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ON THE EQUATIONS $U_n = 5\square$ AND $V_n = 5\square$

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Abstract. Let $P \geq 3$ be an integer and let (U_n) and (V_n) denote the generalized Fibonacci and Lucas sequences defined by $U_0 = 0, U_1 = 1$; $V_0 = 2, V_1 = P$, and $U_{n+1} = PU_n - U_{n-1}$, $V_{n+1} = PV_n - V_{n-1}$ for $n \geq 1$. The purpose of this study, assuming P is odd, is to determine the values of n such that $V_n = 5\square$ and $U_n = 5\square$. Moreover, we solve the equations $V_n = 5V_m\square$ and $U_n = 5U_m\square$.

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1. INTRODUCTION

Let P and Q be nonzero integers such that $P^2 + 4Q \neq 0$. The generalized Fibonacci sequence (U_n) and Lucas sequence (V_n) are given recursively according to the following relations for $n \geq 1$.

$$U_0 = 0,\ U_1 = 1,\ U_{n+1} = PU_n + QU_{n-1}$$

and

$$V_0 = 2,\ V_1 = P, V_{n+1} = PV_n + QV_{n-1}.$$

Both sequences depend on the initial choice of pair (P, Q), hence we sometimes use $U_n(P, Q)$ and $V_n(P, Q)$ in order to emphasize their dependence on the parameters (P, Q). U_n and V_n are called the nth generalized Fibonacci number and the nth generalized Lucas number, respectively. Furthermore, generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$$U_{-n} = -(-Q)^{-n}U_n \text{ and } V_{-n} = (-Q)^{-n}V_n \ (n \geq 1),$$

respectively. It is well known that

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta) \text{ and } V_n = \alpha^n + \beta^n$$

where $\alpha = \left(P + \sqrt{P^2 + 4Q}\right)/2$ and $\beta = \left(P - \sqrt{P^2 + 4Q}\right)/2$, which are the roots of the equation $x^2 - Px - Q = 0$. The above formulas are known as Binet's formulas.

We will assume that $P^2+4Q>0$. Special cases of the sequences (U_n) and (V_n) are known. For example, the generalized Fibonacci sequence $(U_n(1,1))$ consist of the familiar Fibonacci numbers, whereas its companion, $(V_n(1,1))$ gives so called Lucas numbers. When $P=2$ and $Q=1$, $(U_n)=(P_n)$ and $(V_n)=(Q_n)$ are the familiar sequences of Pell and Pell-Lucas numbers. For more information about generalized Fibonacci and Lucas sequences, see [8].

There has been much interest in when the terms of generalized Fibonacci and Lucas sequences are perfect square($=\square$) or $k\square$. When P is odd and $Q=\pm 1$, by using elementary arguments, many authors solved the equations $U_n=k\square$ and $V_n=k\square$ for some specific values of k (see [2–4, 9, 10]). Interested readers can also consult [12] and [6] for a brief history of this subject.

In [6], the authors determined all indices n such that $U_n(P,1)=5\square$ and $U_n(P,1)=5U_m(P,1)\square$ under some assumptions on P. When P is odd, the authors solved the equation $V_n(P,1)=5\square$. Moreover, they showed that the equation $V_n(P,1)=5V_m(P,1)\square$ has no solutions. In this study, using congruences, with extensive reliance upon the Jacobi symbol, we determine that the five times square terms of the generalized Fibonacci sequence $(U_n(P,Q))$ for which $P\geq 3$ is odd and $Q=-1$ may occur only for $n=2$ or 3. We obtain a similar result for the generalized Lucas sequence $(V_n(P,Q))$. Moreover, when $P\geq 3$ is odd and $Q=-1$, we solve the equations $V_n=5V_m\square$ and $U_n=5U_m\square$.

In section 2, we give some identities, lemmas, and theorems needed later. Then in section 3, we present our main theorems. Throught this study, $\left(\frac{*}{*}\right)$ will denote the Jacobi symbol. Our method of proof is similar to that presented by Cohn, McDaniel and Ribenboim [2–4, 9].

2. PRELIMINARY FACTS

From now on, we assume that $Q=-1$. We omit the proofs of the following two lemmas, as they are based a straightforward induction.

Lemma 1. *If n is even, then $V_n\equiv \pm 2 \ (mod\ \ P^2)$ and if n is odd, then $V_n\equiv \pm nP$ $(mod\ \ P^2)$.*

Lemma 2. *If n is even, then $U_n\equiv \pm\frac{n}{2}P \ (mod\ \ P^2)$ and if n is odd, then $U_n\equiv \pm 1$ $(mod\ \ P^2)$.*

Lemma 3.

$$3|U_n \Leftrightarrow \begin{cases} n\equiv 0 \ (mod\ 2) \ \text{if}\ 3|P, \\ n\equiv 0 \ (mod\ 3) \ \text{if}\ 3\nmid P. \end{cases}$$

One can see the proofs of the following two theorems in [5].

