (n+1)\sum_{j=0}^{n} p_{n,j}(y)\int_0^1 p_{n,j}(t) f\left(x, \frac{nt+\alpha_2}{n+\beta_2}\right) dt \tag{3.3}$$

for any $x \in [0,1]$.

The GBS operator of Durrmeyer–Stancu is the “boolean sum” of the parametric extensions of the operators ${}^{x}D_m^{(\alpha_1,\beta_1)}$ and ${}^{y}D_n^{(\alpha_2,\beta_2)}$, so

$$U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}: L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$$

is defined for any positive integers m and n by the relation

$$\begin{aligned} U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} &= {}^{x}D_m^{(\alpha_1,\beta_1)} \oplus {}^{y}D_n^{(\alpha_2,\beta_2)} = \\ &= {}^{x}D_m^{(\alpha_1,\beta_1)} + {}^{y}D_n^{(\alpha_2,\beta_2)} - {}^{x}D_m^{(\alpha_1,\beta_1)} \circ {}^{y}D_n^{(\alpha_2,\beta_2)}, \end{aligned} \tag{3.4}$$

where

$${}^{x}D_m^{(\alpha_1,\beta_1)} \circ {}^{y}D_n^{(\alpha_2,\beta_2)} = D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}.$$

Lemma 3.1. *The Durrmeyer–Stancu type GBS operator is defined, for any $f \in L_1([0,1]\times[0,1])$ and any positive integers m and n, by the relation*

$$\begin{aligned} \left(U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) &= (m+1)(n+1)\sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) \\ &\times \int_0^1\int_0^1 p_{m,k}(s) p_{n,j}(t)\Bigg(f\left(\frac{ms+\alpha_1}{m+\beta_1}, y\right) + f\left(x, \frac{nt+\alpha_2}{n+\beta_2}\right) \\ &- f\left(\frac{ms+\alpha_1}{m+\beta_1}, \frac{nt+\alpha_2}{n+\beta_2}\right)\Bigg) ds\,dt \end{aligned} \tag{3.5}$$

for all $(x,y) \in [0,1]\times[0,1]$.

Proof. The assertion follows from definitions (3.1)–(3.4). □

Lemma 3.2. *For any positive integers m,n and any $(x,y) \in [0,1]\times[0,1]$, the parametric extensions ${}^{x}D_m^{(\alpha_1,\beta_1)}$ and ${}^{y}D_n^{(\alpha_2,\beta_2)}$ are linear and positive operators.*

Proof. The assertion is obtained by direct computation. □

Lemma 3.3. *operators Let m and n be given positive integers. The operator $D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_1)}$ is linear and positive.*

Proof. The proof is immediate. □

The operator $D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_1)}$ is called in what follows the *bivariate Durrmeyer–Stancu operator*.

Lemma 3.4. *For any nonzero natural numbers m, n and arbitrary $(x,y)\in[0,1]\times[0,1]$, we have*

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{00}\right)(x,y)=1, \tag{3.6}$$

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{10}\right)(x,y)=\frac{m^2}{(m+\beta_1)(m+2)}x+\frac{(\alpha_1+1)m+2\alpha_1}{(m+\beta_1)(m+2)}, \tag{3.7}$$

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{01}\right)(x,y)=\frac{n^2}{(n+\beta_2)(n+2)}y+\frac{(\alpha_2+1)n+2\alpha_2}{(n+\beta_2)(n+2)}, \tag{3.8}$$

$$\begin{aligned}\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{22}\right)(x,y)&=\frac{m^3(m-1)}{(m+\beta_1)^2(m+2)(m+3)}x^2\\&+\frac{4m^3+2\alpha_1m^2(m+3)}{(m+\beta_1)^2(m+2)(m+3)}x+\frac{2m^2+2\alpha_1m(m+3)+\alpha_1^2(m+2)(m+3)}{(m+\beta_1)^2(m+2)(m+3)}\\&+\frac{n^3(n-1)}{(n+\beta_2)^2(n+2)(n+3)}y^2\frac{4n^3+2\alpha_2n^2(n+3)}{(n+\beta_2)^2(n+2)(n+3)}y\\&+\frac{2n^2+2\alpha_2n(n+3)+\alpha_2^2(n+2)(n+3)}{(n+\beta_2)^2(n+2)(n+3)},\end{aligned} \tag{3.9}$$

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}(\cdot-x)^2\right)(x,y)=\left(D_m^{(\alpha_1,\beta_1)}\varphi_x^2\right)(x)=\left(\delta_m^{(\alpha_1,\beta_1)}(x)\right)^2, \tag{3.10}$$

