Miskolc Mathematical Notes
Vol. 16 (2015), No. 2, pp. 1055–1062
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1329

BEST PROXIMITY POINTS FOR GENERALIZED MULTIVALUED CONTRACTIONS IN METRIC SPACES

ESMAEIL NAZARI

Received 15 September, 2014

Abstract. In the present paper, we prove a best proximity point theorem for multivalued non-self-contractive type mappings which is a generalization of recent best proximity point theorems and some famous fixed point theorems.

2010 *Mathematics Subject Classification:* 47H10; 47H06

Keywords: best proximity point, fixed point, optimal approximate solution, multivalued mapping

1. Introduction

Let A, B be nonempty subsets of a metric space (X,d) and $T : A \to B$ be a non-self-mapping. Clearly, the set of fixed points of T can be empty. Therefore, it is of primary importance to seek an element x that in some sense is closest to Tx. That is, if there is no solution to the fixed point equation $Tx = x$, one tries to determine an approximate solution x subject to the condition that the distance between x and Tx is minimal. A classical best approximation theorem was introduced by Fan [4]. It states that if A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space X and $T : A \to X$ is a continuous mapping, Then there exists $x \in A$ such that $d(x,Tx) = d(Tx,A)$. Recently, there have been many subsequent extensions of Fan's theorem, see [7, 8, 12] and references therein. A point $x \in A$ is called a best proximity point for T if distance of x to Tx is equal to the distance of A to B. In fact best proximity point theorems have been studied to find necessary conditions such that the minimization problem,

$$\min_{x\in A} d(x,Tx) \tag{1.1}$$

has at least one solution. Investigation of several variants of contractions for the existence of a best proximity point can be found in [2, 3, 5, 9–11, 13, 14].

In this article, we consider a classes of multivalued non-self-mapping which called (ϕ,θ) contractive mappings and we present some best proximity point theorems for these classes of non-self-mappings in metric spaces.

© 2015 Miskolc University Press

Let A and B be two nonempty subsets of a metric space. We will use the following notations:

$$d(A,B)=\inf\{d(x,y):x\in A,y\in B\},$$
$$A_0=\{x\in A:d(x,y)=d(A,B)\quad \text{for some}\quad y\in B\},$$
$$B_0=\{y\in B:d(x,y)=d(A,B)\quad \text{for some}\quad y\in A\},$$
$$D(x,B)=\inf\{d(x,y):y\in B\},\quad \forall x\in X,$$
$$H(A,B)=\max\{\sup_{x\in A}D(x,B),\sup_{y\in B}D(y,A)\}.$$

Let A and B be nonempty subsets of a metric space (X,d). Assume that $T:A\to 2^B$ is a multivalued non-self-mapping. A point $x\in A$ is said to be a fixed point of T if $x\in Tx$. In case $A\cap B=\varnothing$, the multifunction T has not fixed point. Then $D(x,Tx)>0$ for all $x\in A$. Therefore, we can explore to find necessary conditions so that the minimization problem

$$\min_{x\in A}D(x,Tx) \tag{1.2}$$

has at least one solution. Since $D(x,Tx)\geq d(A,B)$ for all $x\in A$, the optimal solution to the problem (1.2) is obtained in some points of A for which the value $d(A,B)$ is attained. A point $x\in A$ is called a best proximity point of a multivalued non-self-mapping T, if $D(x,Tx)=d(A,B)$. We note that if $d(A,B)=0$, then we get a fixed point of T.

Definition 1 ([11])**.** Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0\neq\varnothing$. Then the pair (A,B) is said to have the P-property iff

$$\begin{cases} d(x_1,y_1)=d(A,B)\\ d(x_2,y_2)=d(A,B)\end{cases}\Rightarrow d(x_1,x_2)=d(y_1,y_2),$$

where $x_1,x_2\in A$ and $y_1,y_2\in B$

Definition 2 ([15])**.** Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0\neq\varnothing$. Then the pair (A,B) is said to have the weak P-property iff

$$\begin{cases} d(x_1,y_1)=d(A,B)\\ d(x_2,y_2)=d(A,B)\end{cases}\Rightarrow d(x_1,x_2)\leq d(y_1,y_2),$$

where $x_1,x_2\in A$ and $y_1,y_2\in B$.

Definition 3. We say that $\varphi:[0,\infty[\to[0,\infty[$ is a (c)-comparison function if and only if the following conditions hold:
(i) φ is a nondecreasing function,
(ii) for any $t>0$, $\sum_{n=0}^{\infty}\varphi^n(t)$ is a convergent series.