Theorem 1. *Let $P\geq 3$ be odd. If $V_n=kx^2$ for some $k|P$ with $k>1$, then $n=1$.*

Theorem 2. *Let $P\geq 3$ be odd. If $U_n=kx^2$ for some $k|P$ with $k>1$, then $n=2$ or $n=6$ and $3|P$.*

The proofs of the following two theorems can be found in [11].

Theorem 3. *Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$ and m be a nonzero integer. Then*

$$U_{2mn+r} \equiv U_r \ (mod\ U_m), \tag{2.1}$$

$$V_{2mn+r} \equiv V_r \ (mod\ U_m). \tag{2.2}$$

Theorem 4. *Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$. Then*

$$U_{2mn+r} \equiv (-1)^n U_r \ (mod\ V_m), \tag{2.3}$$

$$V_{2mn+r} \equiv (-1)^n V_r \ (mod\ V_m). \tag{2.4}$$

Now we state the following theorem from [9].

Theorem 5. *Let $P \geq 3$ be odd. If $V_n = x^2$ for some integer x, then $n = 1$. If $V_n = 2x^2$ for some integer x, then $n = 3$, $P = 3, 27$.*

We state the following theorem due to Ribenboim and McDaniel [9].

Theorem 6. *Let $P \geq 3$ be odd. If $U_n = x^2$, then $n = 1$ or $n = 6$ and $P = 3$.*

The following theorem can be obtained from Theorem 9 given in [4].

Theorem 7. *Let $P \geq 3$ be odd, $m, n > 1$ be integers. The equation $U_n = 2U_m x^2$ has no solutions except for the cases $n = 6$, $m = 3$, $P = 3, 27$.*

The following two theorems can be obtained from Theorems 14 and 15 given in [4].

Theorem 8. *The equation $V_n = V_m x^2$, where $P \geq 3$, and P is odd, and $n \geq m > 0$ has only the trivial solution $n = m$.*

Theorem 9. *The equation $V_n = 2V_m x^2$, where $P \geq 3$, and P is odd, and $m, n > 0$ has no solutions.*

Now we give some identities concerning generalized Fibonacci and Lucas numbers:

$$U_{-n} = -U_n \text{ and } V_{-n} = V_n, \tag{2.5}$$

$$U_{2n} = U_n V_n, \tag{2.6}$$

$$V_{2n} = V_n^2 - 2, \tag{2.7}$$

$$V_{3n} = V_n(V_n^2 - 3), \tag{2.8}$$

$$U_{3n} = U_n\left((P^2-4)U_n^2 + 3\right) = U_n(V_n^2 - 1), \tag{2.9}$$

$$V_n^2 - \left(P^2 - 4\right) U_n^2 = 4, \tag{2.10}$$

$$\text{if } P \text{ is odd, then } 2|V_n \Leftrightarrow 2|U_n \Leftrightarrow 3|n, \tag{2.11}$$

$$V_m | V_n \Leftrightarrow m | n \text{ and } n/m \text{ is odd}, \tag{2.12}$$

$$U_m | U_n \Leftrightarrow m | n. \tag{2.13}$$

Let $m = 2^a k$, $n = 2^b l$, k and l are odd, $a, b \geq 0$, and $d = (m, n)$. Then

$$(U_m, V_n) = \begin{cases} V_d \text{ if } a > b, \\ 1 \text{ or } 2 \text{ if } a \leq b. \end{cases} \tag{2.14}$$

$$U_{5n} = U_n \left((P^2 - 4)^2 U_n^4 + 5(P^2 - 4) U_n^2 + 5 \right). \tag{2.15}$$

If $5 | U_n$, then from (2.15), we have

$$U_{5n} = 5 U_n (5a + 1) \tag{2.16}$$

for some $a \geq 0$.

$$V_{5n} = V_n (V_n^4 - 5 V_n^2 + 5). \tag{2.17}$$

If $5 | P$ and n is odd, then $5 | V_n$ and therefore from (2.17), it follows that

$$V_{5n} = 5 V_n (5a + 1), \tag{2.18}$$

for some $a \geq 0$.
From Lemma 1 and the identity (2.7), we have

$$5 | V_n \Leftrightarrow 5 | P \text{ and } n \text{ is odd}. \tag{2.19}$$

When P is odd, it is clear that

$$\left(\frac{-1}{V_{2^r}} \right) = -1. \tag{2.20}$$

If P is odd and $r \geq 2$, then $V_{2^r} \equiv -1 \left(mod\ \frac{P^2 - 3}{2} \right)$ and thus

$$\left(\frac{(P^2 - 3)/2}{V_{2^r}} \right) = \left(\frac{P^2 - 3}{V_{2^r}} \right) = 1. \tag{2.21}$$

$$V_{2^r} \equiv \begin{cases} -2\ (mod\ P), \text{ if } r = 1, \\ 2\ (mod\ P), \text{ if } r \geq 2. \end{cases} \tag{2.22}$$

