ON THE EQUATIONS $U_n = 5\Box$ AND $V_n = 5\Box$

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Abstract. Let $P \geq 3$ be an integer and let $(U_n)$ and $(V_n)$ denote the generalized Fibonacci and Lucas sequences defined by $U_0 = 0, U_1 = 1, V_0 = 2, V_1 = P$, and $U_{n+1} = PU_n - U_{n-1}$, $V_{n+1} = PV_n - V_{n-1}$ for $n \geq 1$. The purpose of this study, assuming $P$ is odd, is to determine the values of $n$ such that $V_n = 5\Box$ and $U_n = 5\Box$. Moreover, we solve the equations $V_n = 5V_m\Box$ and $U_n = 5U_m\Box$.

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1. INTRODUCTION

Let $P$ and $Q$ be nonzero integers such that $P^2 + 4Q \neq 0$. The generalized Fibonacci sequence $(U_n)$ and Lucas sequence $(V_n)$ are given recursively according to the following relations for $n \geq 1$.

$$U_0 = 0, \ U_1 = 1, \ U_{n+1} = PU_n + QU_{n-1}$$

and

$$V_0 = 2, \ V_1 = P, \ V_{n+1} = PV_n + QV_{n-1}.$$ 

Both sequences depend on the initial choice of pair $(P, Q)$, hence we sometimes use $U_n(P, Q)$ and $V_n(P, Q)$ in order to emphasize their dependence on the parameters $(P, Q)$. $U_n$ and $V_n$ are called the $n$th generalized Fibonacci number and the $n$th generalized Lucas number, respectively. Furthermore, generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$$U_{-n} = (-Q)^{-n} U_n \text{ and } V_{-n} = (-Q)^{-n} V_n \ (n \geq 1),$$

respectively. It is well known that

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta) \text{ and } V_n = \alpha^n + \beta^n$$

where $\alpha = (P + \sqrt{P^2 + 4Q})/2$ and $\beta = (P - \sqrt{P^2 + 4Q})/2$, which are the roots of the equation $x^2 - Px - Q = 0$. The above formulas are known as Binet’s formulas.
We will assume that \( P^2 + 4Q > 0 \). Special cases of the sequences \((U_n)\) and \((V_n)\) are known. For example, the generalized Fibonacci sequence \((U_n(1, 1))\) consist of the familiar Fibonacci numbers, whereas its companion, \((V_n(1, 1))\) gives so called Lucas numbers. When \( P = 2 \) and \( Q = 1 \), \((U_n) = (P_n)\) and \((V_n) = (Q_n)\) are the familiar sequences of Pell and Pell-Lucas numbers. For more information about generalized Fibonacci and Lucas sequences, see [8].

There has been much interest in when the terms of generalized Fibonacci and Lucas sequences are perfect square (= \( \square \)) or \( k \). When \( P \) is odd and \( Q = 1 \), by using elementary arguments, many authors solved the equations \( U_n = k \square \) and \( V_n = k \square \) for some specific values of \( k \) (see [2–4, 9, 10]). Interested readers can also consult [12] and [6] for a brief history of this subject.

In [6], the authors determined all indices \( n \) such that \( U_n(P, 1) = 5 \square \) and \( U_n(P, 1) = 5U_m(P, 1) \square \) under some assumptions on \( P \). When \( P \) is odd, the authors solved the equation \( V_n(P, 1) = 5 \square \). Moreover, they showed that the equation \( V_n(P, 1) = 5V_m(P, 1) \square \) has no solutions. In this study, using congruences, with extensive reliance upon the Jacobi symbol, we determine that the five times square terms of the generalized Fibonacci sequence \((U_n(P, Q))\) for which \( P \equiv 3 \) is odd and \( Q = -1 \) may occur only for \( n = 2 \) or 3. We obtain a similar result for the generalized Lucas sequence \((V_n(P, Q))\). Moreover, when \( P \geq 3 \) is odd and \( Q = -1 \), we solve the equations \( V_n = 5V_m \square \) and \( U_n = 5U_m \square \).

In section 2, we give some identities, lemmas, and theorems needed later. Then in section 3, we present our main theorems. Throughout this study, \( \left( \frac{z}{P} \right) \) will denote the Jacobi symbol. Our method of proof is similar to that presented by Cohn, McDaniel and Ribenboim [2–4, 9].

2. Preliminary Facts

From now on, we assume that \( Q = -1 \). We omit the proofs of the following two lemmas, as they are based a straightforward induction.

**Lemma 1.** If \( n \) is even, then \( V_n = \pm 2 \mod P^2 \) and if \( n \) is odd, then \( V_n = \pm nP \mod P^2 \).

**Lemma 2.** If \( n \) is even, then \( U_n = \pm \frac{n}{2} P \mod P^2 \) and if \( n \) is odd, then \( U_n = \pm 1 \mod P^2 \).

**Lemma 3.**

\[
3|U_n \iff \begin{cases} n \equiv 0 \mod 2 \text{ if } 3|P, \\ n \equiv 0 \mod 3 \text{ if } 3 \nmid P. \end{cases}
\]

One can see the proofs of the following two theorems in [5].

**Theorem 1.** Let \( P \geq 3 \) be odd. If \( V_n = kx^2 \) for some \( k \) \( \mid P \) with \( k > 1 \), then \( n = 1 \).

