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PARAMETRIC MARCINKIEWICZ INTEGRALS ON WEIGHTED HERZ SPACES

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Abstract. Let $0<\rho<n$ and μ_Ω^ρ be the parametric Marcinkiewicz integral. In this paper we shall obtain the strong type and weak type estimates of μ_Ω^ρ on the weighted Herz spaces $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ with general weights. The boundedness of the commutators generated by BMO functions and parametric Marcinkiewicz integral is also obtained.

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1. Introduction and results

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n (n\geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega\in L^1(\mathbb{S}^{n-1})$ and

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x')=0, \tag{1.1}$$

where $x'=x/|x|$ for any $x\neq 0$. For $0<\rho<n$, Hörmander in [6] defined the parametric Marcinkiewicz integral operator μ_Ω^ρ of higher dimension as follows.

$$\mu_\Omega^\rho(f)(x)=\left(\int_0^\infty |F_{\Omega,t}^\rho(x)|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2},$$

where

$$F_{\Omega,t}^\rho(x)=\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f(y)dy.$$

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Let b be a locally integrable function, the commutator generated by parametric Marcinkiewcz integral μ_Ω^ρ and b is defined by

$$[b,\mu_\Omega^\rho](f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}(b(x)-b(y))f(y)dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{\frac{1}{2}}.$$

When $\rho=1$, we shall denote μ_Ω^1 simply by μ_Ω.

The area of the Marcinkiewcz integrals have been under intensive research. This operator μ_Ω was first introduced by Stein in [15]. He proved that if $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})(0<\alpha\le 1)$, then μ_Ω is the operator of strong type (p,p) for $1<p\le 2$ and of weak type $(1,1)$. Here, we say that $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})$ if

$$|\Omega(x')-\Omega(y')|\le |x'-y'|^\alpha,\ \ x',y'\in\mathbb{S}^{n-1}.$$

In 1990, Torchinsky and Wang in [16] considered the weighted case and proved that If $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})(0<\alpha\le 1)$, and $b\in BMO(\mathbb{R}^n)$, then for all $1<p<\infty$, and $\omega\in A_p$ (Muckenhoupt weight class), μ_Ω and $[b,\mu_\Omega]$ are all bounded in $L^p(\omega)$. On the other hand, in 1960, Hörmander [6] showed that if $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})(0<\alpha\le 1)$, then for $0<\rho<n$, μ_Ω^ρ is of strong type (p,p) for all $1<p<\infty$. In [14] Shi and Jiang obtained the weighted L^p-boundedness of parametric Marcinkiewicz integral and its commutator.

Theorem A ([14]). *Let $0<\rho<n$ and $\Omega\in L^\infty(\mathbb{S}^{n-1})$. If $\omega\in A_p(1<p<\infty)$ and $b\in BMO(\mathbb{R}^n)$, then there exists a constant $C>0$ independent of f such that*

$$\|\mu_\Omega^\rho(f)\|_{L^p(\omega)}\le C\|f\|_{L^p(\omega)}$$

and

$$\|[b,\mu_\Omega^\rho](f)\|_{L^p(\omega)}\le C\|b\|_*\|f\|_{L^p(\omega)}.$$

Notice that $Lip_\alpha(\mathbb{S}^{n-1})(0<\alpha\le 1)\subsetneqq L^\infty(\mathbb{S}^{n-1})\subsetneqq L^1(\mathbb{S}^{n-1})$, then the results of Theorem A are extend and improve the results of Torchinsky and Wang in [16].

Let us recall the definition of Littlewood-Paley function

$$S_\Phi(f)(x)=\left(\int_{\mathbb{R}^n}|\Phi_t*f(x)|^2\frac{dt}{t}\right)^{1/2},$$

where $\Phi_t(x)=t^{-n}\Phi(x/t)$ and $\Phi\in L^1(\mathbb{R}^n)$ which satisfies $\int_{\mathbb{R}^n}\Phi(x)dx=0$. It is well known that the Littlewood-Paley function have long played important roles in harmonic analysis. When Φ is given by

$$\Phi(x)=|x|^{-n+\rho}\Omega(x)\chi_{[0,1]}(|x|),$$

where Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying (1.1), then $\Phi(x)$ becomes the parametric Marcinkiewicz integral operator. Therefore, many authors have been interested in studying the boundedness properties of μ_Ω^ρ on various function spaces, it can be seen in [1–3, 18].

In [13], Lu, Yabuta and Yang obtained the boundedness results for sublinear operators on weighted Herz spaces with general Muckenhoupt weights. Recently, many authors considered the boundedness of operators on weighted Herz spaces with general Muckenhoupt weights. In [8], Komori and Matsuoka showed the boundedness of singular integral operators and fractional integrals on weighted Herz spaces. In [5], Guo and Jiang discussed the boundedness of commutators of singular integral operators on weighted Herz spaces, and as an application, they obtained the interior estimates on weighted Herz spaces for the solutions of some nondivergence elliptic equations. Hu , He and Wang [19] studied the boundedness of commutators of fractional integrals in generalized Herz spaces. More results concerning the boundedness of operators on Herz spaces can be seen in [12, 17].