If $3 \nmid P$ and P is odd, then $V_{2^r} \equiv -1\ (mod\ 3)$ for $r \geq 1$ and therefore

$$\left(\frac{3}{V_{2^r}} \right) = 1. \tag{2.23}$$

If $3 | P$ and P is odd, then $V_{2^r} \equiv -1\ (mod\ 3)$ for $r \geq 2$ and therefore

$$\left(\frac{3}{V_{2^r}} \right) = 1. \tag{2.24}$$

Let P be odd. Then

$$\left(\frac{5}{V_{2^r}}\right) = \begin{cases} -1, \text{ if } 5|P, \\ 1, \text{ if } P^2 \equiv 1 \ (mod\ 5), \\ -1, \text{ if } P^2 \equiv -1 (mod\ 5), \end{cases} \tag{2.25}$$

for every $r \geq 1$.

Most of the properties above are well–known; properties between (2.5)–(2.10) can be found in [8], [9], [10], [2]; properties between (2.11)–(2.14) can be found in [7], [9], [10], [2]. Since the others are fairly easy to prove, we omit their proofs.

The following lemma can be proved by using (2.1).

Lemma 4.

$$5|U_n \Leftrightarrow \begin{cases} 2|n, \text{ if } 5|P, \\ 3|n, \text{ if } P^2 \equiv 1 \ (mod\ 5), \\ 5|n, \text{ if } P^2 \equiv -1 \ (mod\ 5). \end{cases}$$

3. Main theorems

From now on, we assume that n and m are positive integers, $P \geq 3$, and P is odd.

Theorem 10. *The equation $V_n = 5x^2$ has a solution only if $n = 1$.*

Proof. Assume that $V_n = 5x^2$ for some integer x. Since $5|V_n$, it follows from (2.19) that $5|P$. This implies by Theorem 1 that $n = 1$. This completes the proof. □

Theorem 11. *There is no integer x such that $V_n = 5V_m x^2$.*

Proof. Assume that $V_n = 5V_m x^2$. Then by (2.19), it is seen that $5|P$ and n is odd. Moreover, since $V_m|V_n$, there exists an odd integer t such that $n = mt$ by (2.12). Since n and t are odd and $n = mt$, m is also odd. Hence, we have from Lemma 1 that

$$V_n \equiv \pm nP \ (mod\ P^2) \text{ and } V_m \equiv \pm mP \ (mod\ P^2).$$

This implies that

$$\pm nP \equiv \pm 5mPx^2 \ (mod\ P^2),$$

i.e.,

$$n \equiv 5mx^2 \ (mod\ P).$$

Using the fact that $5|P$, it follows that $5|n$. Firstly, assume that $5|t$. Then $t = 5s$ for some positive odd integer s and therefore $n = mt = 5ms$. By (2.17), we immediately have

$$V_n = V_{5ms} = V_{ms}(V_{ms}^4 - 5V_{ms}^2 + 5).$$

Since ms is odd and $5|P$, it follows that $5|V_{ms}$ by (2.19) and therefore

$$\frac{V_{ms}}{V_m}\left(\frac{V_{ms}^4-5V_{ms}^2+5}{5}\right)=x^2.$$

Clearly,

$$\left(V_{ms}/V_m,(V_{ms}^4-5V_{ms}^2+5)/5\right)=1.$$

This implies that

$$V_{ms}^4-5V_{ms}^2+5=5b^2$$

for some $b\geq 0$. But the integral points on $5Y^2=X^4-5X^2+5$ are immediately determined by using MAGMA [1] to be $(X,\pm Y)=(0,1)$, which gives $V_{ms}=0$, which is impossible. Secondly, assume that $5\nmid t$. Since $n=mt$ and $5|n$, it is seen that $5|m$. Then we can write $m=5^r a$ with $5\nmid a$ and $r\geq 1$. By (2.18), we obtain

$$V_m=V_{5^r a}=5V_{5^{r-1}a}(5a_1+1)$$

for some positive integer a_1. Thus, we conclude that

$$V_m=V_{5^r a}=5^r V_a(5a_1+1)(5a_2+1)...(5a_r+1)$$

for some positive integers a_i with $1\leq i\leq r$. Let $A=(5a_1+1)(5a_2+1)...(5a_r+1)$. Thus, we have $V_m=5^r V_a A$, where $5\nmid A$. In a similar manner, we see that

$$V_n=V_{5^r at}=5^r V_{at}(5b_1+1)(5b_2+2)...(5b_r+1)$$

for some positive integers b_j with $1\leq j\leq r$. Thus, we have $V_n=5^r V_{at}B$, where $5\nmid B$. As a consequence, we get

$$5^r V_{at}B=5\cdot 5^r V_a Ax^2,$$

implying that

$$V_{at}B=5V_a Ax^2.$$

By Lemma 1, it is seen that

$$\pm atPB\equiv\pm 5aPAx^2\ (mod\ \ P^2),$$

i.e.,

$$atB\equiv 5aAX^2\ (mod\ \ P).$$

Since $5|P$, it follows that $5|atB$. However, this is impossible since $5\nmid a$, $5\nmid t$, and $5\nmid B$. This completes the proof. □