**Theorem 2.** Let \( P \geq 3 \) be odd. If \( U_n = kx^2 \) for some \( k \) \( \mid P \) with \( k > 1 \), then \( n = 2 \) or \( n = 6 \) and \( 3 \mid P \).
The proofs of the following two theorems can be found in [11].

**Theorem 3.** Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$ and $m$ be a nonzero integer. Then
\begin{align*}
U_{2mn+r} &\equiv U_r \pmod{U_m}, \quad (2.1) \\
V_{2mn+r} &\equiv V_r \pmod{V_m}. \quad (2.2)
\end{align*}

**Theorem 4.** Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$. Then
\begin{align*}
U_{2mn+r} &\equiv (-1)^n U_r \pmod{V_m}, \quad (2.3) \\
V_{2mn+r} &\equiv (-1)^n V_r \pmod{V_m}. \quad (2.4)
\end{align*}

Now we state the following theorem from [9].

**Theorem 5.** Let $P \geq 3$ be odd. If $V_n = x^2$ for some integer $x$, then $n = 1$. If $V_n = 2x^2$ for some integer $x$, then $n = 3$, $P = 3, 27$.

We state the following theorem due to Ribenboim and McDaniel [9].

**Theorem 6.** Let $P \geq 3$ be odd. If $U_n = x^2$, then $n = 1$ or $n = 6$ and $P = 3$.

The following theorem can be obtained from Theorem 9 given in [4].

**Theorem 7.** Let $P \geq 3$ be odd, $m, n > 1$ be integers. The equation $U_n = 2U_m x^2$ has no solutions except for the cases $n = 6$, $m = 3$, $P = 3, 27$.

The following two theorems can be obtained from Theorems 14 and 15 given in [4].

**Theorem 8.** The equation $V_n = V_m x^2$, where $P \geq 3$, and $P$ is odd, and $n \geq m > 0$ has only the trivial solution $n = m$.

**Theorem 9.** The equation $V_n = 2V_m x^2$, where $P \geq 3$, and $P$ is odd, and $m, n > 0$ has no solutions.

Now we give some identities concerning generalized Fibonacci and Lucas numbers:
\begin{align*}
U_{-n} &= -U_n \text{ and } V_{-n} = V_n, \quad (2.5) \\
U_{2n} &= U_n V_n, \quad (2.6) \\
V_{2n} &= V_n^2 - 2, \quad (2.7) \\
V_{3n} &= V_n(V_n^2 - 3). \quad (2.8) \\
U_{3n} &= U_n((P^2 - 4)U_n^2 + 3) = U_n(V_n^2 - 1). \quad (2.9)
\end{align*}
\begin{align*}
V_n^2 - (P^2 - 4) U_n^2 &= 4, \quad (2.10)
\text{if } P \text{ is odd, then } 2|V_n \iff 2|U_n \iff 3|n, \quad (2.11)
\end{align*}
\[ V_m | V_n \Leftrightarrow m | n \text{ and } n / m \text{ is odd}, \] (2.12)

\[ U_m | U_n \Leftrightarrow m | n. \] (2.13)

Let \( m = 2^a k, n = 2^b l, k \text{ and } l \text{ are odd, } a, b \geq 0, \text{ and } d = (m,n). \) Then

\[ (U_m, V_n) = \begin{cases} V_d \text{ if } a > b, \\ 1 \text{ or } 2 \text{ if } a \leq b. \end{cases} \] (2.14)

\[ U_{5n} = U_n \left( (P^2 - 4)U_n^4 + 5(P^2 - 4)U_n^2 + 5 \right). \] (2.15)

If \( 5 | U_n, \) then from (2.15), we have

\[ U_{5n} = 5U_n (5a + 1) \] (2.16)

for some \( a \geq 0. \)

\[ V_{5n} = V_n (V_n^4 - 5V_n^2 + 5). \] (2.17)

If \( 5 | P \text{ and } n \text{ is odd}, \) then \( 5 | V_n \) and therefore from (2.17), it follows that

\[ V_{5n} = 5V_n (5a + 1), \] (2.18)

for some \( a \geq 0. \)

From Lemma 1 and the identity (2.7), we have

\[ 5 | V_n \Leftrightarrow 5 | P \text{ and } n \text{ is odd}. \] (2.19)

When \( P \text{ is odd}, \) it is clear that

\[ \left( \frac{-1}{V_{2r}} \right) = -1. \] (2.20)

If \( P \text{ is odd and } r \geq 2, \) then \( V_{2r} \equiv -1 \left( mod \frac{P^2 - 3}{2} \right) \) and thus

\[ \left( \frac{P^2 - 3}{V_{2r}} \right) = \left( \frac{P^2 - 3}{V_{2r}} \right) = 1. \] (2.21)

\[ V_{2r} \equiv \begin{cases} -2 \left( mod \ P \right), & \text{if } r = 1, \\ 2 \left( mod \ P \right), & \text{if } r \geq 2. \end{cases} \] (2.22)

If \( 3 \nmid P \) and \( P \text{ is odd}, \) then \( V_{2r} \equiv -1 \left( mod \ 3 \right) \) for \( r \geq 1 \) and therefore

\[ \left( \frac{3}{V_{2r}} \right) = 1. \] (2.23)

If \( 3 | P \) and \( P \text{ is odd}, \) then \( V_{2r} \equiv -1 \left( mod \ 3 \right) \) for \( r \geq 2 \) and therefore

\[ \left( \frac{3}{V_{2r}} \right) = 1. \] (2.24)
Let \( P \) be odd. Then

\[
\left( \frac{5}{V_{2r}} \right) = \begin{cases} 
-1, & \text{if } 5|P, \\
1, & \text{if } P^2 \equiv 1 \pmod{5}, \\
-1, & \text{if } P^2 \equiv -1 \pmod{5},
\end{cases}
\]

for every \( r \geq 1 \).