In what follows, we will denote:

$$\Theta = \{\theta : [0,+\infty)^4 \to [0,+\infty) :$$
$$\theta \text{ is continuous and } \theta(t_1,t_2,t_3,t_4) = 0 \Leftrightarrow t_1t_2t_3t_4 = 0\}.$$

Example 1. The following functions belong to Θ :
(1) $\theta(t_1,t_2,t_3,t_4) = L\min\{t_1,t_2,t_3,t_4\}, L > 0$
(2) $\theta(t_1,t_2,t_3,t_4) = t_1t_2t_3t_4$,
(3) $\theta(t_1,t_2,t_3,t_4) = \ln(1+t_1t_2t_3t_4)$,
(4) $\theta(t_1,t_2,t_3,t_4) = \exp(t_1t_2t_3t_4)-1$.

The notion of almost (φ,θ)-contraction for single valued non-self mapping was introduced by Bessem Samet as follows.

Definition 4 ([10])**.** A mapping $T : A \to B$ is said to be an almost (φ,θ)-contraction if and only if there exist $\varphi \in \Phi$ and $\theta \in \Theta$ such that, for all $x, y \in A$,

$$d(Tx,Ty) \leq \varphi\Big(d(x,y)\Big) + \theta\Big(d(y,Tx) - d(A,B), d(x,Ty) - d(A,B), d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\Big)$$

He proved the following result.

Theorem 1 ([10])**.** *Let A and B be closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Suppose that $T : A \to B$ satisfies the following conditions:*
(i) T is an almost (φ,θ)-contraction,
(ii) $T(A_0) \subseteq B_0$,
(iii) the pair (A,B) has the P-property.

Then, there exists a unique element $x^* \in A$ such that

$$d(x^*,Tx^*) = d(A,B)$$

Moreover, for any fixed element $x_0 \in A_0$, any iterative sequence $\{x_n\}$ satisfying

$$d(x_{n+1},Tx_n) = d(A,B)$$

converges to x^*.

Now, in the following we defined the notion of (φ,θ)- contraction for multivalued mappings.

Definition 5. A mapping $T: A \to 2^B$ is said to be an almost (φ, θ)-contraction if and only if there exist $\varphi \in \Phi$ and $\theta \in \Theta$ such that, for all $x, y \in A$,

$$H(Tx, Ty) \le \varphi\Big(d(x,y)\Big) + \theta\Big(D(y,Tx) - d(A,B), D(x,Ty) - d(A,B), D(x,Tx) - d(A,B), D(y,Ty) - d(A,B)\Big)$$

2. Main Results

Our first main result is the following theorem.

Theorem 2. *Let A and B be closed subsets of a complete metric space (X,d) such that $A_0 \neq \varnothing$ and the pair (A,B) satisfies the weak P-property. Suppose that $T: A \to 2^B$ be a multi-valued almost (φ,θ)-contraction non-self mapping. If $T(x)$ is bounded and closed in B for all $x \in A$, and $T(x_0) \subseteq B_0$ for each $x_0 \in A_0$, then T has a best proximity point in A.*

Proof. Select $x_0 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$. By the definition of the set B_0, we can fined an element x_1 in A_0 such that $d(x_1,y_0) = d(A,B)$. If $y_0 \in Tx_1$, then $d(A,B) \le D(x_1,Tx_1) \le d(x_1,y_0) = d(A,B)$, therefore $D(x_1,Tx_1) = d(A,B)$ and x_1 is a best proximity point of T. If $y_0 \notin Tx_1$ and $q > 1$ be given. Then

$$0 < d(y_0,Tx_1) \le H(Tx_0,Tx_1) < qH(Tx_0,Tx_1).$$

Hence, there exists $y_1 \in Tx_1$ such that

$$0 < d(y_0,y_1) < qH(Tx_0,Tx_1) \le q\varphi\Big(d(x_0,x_1)\Big) + q\theta\Big(D(x_1,Tx_0) - d(A,B), D(x_0,Tx_1) - d(A,B), D(x_0,Tx_0) - d(A,B), D(x_1,Tx_1) - d(A,B)\Big)$$

Since $D(x_1,Tx_0) = d(A,B)$, we have

$$\begin{aligned} 0 < d(y_0,y_1) &< q\varphi\Big(d(x_0,x_1)\Big) + q\theta\Big(0, D(x_0,Tx_1) - d(A,B), \\ &\qquad D(x_0,Tx_0) - d(A,B), D(x_1,Tx_1) - d(A,B)\Big) \qquad (2.1) \\ &= q\varphi\Big(d(x_0,x_1)\Big). \end{aligned}$$