Theorem 12. *If $P\geq 3$ is odd, then the equation $U_n=5x^2$ has the solution $n=2$ when $5|P$ and $n=3$ when $P^2\equiv 1\ (mod\ 5)$. The equation $U_n=5x^2$ has no solutions when $P^2\equiv -1\ (mod\ 5)$.*

Proof. Assume that $U_n = 5x^2$ for some integer x. Now we distinguish three cases.
Case I : Let $5|P$. Then by Theorem 2, we see that $n = 2$ or $n = 6$ and $3|P$. But, it can be easily shown that for the case when $n = 6$ and $3|P$, the equation $U_n = 5x^2$ has no solutions.
Case II : Let $P^2 \equiv 1 \ (mod \ 5)$. Since $5|U_n$, it follows from Lemma 4 that $3|n$. Hence, $n = 3m$ for some positive integer m. Assume that m is even. Then $m = 2s$ for some positive integer s and therefore $n = 6s$. And so by (2.6), we get $U_n = U_{6s} = U_{3s}V_{3s} = 5x^2$. Clearly, $(U_{3s}, V_{3s}) = 2$ by (2.14) and (2.11). Then either

$$U_{3s} = 2a^2, \ V_{3s} = 10b^2 \tag{3.1}$$

or

$$U_{3s} = 10a^2, \ V_{3s} = 2b^2 \tag{3.2}$$

for some positive integers a and b. Assume that (3.1) is satisfied. Since $5|V_{3s}$, it follows from (2.19) that $5|P$. But this contradicts the fact that $P^2 \equiv 1(mod 5)$. Now assume that (3.2) is satisfied. Then by Theorem 5, we have $3s = 3$ and $P = 3, 27$. Therefore $s = 1$. If $P = 3$, then $U_3 = P^2 - 1 = 8 = 10a^2$, which is impossible. If $P = 27$, then $U_3 = P^2 - 1 = 27^2 - 1 = 10a^2$, which is also impossible. Now assume that m is odd. Then by (2.9), we get $U_{3m} = U_m\left((P^2-4)U_m + 3\right)$. Clearly, $\left(U_m, (P^2-4)U_m^2 + 3\right) = 1$ or 3. Then it follows that $(P^2-4)U_m^2 + 3 = wa^2$ for some $w \in \{1, 3, 5, 15\}$. Since $(P^2-4)U_m^2 + 3 = V_{2m} + 1$ by (2.7) and (2.10), it is seen that $V_{2m} + 1 = wa^2$. Assume that $m > 1$. Then $m = 4q \pm 1 = 2^r a \pm 1$ with a odd and $r \geq 2$. Thus,

$$wa^2 = V_{2m} + 1 \equiv 1 - V_2 \equiv -(P^2 - 3) \ (mod \ V_{2^r})$$

by (2.4). This shows that

$$\left(\frac{w}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right).$$

By using (2.23), (2.24), and (2.25), it can be seen that $\left(\frac{w}{V_{2^r}}\right) = 1$ for $w = 3, 5, 15$. Moreover, $\left(\frac{-1}{V_{2^r}}\right) = -1$ and $\left(\frac{P^2-3}{V_{2^r}}\right) = 1$ by (2.20) and (2.21), respectively. Thus, we get

$$1 = \left(\frac{w}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right) = -1,$$

which is impossible. Therefore $m = 1$ and thus $n = 3$.
Case III : Let $P^2 \equiv -1 \ (mod \ 5)$. Since $5|U_n$, it follows that $5|n$ by Lemma 4. Thus $n = 5t$ for some positive integer t. Since $P^2 \equiv -1 \ (mod \ 5)$, it is obvious that $5|P^2 - 4$ and therefore there exists a positive integer A such that $P^2 - 4 = 5A$. By (2.15), we get $U_n = U_{5t} = U_t\left((P^2-4)^2U_t^4 + 5(P^2-4)U_t^2 + 5\right)$. Substituting $P^2 - 4 = 5A$

into the preceding equation gives $U_n = U_{5t} = 5U_t\left(5A^2U_t^4 + 5AU_t^2 + 1\right)$. Let $B = A^2U_t^4 + AU_t^2$. As a consequence, we have