Most of the properties above are well-known; properties between (2.5)–(2.10) can be found in [8], [9], [10], [2]; properties between (2.11)–(2.14) can be found in [7], [9], [10], [2]. Since the others are fairly easy to prove, we omit their proofs.

The following lemma can be proved by using (2.1).

**Lemma 4.**

\[
5|U_n \iff \begin{cases} 
2|n, & \text{if } 5|P, \\
3|n, & \text{if } P^2 \equiv 1 \pmod{5}, \\
5|n, & \text{if } P^2 \equiv -1 \pmod{5}.
\end{cases}
\]

3. **Main Theorems**

From now on, we assume that \( n \) and \( m \) are positive integers, \( P \geq 3 \), and \( P \) is odd.

**Theorem 10.** The equation \( V_n = 5x^2 \) has a solution only if \( n = 1 \).

**Proof.** Assume that \( V_n = 5x^2 \) for some integer \( x \). Since \( 5|V_n \), it follows from (2.19) that \( 5|P \). This implies by Theorem 1 that \( n = 1 \). This completes the proof. \( \Box \)

**Theorem 11.** There is no integer \( x \) such that \( V_n = 5V_m x^2 \).

**Proof.** Assume that \( V_n = 5V_m x^2 \). Then by (2.19), it is seen that \( 5|P \) and \( n \) is odd. Moreover, since \( V_m|V_n \), there exists an odd integer \( t \) such that \( n = mt \) by (2.12). Since \( n \) and \( t \) are odd and \( n = mt \), \( m \) is also odd. Hence, we have from Lemma 1 that

\[
V_n \equiv \pm nP \pmod{P^2} \quad \text{and} \quad V_m \equiv \pm mP \pmod{P^2}.
\]

This implies that

\[
\pm nP \equiv \pm 5mP x^2 \pmod{P^2},
\]

i.e.,

\[
n \equiv 5mx^2 \pmod{P}.
\]

Using the fact that \( 5|P \), it follows that \( 5|n \). Firstly, assume that \( 5|t \). Then \( t = 5s \) for some positive odd integer \( s \) and therefore \( n = mt = 5ms \). By (2.17), we immediately have

\[
V_n = V_{5ms} = V_{ms}(V_{ms}^4 - 5V_{ms}^2 + 5).
\]
Since $ms$ is odd and $5|P$, it follows that $5|V_{ms}$ by (2.19) and therefore

$$\frac{V_{ms}}{V_m} \left( \frac{V_{ms}^4 - 5V_{ms}^2 + 5}{5} \right) = x^2.$$  

Clearly,

$$\left( V_{ms}/V_m, (V_{ms}^4 - 5V_{ms}^2 + 5)/5 \right) = 1.$$  

This implies that

$$V_{ms}^4 - 5V_{ms}^2 + 5 = 5b^2$$  

for some $b \geq 0$. But the integral points on $5Y^2 = X^4 - 5X^2 + 5$ are immediately determined by using MAGMA [1] to be $(X, \pm Y) = (0, 1)$, which gives $V_{ms} = 0$, which is impossible. Secondly, assume that $5 \nmid t$. Since $n = mt$ and $5|n$, it is seen that $5|m$. Then we can write $m = 5^r a$ with $5 \nmid a$ and $r \geq 1$. By (2.18), we obtain

$$V_m = V_{sr} a = 5V_{sr-1} a (5a_1 + 1)$$  

for some positive integer $a_1$. Thus, we conclude that

$$V_m = V_{sr} a = 5^r V_a (5a_1 + 1)(5a_2 + 1) \ldots (5a_r + 1)$$  

for some positive integers $a_i$ with $1 \leq i \leq r$. Let $A = (5a_1 + 1)(5a_2 + 1) \ldots (5a_r + 1)$. Thus, we have $V_m = 5^r V_a A$, where $5 \nmid A$. In a similar manner, we see that

$$V_n = V_{sr} at = 5^r V_{at} (5b_1 + 1)(5b_2 + 2) \ldots (5b_r + 1)$$  

for some positive integers $b_j$ with $1 \leq j \leq r$. Thus, we have $V_n = 5^r V_{at} B$, where $5 \nmid B$. As a consequence, we get

$$5^r V_{at} B = 5 \cdot 5^r V_a A x^2,$$

implying that

$$V_{at} B = 5V_a A x^2.$$

By Lemma 1, it is seen that

$$\pm at PB \equiv \pm 5a PAx^2 \pmod{P^2},$$

i.e.,

$$at B \equiv 5a AX^2 \pmod{P}.$$  

Since $5|P$, it follows that $5|at B$. However, this is impossible since $5 \nmid a$, $5 \nmid t$, and $5 \nmid B$. This completes the proof. \qed

**Theorem 12.** If $P \geq 3$ is odd, then the equation $U_n = 5x^2$ has the solution $n = 2$ when $5|P$ and $n = 3$ when $P^2 \equiv 1 \pmod{5}$. The equation $U_n = 5x^2$ has no solutions when $P^2 \equiv -1 \pmod{5}$. 