One the other hand since $y_1 \in Tx_1 \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2,y_1) = d(A,B)$. By using the weak P-property of (A,B) we obtain $d(x_2,x_1) \le d(y_0,y_1)$. Now, put $t_0 = d(x_0,x_1)$, then $t_0 > 0$ and by (2.1) we have $d(x_1,x_2) < q\varphi(t_0)$. Since φ is strictly increasing, $\varphi\Big(d(x_1,x_2)\Big) < \varphi\Big(q\varphi(t_0)\Big)$. Set $q_1 = \frac{\varphi\big(q\varphi(t_0)\big)}{\varphi\big(d(x_1,x_2)\big)} > 1$. If $y_1 \in Tx_2$ then x_2 is a best proximity point of T. suppose that $y_1 \notin Tx_2$, then

$$0 < d(y_1,Tx_2) \le H(Tx_1,Tx_2) < qH(Tx_1,Tx_2).$$

Therefore, there exits $y_2 \in Tx_2$ such that

$$\begin{aligned} 0 < d(y_2,y_1) &< q_1 H(Tx_2,Tx_1) \\ &\leq q_1\varphi\Big(d(x_1,x_2)\Big) + q_1\theta\Big(D(x_2,Tx_1)-d(A,B), D(x_1,Tx_2) \\ &\quad -d(A,B), D(x_1,Tx_1)-d(A,B), D(x_2,Tx_2)-d(A,B)\Big) \end{aligned}$$

Since $D(x_2,Tx_1) = d(A,B)$, we have

$$\begin{aligned} 0 < d(y_2,y_1) &< q_1\varphi\Big(d(x_1,x_2)\Big) + q_1\theta\Big(0, D(x_1,Tx_2)-d(A,B), D(x_1,Tx_1) \\ &\quad -d(A,B), D(x_2,Tx_2)-d(A,B)\Big) \\ &= q_1\varphi\Big(d(x_1,x_2)\Big) = \varphi\Big(q\varphi(t_0)\Big). \end{aligned} \tag{2.2}$$

Again, since $y_2 \in Tx_2 \subseteq B_0$, there exist $x_3 \in A_0$ such that $d(x_3,y_2) = d(A,B)$. By using the weak P-property of (A,B) we obtain $d(x_3,x_2) \leq d(y_2,y_1)$. Since φ is in strictly increasing by using (2.2) we have $\varphi\Big(d(x_3,x_2)\Big) < \varphi^2(q\varphi(t_0))$. Set $q_2 = \dfrac{\varphi^2\Big(q\varphi(t_0)\Big)}{\varphi\Big(d(x_3,x_2)\Big)} > 1$. If $y_2 \in Tx_3$ then x_3 is a best proximity point of T. Suppose that $y_2 \notin Tx_3$ then we have,

$$0 < d(y_2,Tx_3) \leq H(Tx_2,Tx_3) < q_2 H(Tx_2,Tx_3).$$

Then there is $y_3 \in Tx_3$ such that

$$\begin{aligned} 0 < d(y_3,y_2) &< q_2 H(Tx_3,Tx_2) \leq q_2\varphi\Big(d(x_3,x_2)\Big) \\ &\quad + q_2\theta\Big(D(x_3,Tx_2)-d(A,B), d(x_2,Tx_3)-d(A,B), D(x_3,Tx_3) \\ &\quad -d(A,B), D(x_2,Tx_2)-d(A,B)\Big) \end{aligned}$$

Since $D(x_3,Tx_2) = d(A,B)$ we have

$$\begin{aligned} 0 < d(y_3,y_2) &< \varphi\Big(d(x_3,x_2)\Big) + q_2\theta\Big(0, d(x_2,Tx_3)-d(A,B), D(x_3,Tx_3) \\ &\quad -d(A,B), D(x_2,Tx_2)-d(A,B)\Big) \\ &= q_2\varphi\Big(d(x_3,x_2)\Big) = \varphi^2(q\varphi(t_0)) \end{aligned}$$

By continuing this process, for each $n \in N$, we can find a sequences $\{x_n\}$ and $\{y_n\}$ in A_0 and B_0 respectively, such that,