$$U_n = U_{5t} = 5U_t(5B+1) = 5x^2,$$

implying that

$$U_t(5B+1) = x^2.$$

It can be easily seen that $(U_t, 5B+1) = 1$. This shows that $U_t = a^2$ and $5B+1 = b^2$ for some $a, b > 0$. By Theorem 6, we see that the only possible values of t and P in which $U_t = a^2$ are $t = 1$ or $t = 6$ and $P = 3$. If $t = 1$, then $n = 5$ and therefore we get $U_n = U_{5t} = U_5 = P^4 - 3P^2 + 1 = 5x^2$. With MAGMA [1], we get $P = 2$, which is impossible since P is odd. If $t = 6$, then $n = 30$. A simple computation shows that there is no integer x such that $U_{30} = 5x^2$ for $P = 3$. □

Theorem 13. *Let $P \geq 3$ and $m > 1$. The equation $U_n = 5U_mx^2$ has no solutions in any of the following cases:*

(i) : $P^2 \equiv -1 \ (mod\ 5)$;
(ii) : P is odd and $5|P$;
(iii) : $P^2 \equiv 1 \ (mod\ 5)$, n is odd, and P is odd;
(iv) : $P^2 \equiv 1 \ (mod\ 5)$, n is even, and P is odd.

Proof. Assume that $U_n = 5U_mx^2$ for some $x > 0$. Since $U_m|U_n$, it follows that $m|n$ by (2.13). Thus, $n = mt$ for some $t > 0$. Since $n \neq m$, we have $t > 1$.
Case I : Let $P^2 \equiv -1 \ (mod\ 5)$. It is obvious that $5|P^2-4$. On the other hand, since $5|U_n$, it follows that $5|n$ by Lemma 4. Dividing the proof into two subcases, we have **Subcase** (i) : Assume that $5|t$. Then $t = 5s$ for some $s > 0$ and therefore $n = mt = 5ms$. By (2.15), we obtain

$$U_n = U_{5ms} = U_{ms}\left((P^2-4)^2U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5\right) = 5U_mx^2. \tag{3.3}$$

Since $5|P^2-4$, it is seen that $5|(P^2-4)^2U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5$. Also, we have $(P^2-4)^2U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5 = V_{ms}^4 - 3V_{ms}^2 + 1$ by (2.10). Rearranging the equation (3.3), we readily obtain

$$x^2 = (U_{ms}/U_m)\left(\left(V_{ms}^4 - 3V_{ms}^2 + 1\right)/5\right),$$

where $\left(U_{ms}/U_m, \left(V_{ms}^4 - 3V_{ms}^2 + 1\right)/5\right) = 1$. Hence, $V_{ms}^4 - 3V_{ms}^2 + 1 = 5b^2$ for some

$b > 0$. But the integral points on $5Y^2 = X^4 - 3X^2 + 1$ are immediately determined by using MAGMA [1] to be $(\pm X, \pm Y) = (2, 1)$, which gives $V_{ms} = 2$, implying that $ms = 0$, which is impossible.
Subcase (ii) : Assume that $5 \nmid t$. Since $5|n$, it follows that $5|m$. Then we can write $m = 5^ra$ with $5 \nmid a$ and $r \geq 1$. By (2.16), it is sen that $U_m = U_{5^ra} = 5U_{5^{r-1}a}(5a_1 + 1)$ for some positive integer a_1. Thus, we conclude that $U_m = U_{5^ra} = 5^rU_a(5a_1 + 1)(5a_2+1)...(5a_r+1)$ for some positive integers a_i with $1 \leq i \leq r$. Let $A = (5a_1 +$

$1)(5a_2+1)...(5a_r+1)$. Then, we have $U_m = 5^r U_a A$, where $5 \nmid A$. In a similar manner, we get $U_n = U_{5^r at} = 5^r U_{at}(5b_1+1)(5b_2+1)...(5b_r+1)$ for some positive integers b_i with $1 \le i \le r$. Let $B = (5b_1+1)(5b_2+1)...(5b_r+1)$. Hence, we have $U_n = 5^r U_{at} B$, where $5 \nmid B$. As a consequence, we get

$$5^r U_{at} B = 5 \cdot 5^r U_a A x^2$$

i.e.,

$$U_{at} B = 5 U_a A x^2.$$

Since $5 \nmid B$, it follows that $5|U_{at}$, implying that $5|at$ by Lemma 4. This contradicts the fact that $5 \nmid a$ and $5 \nmid t$. This concludes the proof of the case when $P^2 \equiv -1$ $(mod\ 5)$.