Proof. Assume that \( U_n = 5x^2 \) for some integer \( x \). Now we distinguish three cases.

**Case I**: Let \( 5|P \). Then by Theorem 2, we see that \( n = 2 \) or \( n = 6 \) and \( 3|P \). But, it can be easily shown that for the case when \( n = 6 \) and \( 3|P \), the equation \( U_n = 5x^2 \) has no solutions.

**Case II**: Let \( P^2 \equiv 1 \pmod{5} \). Since \( 5|U_n \), it follows from Lemma 4 that \( 3|n \). Hence, \( n = 3m \) for some positive integer \( m \). Assume that \( m \) is even. Then \( m = 2s \) for some positive integer \( s \) and therefore \( n = 6s \). And so by (2.6), we get \( U_n = U_{6s} = U_{3s}V_{3s} = 5x^2 \). Clearly, \( (U_{3s}, V_{3s}) = 2 \) by (2.14) and (2.11). Then either

\[
U_{3s} = 2a^2, \quad V_{3s} = 10b^2 \quad (3.1)
\]

or

\[
U_{3s} = 10a^2, \quad V_{3s} = 2b^2 \quad (3.2)
\]

for some positive integers \( a \) and \( b \). Assume that (3.1) is satisfied. Since \( 5|V_{3s} \), it follows from (2.19) that \( 5|P \). But this contradicts the fact that \( P^2 \equiv 1 \pmod{5} \). Now assume that (3.2) is satisfied. Then by Theorem 5, we have \( 3s = 3 \) and \( P = 3, 27 \). Therefore \( s = 1 \). If \( P = 3 \), then \( U_3 = P^2 - 1 = 8 = 10a^2 \), which is impossible. If \( P = 27 \), then \( U_3 = P^2 - 1 = 27^2 - 1 = 10a^2 \), which is also impossible. Now assume that \( m \) is odd. Then by (2.9), we get \( U_{3m} = U_m \left( (P^2 - 4)U_m^3 + 3 \right) \). Clearly, \( (U_m, (P^2 - 4)U_m + 3) = 1 \) or \( 3 \). Then it follows that \( (P^2 - 4)U_m^3 + 3 = wa^2 \) for some \( w \in \{1, 3, 5, 15\} \). Since \( (P^2 - 4)U_m^3 + 3 = V_{2m} + 1 \) by (2.7) and (2.10), it is seen that \( V_{2m} + 1 = wa^2 \). Assume that \( m > 1 \). Then \( m = 4q \pm 1 = 2^r a \pm 1 \) with \( a \) odd and \( r \geq 2 \). Thus,

\[
w^2 = V_{2m} + 1 \equiv 1 - V_2 \equiv -(P^2 - 3) \pmod{V_{2r}}
\]

by (2.4). This shows that

\[
\left( \frac{w}{V_{2r}} \right) = \left( \frac{-1}{V_{2r}} \right) \left( \frac{P^2 - 3}{V_{2r}} \right).
\]

By using (2.23), (2.24), and (2.25), it can be seen that \( \left( \frac{w}{V_{2r}} \right) = 1 \) for \( w = 3, 5, 15 \).

Moreover, \( \left( \frac{-1}{V_{2r}} \right) = -1 \) and \( \left( \frac{P^2 - 3}{V_{2r}} \right) = 1 \) by (2.20) and (2.21), respectively. Thus, we get

\[
1 = \left( \frac{w}{V_{2r}} \right) = \left( \frac{-1}{V_{2r}} \right) \left( \frac{P^2 - 3}{V_{2r}} \right) = -1,
\]

which is impossible. Therefore \( m = 1 \) and thus \( n = 3 \).

**Case III**: Let \( P^2 \equiv -1 \pmod{5} \). Since \( 5|U_n \), it follows that \( 5|n \) by Lemma 4. Thus \( n = 5t \) for some positive integer \( t \). Since \( P^2 \equiv -1 \pmod{5} \), it is obvious that \( 5|P^2 - 4 \) and therefore there exists a positive integer \( A \) such that \( P^2 - 4 = 5A \). By (2.15), we get \( U_n = U_{5t} = U_t \left( (P^2 - 4)U_t^4 + 5(P^2 - 4)U_t^2 + 5 \right) \). Substituting \( P^2 - 4 = 5A \)
into the preceding equation gives $U_n = U_{5t} = 5U_t (5A^2 U_t^4 + 5A U_t^2 + 1)$. Let $B = A^2 U_t^4 + A U_t^2$. As a consequence, we have

$$U_n = U_{5t} = 5U_t (5B + 1) = 5x^2,$$

implying that

$$U_t (5B + 1) = x^2.$$

It can be easily seen that $(U_t, 5B + 1) = 1$. This shows that $U_t = a^2$ and $5B + 1 = b^2$ for some $a, b > 0$. By Theorem 6, we see that the only possible values of $t$ and $P$ in which $U_t = a^2$ are $t = 1$ or $t = 6$ and $P = 3$. If $t = 1$, then $n = 5$ and therefore we get $U_n = U_{5t} = U_5 = P^4 - 3P^2 + 1 = 5x^2$. With MAGMA [1], we get $P = 2$, which is impossible since $P$ is odd. If $t = 6$, then $n = 30$. A simple computation shows that there is no integer $x$ such that $U_{30} = 5x^2$ for $P = 3$. $\Box$