(1) $y_n \in Tx_n \subseteq B_0$,

(2) $d(x_{n+1},y_n) = d(A,B)$

(3) $d(y_{n+1}, y_n) \leq \varphi^n\Big(q\varphi(t_0)\Big)$.
Since (A, B) satisfies the weak p-property, we conclude that

$$d(x_n, x_{n+1}) \leq d(y_{n-1}, y_n) \qquad \forall n \in N$$

we now have

$$d(x_n, x_{n+1}) \leq d(y_{n-1}, y_n) \leq \varphi^{n-1}\Big(q\varphi(t_0\Big)$$

Let $m > n$. Then

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \varphi^{i-1}\Big(q\varphi(t_0)\Big)$$

and so $\{x_n\}$ is a Cauchy sequence in A. Hence, there exists $x^* \in A$ such that $x_n \to x^*$. Similarly, by using (3) we can show that the sequence $\{y_n\}$ in B is Cauchy and hence is convergent. Suppose that $y_n \to y^*$. By the relation $d(x_{n+1}, y_n) = d(A, B)$, for all $n \in N$, we conclude that $d(x^*, y^*) = d(A, B)$. Now we show that $y^* \in Tx^*$. Since $y_n \in Tx_n$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} D(y_n, Tx^*)\\
&\leq \lim_{n\to\infty} H(Tx_n, Tx^*)\\
&\leq \lim_{n\to\infty} \Big[\varphi\Big(d(x_n, x^*)\Big) + \theta\Big(D(x^*, Tx_n)\\
&-d(A, B), D(x_n, Tx^*) - d(A, B), D(x_n, Tx_n) - d(A, B),\\
&D(x^*, Tx^*) - d(A, B)\Big)\Big]\\
&= 0 + \theta\Big(\lim_{n\to\infty} d(x^*, y_n) - d(A, B), \lim_{n\to\infty} \big(D(x_n, Tx^*)\\
&-d(A, B)\big), \lim_{n\to\infty} \big(D(x_n, Tx_n) - d(A, B)\big), D(x^*, Tx^*) - d(A, B)\Big)\\
&= 0 + \theta\Big(0, \lim_{n\to\infty} \big(D(x_n, Tx^*)\\
&-d(A, B)\big), \lim_{n\to\infty} \big(D(x_n, Tx_n) - d(A, B)\big), D(x^*, Tx^*) - d(A, B)\Big)\\
&= 0.
\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} D(y_n, Tx^*) = 0.$$

Hence $D(y^*, Tx^*) = 0$. Since Tx^* is closed, We conclude that $y^* \in Tx^*$. Now we have,

$$d(A, B) \leq D(x^*, Tx^*) \leq d(x^*, y^*) = d(A, B),$$

which implies that $D(x^*, Tx^*) = d(A, B)$, that is $x^* \in A$ is a best proximity point of T. This completes the proof of theorem. □

Taking $\varphi(t) = \alpha t$ we have the following result which an extension of theorem 2.1 in [1].

Corollary 1. *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \varnothing$ and (A,B) satisfies the weak P-property. Let $T : A \to 2^B$ be a multivalued non-self-mapping, for which there exist a constant $\alpha \in [0,1)$ and $\theta \in \Theta$ such that for all $x, y \in X$*

$$H(Tx,Ty) \leq \alpha d(x,y) + \theta\Big(D(y,Tx) - d(A,B), D(x,Ty) - d(A,B), D(x,Tx) - d(A,B), D(y,Ty) - d(A,B)\Big)$$

Suppose also that $T(x)$ is bounded and closed in B for all $x \in A$, and $T(x_0) \subseteq B_0$ for each $x_0 \in A_0$, then T has a best proximity point in A.

Example 2. Let $X = \Re$ with the usual metric. Suppose $A := \{0,3,6,9\}$ and $B := \{-1,2,5,8\}$. Then, A and B are nonempty and closed subsets of X and $A_0 = A$ and $B_0 = B$. We note that, $d(A,B) = 1$. It is easy to show that the pair (A,B) has the weak P-property. Let $T : A \to 2^B$ ba a mapping defined by $T0 = \{8\}$ and $Tx = \{5,8\}$, if $x \neq 0$. Consider the functions $\theta(t_1,t_2,t_3,t_4) = t_1t_2t_3t_4$ and $\varphi(t) = \frac{t}{2}$ for all $t \geq 0$. Then T is (φ,θ)- multivalued contraction. Thus T has a best proximity point Note that $x = 6$ and $x = 9$ are best proximity point of T. It is interesting to note that the non-self mapping T is not a non-self contraction.

Taking $B = A$ in Theorem 2, we obtain the following result.