Case II : Let P be odd and $5|P$. Since $5|U_n$, it is seen from Lemma 4 that n is even. On the other hand, we have $n = mt$. So, we first assume that t is even. Then $t = 2s$ for some $s > 0$. By (2.6), we get $U_n = U_{2ms} = U_{ms}V_{ms} = 5U_m x^2$, implying that $(U_{ms}/U_m)V_{ms} = 5x^2$. Clearly, $d = (U_{ms}/U_m, V_{ms}) = 1$ or 2 by (2.14). If $d = 1$, then

$$U_{ms} = U_m a^2,\ V_{ms} = 5b^2 \tag{3.4}$$

or

$$U_{ms} = 5U_m a^2,\ V_{ms} = b^2 \tag{3.5}$$

for some $a, b > 0$. If (3.4) holds, then the only possible value of ms in which $V_{ms} = 5b^2$ is 1 by Theorem 1, which contradicts the fact that $m > 1$. If (3.5) holds, then by Theorem 5, we have $ms = 1$, which is impossible since $m > 1$.

If $d = 2$, then

$$U_{ms} = 2U_m a^2,\ V_{ms} = 10b^2 \tag{3.6}$$

or

$$U_{ms} = 10U_m a^2,\ V_{ms} = 2b^2 \tag{3.7}$$

for some $a, b > 0$. Suppose (3.6) holds. Then by Theorem 7, we get $ms = 6$, $m = 3$, $P = 3, 27$. There is no integer b such that $V_6 = 10b^2$ for the case when $P = 3$ or 27. Suppose (3.7) holds. Then by Theorem 7, the only possible values of ms and P in which $V_{ms} = 2b^2$ are $ms = 3$ and $P = 3, 27$. Since $m > 1$, it follows that $m = 3$ and therefore we obtain $U_3 = 10U_3 a^2$, which is impossible.

Now assume that t is odd. Since $n = mt$ and n is even, it follows that m is even. Hence, we have $U_n \equiv \pm(n/2)P (mod\ \ P^2)$ and $U_m \equiv \pm(m/2)P\ \ (mod\ \ P^2)$ by Lemma 2. This shows that $\pm\frac{n}{2}P \equiv \pm 5\frac{m}{2}Px^2\ (mod\ \ P^2)$, i.e., $\frac{n}{2} \equiv 5\frac{m}{2}x^2\ (mod$ $P)$. Since $5|P$, it is seen that $5|n$. Dividing remainder of the proof into two subcases, we have

Subcase (i) : Let $5|t$. Then $t = 5s$ for some $s > 0$ and therefore $n = mt = 5ms$. By (2.15), we obtain

$$U_n = U_{5ms} = U_{ms}\left((P^2-4)^2U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5\right). \tag{3.8}$$

Since ms is even and $5|P$, it is seen that $5|U_{ms}$ by Lemma 4. Also, we have $(P^2-4)^2U_{ms}^4+5(P^2-4)U_{ms}^2+5=V_{ms}^4-3V_{ms}^2+1$ by (2.10). Hence, rearranging the equation (3.8) gives

$$x^2=(U_{ms}/U_m)\left(\left(V_{ms}^4-3V_{ms}^2+1\right)/5\right),$$

where $\left((U_{ms}/U_m),\left(V_{ms}^4-3V_{ms}^2+1\right)/5\right)=1$. This implies that $V_{ms}^4-3V_{ms}^2+1=5b^2$ for some $b>0$. But the integral points on $5Y^2=X^4-3X^2+1$ are immediately determined by using MAGMA [1] to be $(\pm X,\pm Y)=(2,1)$, which gives $V_{ms}=2$, implying that $ms=0$, which is impossible.

Subcase (ii) : Let $5\nmid t$. Since $5|n$, it follows that $5|m$. Then we can write $m=5^ra$ with $5\nmid a$ and $r\geq 1$. By (2.16), it is sen that $U_m=U_{5^ra}=5U_{5^{r-1}a}(5a_1+1)$ for some positive integer a_1. Thus, we conclude that $U_m=U_{5^ra}=5^rU_a(5a_1+1)(5a_2+1)...(5a_r+1)$ for some positive integers a_i with $1\leq i\leq r$. Let $A=(5a_1+1)(5a_2+1)...(5a_r+1)$. Then, we have $U_m=5^rU_aA$, where $5\nmid A$. In a way similar, we get $U_n=U_{5^rat}=5^rU_{at}(5b_1+1)(5b_2+1)...(5b_r+1)$ for some positive integers b_i with $1\leq i\leq r$. Let $B=(5b_1+1)(5b_2+1)...(5b_r+1)$. Hence, we have $U_n=5^rU_{at}B$, where $5\nmid B$. Substituting the new values of U_n and U_m into $U_n=5U_mx^2$ gives