**Theorem 13.** Let $P \geq 3$ and $m > 1$. The equation $U_n = 5U_mx^2$ has no solutions in any of the following cases:

(i) : $P^2 \equiv -1$ (mod 5);

(ii) : $P$ is odd and $5|P$;

(iii) : $P^2 \equiv 1$ (mod 5), $n$ is odd, and $P$ is odd;

(iv) : $P^2 \equiv 1$ (mod 5), $n$ is even, and $P$ is odd.

**Proof.** Assume that $U_n = 5U_mx^2$ for some $x > 0$. Since $U_m|U_n$, it follows that $m|n$ by (2.13). Thus, $n = mt$ for some $t > 0$. Since $n \neq m$, we have $t > 1$.

**Case I :** Let $P^2 \equiv -1$ (mod 5). It is obvious that $5|P^2 - 4$. On the other hand, since $5|U_n$, it follows that $5|n$ by Lemma 4. Dividing the proof into two subcases, we have

**Subcase (i) :** Assume that $5|t$. Then $t = 5s$ for some $s > 0$ and therefore $n = mt = 5ms$. By (2.15), we obtain

$$U_n = U_{5ms} = U_{ms} \left((P^2 - 4)^2 U_{ms}^4 + 5(P^2 - 4) U_{ms}^2 + 5\right) = 5U_mx^2. \quad (3.3)$$

Since $5|P^2 - 4$, it is seen that $5|(P^2 - 4)^2 U_{ms}^4 + 5(P^2 - 4) U_{ms}^2 + 5$. Also, we have $(P^2 - 4)^2 U_{ms}^4 + 5(P^2 - 4) U_{ms}^2 + 5 = V_{ms}^4 - 3V_{ms}^2 + 1$ by (2.10). Rearranging the equation (3.3), we readily obtain

$$x^2 = \left(U_{ms}/U_m\right) \left((V_{ms}^4 - 3V_{ms}^2 + 1)/5\right),$$

where $\left(U_{ms}/U_m, (V_{ms}^4 - 3V_{ms}^2 + 1)/5\right) = 1$. Hence, $V_{ms}^4 - 3V_{ms}^2 + 1 = 5b^2$ for some $b > 0$. But the integral points on $5Y^2 = X^4 - 3X^2 + 1$ are immediately determined by using MAGMA [1] to be $(\pm X, \pm Y) = (2, 1)$, which gives $V_{ms} = 2$, implying that $ms = 0$, which is impossible.

**Subcase (ii) :** Assume that $5 \not| t$. Since $5|n$, it follows that $5|m$. Then we can write $m = 5r$ with $5 \not| a$ and $r \geq 1$. By (2.16), it is seen that $U_m = U_{5r} = 5U_{5r-1} (5a_1 + 1)$ for some positive integer $a_1$. Thus, we conclude that $U_m = U_{5r} = 5U_{5r-1} (5a_1 + 1) (5a_2 + 1) ... (5a_r + 1)$ for some positive integers $a_i$ with $1 \leq i \leq r$. Let $A = (5a_1 +
1)(5a_2 + 1)...(5a_r + 1). Then, we have \( U_m = 5^t U_a A \), where \( 5 \nmid A \). In a similar manner, we get \( U_n = U_{st} = 5^t U_a ((5b_1 + 1)(5b_2 + 1)...(5b_r + 1) \) for some positive integers \( b_i \) with \( 1 \leq i \leq r \). Let \( B = (5b_1 + 1)(5b_2 + 1)...(5b_r + 1) \). Hence, we have \( U_n = 5^t U_a B \), where \( 5 \nmid B \). As a consequence, we get
\[
5^t U_a B = 5 \cdot 5^t U_a A x^2
\]
i.e.,
\[
U_a B = 5U_a A x^2.
\]
Since \( 5 \nmid B \), it follows that \( 5 \nmid U_a t \), implying that \( 5 \nmid at \) by Lemma 4. This contradicts the fact that \( 5 \nmid a \) and \( 5 \nmid t \). This concludes the proof of the case when \( P^2 \equiv -1 \pmod{5} \).