Corollary 2. *Let (X,d) be a complete metric space, and A be a nonempty and closed subset of X. Let $T : A \to 2^A$ be an almost (φ,θ)-contraction self-mapping. Then T has a fixed point $x \in A$.*

Taking $\varphi(t) = \alpha t$ and $\theta(t_1,t_2,t_3,t_4) = L\min\{t_1,t_2,t_3,t_4\}$, we obtain from Corollary 2 the following result which is a generalization of Nadler fixed point theorem [6].

Corollary 3. *Let (X,d) be a complete metric space, and A be a nonempty closed subset of X. Let $T : A \to 2^A$ be a mapping such that there exist $\alpha \in [0,1)$ and $L > 0$ such that, for all $x, y \in A$,*

$$H(Tx,Ty) \leq \alpha d(x,y) + L\min\{D(y,Tx), D(x,Ty), D(x,Tx), D(y,Ty)\}$$

Then T has a unique fixed point $x \in A$.

Acknowledgement

The authors are grateful to the referees for their helpful suggestions contributing to the improvement of the paper.

REFERENCES

[1] A. Abkar and M. Gabeleh, "The existence of best proximity points for multivalued nonself-mappings," *RACSAM*, vol. 107, no. 2, pp. 319–325, 2013, doi: 10.1007/s13398-012-0074-6.

[2] A. Abkar and M. Gabeleh, "A note on some best proximity point theorems proved under p-property," *abstract Appl. Anal*, vol. Article ID 189567, 2013.

[3] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *J. Math. Anal. Appl*, vol. 323, no. 2, pp. 1001–1006, 2006, doi: 10.1016/j.jmaa.2005.10.081.

[4] K. Fan, "Extensions of two fixed point theorems of F.E. Browder," *Math. Z*, vol. 112, no. 3, pp. 234–240, 1969.

[5] W. A. Kirk, S. Reich, and P. Veeramani, "Proximinal retracts and best proximity pair theorems," *Numer. Funct. Anal. Optim*, vol. 24, no. 7-8, pp. 851–862, 2003, doi: 10.1081/NFA-120026380.

[6] S. B. J. Nadler, "Multivalued contraction mappinsg," *Pac. J. Math*, vol. 30, no. 2, pp. 475–488, 1969, doi: 10.2140/pjm.1969.30.475.

[7] J. B. Prolla, "Fixed point theorems for set valued mappings and existence of best approximations," *Numer. Funct. Anal. Optim.*, vol. 5, no. 4, pp. 449–455, 2010.

[8] S. Reich, "Approximate selections, best approximations, fixed points and invariant sets," *J.Math. Anal. Appl*, vol. 62, no. 1, pp. 104–113, 1978, doi: 10.1016/0022-247X(78)90222-6.

[9] S. Sadiq Basha, "Extensions of banach-s contraction principle," *Numer. Funct. Anal. Optim*, vol. 31, no. 5, pp. 569–576, 2010, doi: 10.1080/01630563.2010.485713.

[10] B. Samet, "Some results on best proximity points," *J. Optim. Theory Appl.*, vol. 159, no. 1, pp. 281–291, 2013, doi: 10.1007/s10957-013-0269-9.

[11] V. Sankar Raj, "A best proximity point theorem for weakly contractive non-self-mappings," *Nonlinear Anal.*, vol. 74, no. 14, pp. 4804–4808, 2011, doi: 10.1016/j.na.2011.04.052.

[12] V. M. Seghal and S. P. Singh, "A generalization of multifunctions of Fans best approximation theorem," *Proc. Amer.Math. Soc*, vol. 102, no. 3, pp. 534–537, 1988.

[13] N. Shahzad, S. Sadiq Basha, and R. Jeyaraj, "Common best proximity points: global optimal solutions," *J. Optim. Theory Appl*, vol. 148, no. 148, pp. 69–78, 2011, doi: 10.1007/s10957-010-9745-7.

[14] T. Suzuki, M. Kikkawa, and C. Vetro, "The existence of best proximity points in metric spaces with the property UC," *Nonlinear Anal.*, vol. 71, no. 7, pp. 2918–2926, 2009, doi: 10.1016/j.na.2009.01.173.

[15] J. Zhang, Y. Su, and Q. Cheng, "A note on a best proximity point theorem for Geraghty-contractions," *Fixed Point Theory and Applications*, vol. article 83, 2013, doi: 10.1186/1687-1812-2013-83.

Author's address

Esmaeil Nazari
Department of Mathematics, Tafresh University, Tafresh, Iran
E-mail address: nazari.esmaeil@gmail.com