$$5^rU_{at}B=5\cdot 5^rU_aAx^2$$

i.e.,

$$U_{at}B=5U_aAx^2.$$

On the other hand, since a is even and at is even, it follows from Lemma 2 that $U_{at}\equiv\pm\frac{at}{2}P\ (mod\ P^2)$ and $U_a\equiv\pm\frac{a}{2}P\ (mod\ P^2)$. Hence, we have

$$\pm\frac{at}{2}PB\equiv\pm 5\frac{a}{2}PAx^2\ (mod\ P^2),$$

implying that

$$\frac{at}{2}B\equiv 5\frac{a}{2}Ax^2\ (mod\ P^2).$$

Since $5|P$, it follows that $5|\frac{at}{2}B$, which shows that $5|atB$. This contradicts the fact that $5\nmid a$, $5\nmid b$, and $5\nmid B$. This concludes the proof for the case when $5|P$.

Case III : Let $P^2\equiv 1\ (mod\ 5)$, n is odd, and P is odd. Then, both m and t are odd. Since $5|U_n$, it follows immediately from Lemma 4 that $3|n$. Using the fact that $n=mt$, we have

Subcase (i) : Assume that $3|m$. Since t is odd, we can write $t=4q\pm 1$ for some $q>0$. If $t=4q+1$, then $t=2\cdot 2^ra+1$ with a odd and $r>0$. And so by (2.3), we get $U_n=U_{mt}=U_{2\cdot 2^ram+m}\equiv -U_m\ (mod\ V_{2^r})$, implying that $5U_mx^2\equiv -U_m\ (mod\ V_{2^r})$. Since $(U_m,V_{2^r})=1$ by (2.14), it follows that $5x^2\equiv -1\ (mod\ V_{2^r})$, which is impossible since $\left(\dfrac{5}{V_{2^r}}\right)=1$ by (2.25) and $\left(\dfrac{-1}{V_{2^r}}\right)=-1$ by (2.20). If $t=4q-1$, then by (2.1), we get $U_n=U_{m(4q-1)}=U_{2\cdot 2mq-m}\equiv -U_m\ (mod\ U_{2m})$. This shows that $5U_mx^2\equiv -U_m\ (mod\ U_{2m})$, implying that $5x^2\equiv -1\ (mod\ V_m)$ by

(2.6). Since $3|m$, it is seen by (2.12) that $V_3|V_m$. Hence, we obtain $5x^2 \equiv -1 \ (mod\ V_3)$, i.e., $5x^2 \equiv -1 \ (mod\ P^2-3)$. But this is impossible since

$$\left(\frac{5}{(P^2-3)/2}\right) = \left(\frac{(P^2-3)/2}{5}\right) = \left(\frac{-1}{5}\right) = 1$$

and

$$\left(\frac{-1}{(P^2-3)/2}\right) = (-1)^{\frac{P^2-5}{4}} = -1.$$

Subcase (ii) : Assume that $3 \nmid m$. Since $n = mt$ and $3|n$, it follows that $3|t$ and therefore $t = 3s$ for some $s > 0$. Then by (2.9), we get

$$U_n = U_{3ms} = U_{ms}\left((P^2-4)U_{ms}^2 + 3\right) = 5U_m x^2,$$

implying that

$$(U_{ms}/U_m)\left((P^2-4)U_{ms}^2 + 3\right) = 5x^2.$$

Clearly,

$$d = (U_{ms}/U_m, \left((P^2-4)U_{ms}^2 + 3\right) = 1 \text{ or } 3.$$

If $d = 1$, then either

$$U_{ms} = U_m a^2,\ (P^2-4)U_{ms}^2 + 3 = 5b^2 \tag{3.9}$$

or

$$U_{ms} = 5U_m a^2,\ (P^2-4)U_{ms}^2 + 3 = b^2 \tag{3.10}$$

for some $a, b > 0$. Suppose (3.9) holds. Then by (2.10), we get $V_{ms}^2 - 1 = 5b^2$ and this gives by (2.7) that $V_{2ms} = 5b^2 - 1$. Since $ms > 1$ is odd, $ms = 4q \pm 1$ for some $q > 0$. Thus $ms = 2 \cdot 2^r a \pm 1$ with a odd and $r > 0$. By using (2.4), we get $5b^2 - 1 = V_{2ms} \equiv -V_{\pm 2} \equiv -V_2 \ (mod\ V_{2^r})$. This shows that $5b^2 - 1 \equiv -(P^2-2) \ (mod\ V_{2^r})$, implying that $5b^2 \equiv -(P^2-3) \ (mod\ V_{2^r})$. By using (2.20), (2.25), and (2.21), it is seen that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{5}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right) = -1,$$

a contradiction. Suppose (3.10) holds. It can be easily seen by combining two equations that $b^2 \equiv 3 \ (mod\ 5)$, which is impossible.
If $d = 3$, then either