**Case II:** Let \( P \) be odd and \( 5 \nmid P \). Since \( 5 \nmid U_n \), it is seen from Lemma 4 that \( n \) is even. On the other hand, we have \( n = mt \). So, we first assume that \( t \) is even. Then \( t = 2s \) for some \( s > 0 \). By (2.6), we get \( U_n = U_{2ms} = U_{ms} V_{ms} = 5U_{m} x^2 \), implying that \( (U_{ms} / U_m)V_{ms} = 5x^2 \). Clearly, \( d = (U_{ms} / U_m, V_{ms}) = 1 \) or 2 by (2.14). If \( d = 1 \), then
\[
U_{ms} = U_m a^2, \ V_{ms} = 5b^2 \tag{3.4}
\]
or
\[
U_{ms} = 5U_m a^2, \ V_{ms} = b^2 \tag{3.5}
\]
for some \( a, b > 0 \). If (3.4) holds, then the only possible value of \( ms \) in which \( V_{ms} = 5b^2 \) is 1 by Theorem 1, which contradicts the fact that \( m > 1 \). If (3.5) holds, then by Theorem 5, we have \( ms = 1 \), which is impossible since \( m > 1 \).

If \( d = 2 \), then
\[
U_{ms} = 2U_m a^2, \ V_{ms} = 10b^2 \tag{3.6}
\]
or
\[
U_{ms} = 10U_m a^2, \ V_{ms} = 2b^2 \tag{3.7}
\]
for some \( a, b > 0 \). Suppose (3.6) holds. Then by Theorem 7, we get \( ms = 6, m = 3, P = 3, 27 \). There is no integer \( b \) such that \( V_6 = 10b^2 \) for the case when \( P = 3 \) or \( 27 \). Suppose (3.7) holds. Then by Theorem 7, the only possible values of \( ms \) and \( P \) in which \( V_{ms} = 2b^2 \) are \( ms = 3 \) and \( P = 3, 27 \). Since \( m > 1 \), it follows that \( m = 3 \) and therefore we obtain \( U_3 = 10U_3 a^2 \), which is impossible.

Now assume that \( t \) is odd. Since \( n = mt \) and \( n \) is even, it follows that \( m \) is even. Hence, we have \( U_n \equiv \pm(n/2)P \pmod{P^2} \) and \( U_m \equiv \pm(m/2)P \pmod{P^2} \) by Lemma 2. This shows that \( \pm \frac{n}{2} P \equiv \pm \frac{m}{2} P x^2 \pmod{P^2} \), i.e., \( \frac{n}{2} \equiv \frac{m}{2} x^2 \pmod{P} \). Since \( 5 \mid P \), it is seen that \( 5 \mid n \). Dividing remainder of the proof into two subcases, we have

**Subcase (i):** Let \( 5 \nmid t \). Then \( t = 5s \) for some \( s > 0 \) and therefore \( n = mt = 5ms \). By (2.15), we obtain
\[
U_n = U_{5ms} = U_{ms} ((P^2 - 4)U_{ms} + 5(P^2 - 4)U_{ms}^2 + 5). \tag{3.8}
\]
Since \( ms \) is even and \( 5|P \), it is seen that \( 5|U_{ms} \) by Lemma 4. Also, we have \((P^2 - 4)^2U_{ms}^4 + 5(P^2 - 4)U_{ms}^2 + 5 = V_{ms}^4 - 3V_{ms}^2 + 1 \) by (2.10). Hence, rearranging the equation (3.8) gives

\[
x^2 = \left( \frac{U_{ms}}{U_m} \right) \left( \frac{V_{ms}^4 - 3V_{ms}^2 + 1}{5} \right),
\]

where \((\frac{U_{ms}}{U_m})(\frac{V_{ms}^4 - 3V_{ms}^2 + 1}{5}) = 1\). This implies that \( V_{ms}^4 - 3V_{ms}^2 + 1 = 5b^2 \) for some positive integer \( b \). The integral points on \( 5Y^2 = X^4 - 3X^2 + 1 \) are immediately determined by using MAGMA [1] to be \((\pm X, \pm Y) = (2, 1)\), which gives \( V_{ms} = 2 \), implying that \( ms = 0 \), which is impossible.

**Subcase (i):** Let \( 5 \nmid t \). Since \( 5|n \), it follows that \( 5|m \). Then we can write \( m = 5^fa \) with \( 5 \nmid a \) and \( r \geq 1 \). By (2.16), it is seen that \( U_m = U_{5^fa} = 5U_{5^{r-1}a}(5a_1 + 1) \) for some positive integer \( a_1 \). Thus, we conclude that \( U_m = U_{5^fa} = 5U_a(5a_1 + 1)(5a_2 + 1)...(5a_r + 1) \) for some positive integers \( a_i \) with \( 1 \leq i \leq r \). Let \( A = (5a_1 + 1)(5a_2 + 1)...(5a_r + 1) \). Then, we have \( U_m = 5^rU_aA \), where \( 5 \nmid A \). In a way similar, we get \( U_n = U_{5^rat} = 5^rU_{at}(5b_1 + 1)(5b_2 + 1)...(5b_r + 1) \) for some positive integers \( b_i \) with \( 1 \leq i \leq r \). Let \( B = (5b_1 + 1)(5b_2 + 1)...(5b_r + 1) \). Hence, we have \( U_n = 5^rU_{at}B \), where \( 5 \nmid B \). Substituting the new values of \( U_n \) and \( U_m \) into \( U_n = 5U_mx^2 \) gives

\[
5^rU_{at}B = 5 \cdot 5^rU_aAx^2
\]

i.e.,

\[
U_{at}B = 5U_aAx^2.
\]

On the other hand, since \( a \) is even and \( at \) is even, it follows from Lemma 2 that \( U_{at} \equiv \pm \frac{at}{2}P \pmod{P^2} \) and \( U_a \equiv \pm \frac{a}{2}P \pmod{P^2} \). Hence, we have

\[
\pm \frac{at}{2}PB \equiv \pm \frac{a}{2}PAx^2 \pmod{P^2},
\]

implying that

\[
\frac{at}{2}B \equiv \frac{a}{2}Ax^2 \pmod{P^2}.
\]

Since \( 5|P \), it follows that \( 5|\frac{at}{2}B \), which shows that \( 5|atB \). This contradicts the fact that \( 5 \nmid a \), \( 5 \nmid b \), and \( 5 \nmid B \). This concludes the proof for the case when \( 5|P \).