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}(*-y)^2\right)(x,y)=\left(D_n^{(\alpha_2,\beta_2)}\varphi_y^2\right)(y)=\left(\delta_n^{(\alpha_2,\beta_2)}(y)\right)^2, \tag{3.11}$$

$$\delta_m^{(\alpha_1,\beta_1)}(x)\le\delta_m^{(\alpha_1,\beta_1)} \tag{3.12}$$

and

$$\delta_n^{(\alpha_2,\beta_2)}(y)\le\delta_n^{(\alpha_2,\beta_2)}, \tag{3.13}$$

where the values $\delta_m^{(\alpha_1,\beta_1)}(x)$, $\delta_m^{(\alpha_1,\beta_1)}$, $\delta_n^{(\alpha_2,\beta_2)}(y)$, and $\delta_n^{(\alpha_2,\beta_2)}$ are defined by relations (2.7)–(2.9).

Proof. The assertion follows from Theorem 2.3. □

Lemma 3.5. *The bivariate Durrmeyer–Stancu operators satisfy the relations*

$$\lim_{m,n\to\infty}\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{10}\right)(x,y)=e_{10}(x,y), \tag{3.14}$$

$$\lim_{m,n\to\infty}\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{01}\right)(x,y)=e_{01}(x,y), \tag{3.15}$$

and

$$\lim_{m,n\to\infty}\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{22}\right)(x,y)=e_{22}(x,y) \tag{3.16}$$

uniformly on $[0,1]\times[0,1]$.

Proof. Relations (3.14), (3.15), and (3.16) are consequences of Lemma 3.4. □

Theorem 3.1. *For any* $f \in C_b([0,1]\times[0,1]) \cap L_1([0,1]\times[0,1])$*, the sequence*

$$\left(U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)_{m,n\geq 1}$$

converges to f *uniformly on* $[0,1]\times[0,1]$.

Proof. It is sufficient to apply Theorem 2.1 and Lemma 3.5. □

Theorem 3.2. *For any positive integers* m *and* n*, any function* $f \in C_b([0,1]\times[0,1]) \cap L_1([0,1]\times[0,1])$ *and any* $(x,y) \in [0,1]\times[0,1]$*, the operator* $U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}$ *has the properties*

$$\left|\left(U_{m,n}{}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) - f(x,y)\right| \leq \Bigg(1 + \frac{1}{\delta_1}\delta_m^{(\alpha_1,\beta_1)}(x) + \frac{1}{\delta_2}\delta_n^{(\alpha_2,\beta_2)}(y) + \frac{1}{\delta_1\delta_2}\delta_m^{(\alpha_1,\beta_1)}(x)\delta_n^{(\alpha_2,\beta_2)}(y)\Bigg)\omega_{\text{mixed}}(f;\delta_1,\delta_2) \tag{3.17}$$

for any $\delta_1, \delta_2 \in (0,1]$ *and*

$$\begin{aligned}\left|\left(U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) - f(x,y)\right| &\leq 4\omega_{\text{mixed}}\left(f;\delta_m^{(\alpha_1,\beta_1)}(x),\delta_n^{(\alpha_2,\beta_2)}(y)\right) \\ &\leq 4\omega_{\text{mixed}}\left(f;\delta_m^{(\alpha_1,\beta_1)},\delta_n^{(\alpha_2,\beta_2)}\right).\end{aligned} \tag{3.18}$$

Proof. For the first inequality (3.17), we apply Theorem 2.2, Lemma 3.4, and relations (3.10) and (3.11). For the inequality (3.18), in the inequality (3.17) we choose

$$\delta_1 = \delta_m^{(\alpha_1,\beta_1)}(x),$$
$$\delta_2 = \delta_n^{(\alpha_2,\beta_2)}(y),$$

and afterwards we take the relations (3.12) and (3.13) into account. □

References

[1] C. Badea, I. Badea, and H. H. Gonska, "A test function theorem and approximation by pseudopolynomials," *Bull. Austral. Math. Soc.*, vol. 34, no. 1, pp. 53–64, 1986.

[2] C. Badea and C. Cottin, "Korovkin-type theorems for generalized Boolean sum operators," in *Approximation theory (Kecskemét, 1990)*, ser. Colloq. Math. Soc. János Bolyai. Amsterdam: North-Holland, 1991, vol. 58, pp. 51–68.

[3] J. L. Durrmeyer, "Une formule d'inversion de la transformée de Laplace: Application à la theorie des moments," Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.

[4] O. T. Pop and D. Bărbosu, "Durrmeyer–Stancu type operators," to appear.

[5] D. D. Stancu, "On a generalization of the Bernšteĭn polynomials," *Studia Univ. Babeş-Bolyai Ser. Math.-Phys.*, vol. 14, no. 2, pp. 31–45, 1969.

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