$$U_{ms} = 3U_m a^2,\ (P^2-4)U_{ms}^2 + 3 = 15b^2 \tag{3.11}$$

or

$$U_{ms} = 15U_m a^2,\ (P^2-4)U_{ms}^2 + 3 = 3b^2 \tag{3.12}$$

for some $a, b > 0$. If we combine two equations given in (3.11), then we readily obtain $b^2 \equiv 2 \ (mod\ 3)$, which is impossible. Suppose (3.12) holds. Then by (2.10), we get $V_{ms}^2 - 1 = 3b^2$ and this gives by (2.7) that $V_{2ms} = 3b^2 - 1$. Since $ms > 1$ is odd, $ms = 4q \pm 1$ for some $q > 0$. Thus $ms = 2 \cdot 2^r a \pm 1$ with a odd and $r > 0$. By using (2.4), we get $3b^2 - 1 = V_{2ms} \equiv -V_{\pm 2} \equiv -V_2 \ (mod\ V_{2^r})$. This shows that

$3b^2-1 \equiv -(P^2-2)\ (mod\ V_{2^r})$, implying that $3b^2 \equiv -(P^2-3)\ (mod\ V_{2^r})$. By (2.23), (2.24), (2.20), and (2.21), it is seen that

$$1=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right)=-1,$$

a contradiction.

Case IV : Let $P^2 \equiv 1\ (mod\ 5)$, n is even, and P is odd. Since $n=mt$, we divide the proof into two subcases:

Subcase (i) : Assume that t is even. Then $t=2s$ for some $s>0$. Hence, we immediately have $U_n/U_m=U_{2ms}/U_m=(U_{ms}/U_m)V_{ms}=5x^2$. Clearly, $d=(U_{ms}/U_m,V_{ms})=1$ or 2 by (2.14). If $d=1$, then

$$U_{ms}=U_m a^2,\ V_{ms}=5b^2 \tag{3.13}$$

or

$$U_{ms}=5U_m a^2,\ V_{ms}=b^2 \tag{3.14}$$

for some $a,b>0$. Suppose (3.13) is satisfied. Since $5|V_{ms}$, it follows from (2.19) that $5|P$, which contradicts the fact that $P^2 \equiv 1\ (mod\ 5)$. Now suppose (3.14) is satisfied. By Theorem 5, the only possible value of ms in which $V_{ms}=b^2$ is 1, which is impossible since $m>1$.

If $d=2$, then

$$U_{ms}=2U_m a^2,\ V_{ms}=10b^2 \tag{3.15}$$

or

$$U_{ms}=10U_m a^2,\ V_{ms}=2b^2 \tag{3.16}$$

for some $a,b>0$. Obviously, (3.15) is not satisfied because of the same reason given above for (3.13). If (3.16) holds, then it is seen by Theorem 5 that the only possible values of ms and P in which $V_{ms}=2b^2$ are $ms=3$ and $P=3,27$. But this is impossible since $P^2 \equiv 1\ (mod\ 5)$.

Subcase (ii) : Assume that t is odd. Since $t>1$, we can write $t=4q+1$ for some $q>0$ or $t=4q+3$ for some $q \geq 0$. On the other hand, since n is even and $n=mt$, it follows that m is even. Therefore we can write $m=2^r a$ with a odd and $r>0$. Assume that $t=4q+1$. Then $n=mt=4qm+m=2\cdot 2^{r+k}b+m$ with b odd. Hence, we get

$$5U_m x^2=U_n=U_{2\cdot 2^{r+k}b+m} \equiv -U_m\ (mod\ V_{2^{r+k}})$$

by (2.3). Since $(U_m,V_{2^{r+k}})=(U_{2^r a},V_{2^{r+k}})=1$ by (2.14), it follows that

$$5x^2 \equiv -1\ (mod\ V_{2^{r+k}}).$$

This is impossible. Because $\left(\frac{5}{V_{2^{r+k}}}\right)=1$ and $\left(\frac{-1}{V_{2^{r+k}}}\right)=-1$ by (2.25) and (2.20), respectively. Now assume that $t=4q+3$. Then we have $n=mt=4qm+3m$. And so by (2.1), we get

$$5U_m x^2=U_n=U_{4qm+3m} \equiv U_{3m}\ (mod\ U_{2m}).$$

By using (2.6) and (2.9), we readily obtain

$$5x^2 \equiv V_m^2 - 1 \ (mod\ V_m),$$

which implies that

$$5x^2 \equiv -1 (mod\ V_m).$$

Using the fact that $m = 2^r a$ with a odd, we have

$$5x^2 \equiv -1 \ (mod\ V_{2^r a}),$$

implying that

$$5x^2 \equiv -1 \ (mod\ V_{2^r})$$

by (2.12). But this is impossible since $\left(\frac{5}{V_{2^r}}\right) = 1$ and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.25) and (2.20), respectively. This completes the proof. □

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