**Case III:** Let \( P^2 \equiv 1 \pmod{5} \), \( n \) is odd, and \( P \) is odd. Then, both \( m \) and \( t \) are odd. Since \( 5|U_n \), it follows immediately from Lemma 4 that \( 3|n \). Using the fact that \( n = mt \), we have

**Subcase (i):** Assume that \( 3|m \). Since \( t \) is odd, we can write \( t = 4q \pm 1 \) for some \( q > 0 \). If \( t = 4q + 1 \), then \( t = 2 \cdot 2^fa + 1 \) with \( a \) odd and \( r > 0 \). And so by (2.3), we get \( U_n = U_{mt} = U_{2 \cdot 2^fam + m} \equiv -U_m \pmod{V_{2r}} \), implying that \( 5U_{mx^2} \equiv -U_m \pmod{V_{2r}} \). Since \( (U_m, V_{2r}) = 1 \) by (2.14), it follows that \( 5x^2 \equiv -1 \pmod{V_{2r}} \), which is impossible since \( \left( \frac{5}{V_{2r}} \right) = 1 \) by (2.25) and \( \left( \frac{-1}{V_{2r}} \right) = -1 \) by (2.20). If \( t = 4q - 1 \), then by (2.1), we get \( U_n = U_{m(4q-1)} = U_{2 \cdot 2mq - m} \equiv -U_m \pmod{U_{2m}} \). This shows that \( 5U_{mx^2} \equiv -U_m \pmod{U_{2m}} \), implying that \( 5x^2 \equiv -1 \pmod{V_m} \) by
ON THE EQUATIONS $U_n = 5\square$ AND $V_n = 5\square$

(2.6). Since $3|m$, it is seen by (2.12) that $V_3|V_m$. Hence, we obtain $5x^2 \equiv -1 \pmod{V_3}$, i.e., $5x^2 \equiv -1 \pmod{P^2 - 3}$. But this is impossible since

$$ \left( \frac{5}{(P^2 - 3)/2} \right) = \left( \frac{(P^2 - 3)/2}{5} \right) = \left( \frac{-1}{5} \right) = 1 $$

and

$$ \left( \frac{-1}{(P^2 - 3)/2} \right) = (-1) \frac{p^2 - 4}{4} = -1. $$

**Subcase (ii):** Assume that $3 \not| m$. Since $n = mt$ and $3|n$, it follows that $3|t$ and therefore $t = 3s$ for some $s > 0$. Then by (2.9), we get

$$ U_n = U_{3ms} = U_{ms}((P^2 - 4)U_{ms}^2 + 3) = 5U_mx^2, $$

implying that

$$ (U_{ms}/U_m)((P^2 - 4)U_{ms}^2 + 3) = 5x^2. $$

Clearly,

$$ d = (U_{ms}/U_m)((P^2 - 4)U_{ms}^2 + 3) = 1 \text{ or } 3. $$

If $d = 1$, then either

$$ U_{ms} = U_m a^2, \ (P^2 - 4)U_{ms}^2 + 3 = 5b^2 \quad (3.9) $$

or

$$ U_{ms} = 5U_m a^2, \ (P^2 - 4)U_{ms}^2 + 3 = b^2 \quad (3.10) $$

for some $a, b > 0$. Suppose (3.9) holds. Then by (2.10), we get $V_{ms}^2 - 1 = 5b^2$ and this gives by (2.7) that $V_{2ms} = 5b^2 - 1$. Since $ms > 1$ is odd, $ms = 4q \pm 1$ for some $q > 0$. Thus $ms = 2 \cdot 2' a \pm 1$ with $a$ odd and $r > 0$. By using (2.4), we get $5b^2 - 1 = V_{2ms} \equiv -V_{\pm 2} \equiv -V_2 \pmod{V_2r}$. This shows that $5b^2 - 1 \equiv -(P^2 - 2) \pmod{V_2r}$, implying that $5b^2 \equiv -(P^2 - 3) \pmod{V_2r}$. By using (2.20), (2.25), and (2.21), it is seen that

$$ 1 = \left( \frac{-1}{V_2r} \right) \left( \frac{5}{V_2r} \right) \left( \frac{P^2 - 3}{V_2r} \right) = -1, $$

a contradiction. Suppose (3.10) holds. It can be easily seen by combining two equations that $b^2 \equiv 3 \pmod{5}$, which is impossible.

If $d = 3$, then either

$$ U_{ms} = 3U_m a^2, \ (P^2 - 4)U_{ms}^2 + 3 = 15b^2 \quad (3.11) $$

or

$$ U_{ms} = 15U_m a^2, \ (P^2 - 4)U_{ms}^2 + 3 = 3b^2 \quad (3.12) $$

for some $a, b > 0$. If we combine two equations given in (3.11), then we readily obtain $b^2 \equiv 2 \pmod{3}$, which is impossible. Suppose (3.12) holds. Then by (2.10), we get $V_{ms}^2 - 1 = 3b^2$ and this gives by (2.7) that $V_{2ms} = 3b^2 - 1$. Since $ms > 1$ is odd, $ms = 4q \pm 1$ for some $q > 0$. Thus $ms = 2 \cdot 2' a \pm 1$ with $a$ odd and $r > 0$. By using (2.4), we get $3b^2 - 1 = V_{2ms} \equiv -V_{\pm 2} \equiv -V_2 \pmod{V_2r}$. This shows that
$3b^2 - 1 \equiv -(P^2 - 2) \pmod{V_{2r}}$, implying that $3b^2 \equiv -(P^2 - 3) \pmod{V_{2r}}$. By (2.23), (2.24), (2.20), and (2.21), it is seen that
\[ 1 = \left( -\frac{1}{V_{2r}} \right) \left( \frac{3}{V_{2r}} \right) \left( \frac{P^2 - 3}{V_{2r}} \right) = -1, \]
a contradiction.

**Case IV**: Let $P^2 \equiv 1 \pmod{5}$, $n$ is even, and $P$ is odd. Since $n = mt$, we divide the proof into two subcases:

**Subcase (i)**: Assume that $t$ is even. Then $t = 2s$ for some $s > 0$. Hence, we immediately have $U_n/U_m = U_{2ms}/U_m = (U_{ms}/U_m)V_{ms} = 5x^2$. Clearly, $d = (U_{ms}/U_m, V_{ms}) = 1$ or 2 by (2.14). If $d = 1$, then
\[ U_{ms} = U_m a^2, \quad V_{ms} = 5b^2 \quad (3.13) \]
or
\[ U_{ms} = 5U_m a^2, \quad V_{ms} = b^2 \quad (3.14) \]
for some $a, b > 0$. Suppose (3.13) is satisfied. Since $5|V_{ms}$, it follows from (2.19) that $5|P$, which contradicts the fact that $P^2 \equiv 1 \pmod{5}$. Now suppose (3.14) is satisfied. By Theorem 5, the only possible value of $ms$ in which $V_{ms} = b^2$ is 1, which is impossible since $m > 1$.

If $d = 2$, then
\[ U_{ms} = 2U_m a^2, \quad V_{ms} = 10b^2 \quad (3.15) \]
or
\[ U_{ms} = 10U_m a^2, \quad V_{ms} = 2b^2 \quad (3.16) \]
for some $a, b > 0$. Obviously, (3.15) is not satisfied because of the same reason given above for (3.13). If (3.16) holds, then it is seen by Theorem 5 that the only possible values of $ms$ and $P$ in which $V_{ms} = b^2$ are $ms = 3$ and $P = 3, 27$. But this is impossible since $P^2 \equiv 1 \pmod{5}$.

**Subcase (ii)**: Assume that $t$ is odd. Since $t > 1$, we can write $t = 4q + 1$ for some $q > 0$ or $t = 4q + 3$ for some $q \geq 0$. On the other hand, since $n$ is even and $n = mt$, it follows that $m$ is even. Therefore we can write $n = 2^r a$ with $a$ odd and $r > 0$. Assume that $t = 4q + 1$. Then $n = mt = 4qm + m = 2 \cdot 2^{r+k} b + m$ with $b$ odd.

Hence, we get
\[ 5U_{m}x^2 = U_n = U_{2^{r+k}b + m} \equiv -U_m \pmod{V_{2^r+k}} \]
by (2.3). Since $(U_m, V_{2^r+k}) = (U_{2^r a}, V_{2^r+k}) = 1$ by (2.14), it follows that
\[ 5x^2 \equiv -1 \pmod{V_{2^r+k}}. \]
This is impossible. Because $\left( \frac{5}{V_{2^r+k}} \right) = 1$ and $\left( \frac{-1}{V_{2^r+k}} \right) = -1$ by (2.25) and (2.20), respectively. Now assume that $t = 4q + 3$. Then we have $n = mt = 4qm + 3m$. And so by (2.1), we get
\[ 5U_{m}x^2 = U_n = U_{4qm + 3m} \equiv U_{3m} \pmod{U_{2m}}. \]
By using (2.6) and (2.9), we readily obtain
\[ 5x^2 \equiv V_m^2 - 1 \pmod{V_m}, \]
which implies that
\[ 5x^2 \equiv -1 \pmod{V_m}. \]
Using the fact that \( m = 2^r a \) with \( a \) odd, we have
\[ 5x^2 \equiv -1 \pmod{V_{2^r a}}, \]
implying that
\[ 5x^2 \equiv -1 \pmod{V_{2^r}} \]
by (2.12). But this is impossible since \( \left( \frac{5}{V_{2^r}} \right) = 1 \) and \( \left( \frac{-1}{V_{2^r}} \right) = -1 \) by (2.25) and (2.20), respectively. This completes the proof. \( \square \)

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