The main purpose of this paper is to consider the boundedness of parametric Marcinkiewicz integral on weighted Herz spaces with A_p weights. At the extreme case, we will also prove that μ_Ω^ρ is bounded from the weighted Herz spaces to the weighted weak Herz spaces. The boundedness of commutators generated by parametric Marcinkiewicz integral and BMO functions on weighted Herz spaces is also considered.

Our main results in the paper are formulated as follows.

Theorem 1. *Let* $0 < \rho < n$ *and* $\Omega \in L^\infty(\mathbb{S}^{n-1})$. *Suppose* $0 < p < \infty, 1 < q < \infty, \omega_1 \in A_{q_1}$ *and* $\omega_2 \in A_{q_2}$. *Then* μ_Ω^ρ *is bounded on* $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ *provided that* ω_1 *and* ω_2 *satisfy either of the following*

(i) $\omega_1 = \omega_2, 1 \le q_1 = q_2 \le q$ *and* $-nq_1/q < \alpha q_1 < n(1-q_1/q)$;
(ii) $\omega_1 \neq \omega_2, 1 \le q_1 < \infty, 1 \le q_2 \le q$ *and* $0 < \alpha q_1 < n(1-q_2/q)$.

Theorem 2. *Let* $0 < \rho < n$ *and* $\Omega \in L^\infty(\mathbb{S}^{n-1})$. *Suppose* $0 < p < 1, 1 < q < \infty, \omega_1 \in A_{q_1}$ *and* $\omega_2 \in A_{q_2}$. *If* $1 \le q_1 < \infty, 1 \le q_2 \le q$ *and* $\alpha q_1 = n(1-q_2/q)$, *then* μ_Ω^ρ *is bounded from* $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ *to* $W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$.

Theorem 3. *Let* $0 < \rho < n$ *and* $\Omega \in L^\infty(\mathbb{S}^{n-1})$. *Suppose* $1 < q < \infty, \omega_1 \in A_{q_1}$, $\omega_2 \in A_{q_2}$ *and* $b \in BMO$. *Then* $[b, \mu_\Omega^\rho]$ *is bounded on* $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ *provided that* ω_1 *and* ω_2 *satisfy either of the following*

(i) $\omega_1 = \omega_2, 1 \le q_1 = q_2 \le q$ *and* $-nq_1/q < \alpha q_1 < n(1-q_1/q)$;
(ii) $\omega_1 \neq \omega_2, 1 \le q_1 < \infty, 1 \le q_2 \le q$ *and* $0 < \alpha q_1 < n(1-q_2/q)$.

Throughout this paper, unless otherwise indicated, C will be used to denote a positive constant that is not necessarily the same at each occurrence.

2. Definitions and preliminaries

We begin this section with some properties of A_p weights which play important role in the proofs of our main results.

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B(x_0, r)$ denote the ball with the center x_0 and radius r, and let $\lambda B = B(x_0, \lambda r)$ for any $\lambda > 0$.

For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E) = \int_E \omega(x)dx$.

Definition 1. A weight ω is said to belong to A_p for $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}dx\right)^{p-1} \le C, \tag{2.1}$$

where s' is the dual of s such that $1/s + 1/s' = 1$.

The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|}\int_B \omega(y)dy \le C \cdot \operatorname*{ess\,inf}_{x\in B} w(x).$$

By (2.1), we have

$$\left(\int_B \omega(x)dx\right)\left(\int_B \omega(x)^{1-p'}dx\right)^{p-1} \le C|B|^p. \tag{2.2}$$

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p-boundedness of Hardy-Littlewood maximal function in [10].

Lemma 1 ([4, 10]). *Let $1 \le p < \infty$ and $\omega \in A_p$. Then the following statements are true:*

(i) *There exists constant C such that*

$$\frac{\omega(B_k)}{\omega(B_j)} \le C2^{np(k-j)} \text{ for } k > j; \tag{2.3}$$

(ii) *For any $1 < p < \infty$, there exists $1 < q < p$ such that $\omega \in A_q(\mathbb{R}^n)$;*

(iii) *There exist two constants C and $\delta > 0$ such that for any measurable set $E \subset B$,*

$$\frac{\omega(E)}{\omega(B)} \le C\left(\frac{|E|}{|B|}\right)^{\delta}; \tag{2.4}$$

(iv) *For any $1 < p < \infty$, one has $\omega^{1-p'} \in A_{p'}(\mathbb{R}^n)$.*

If ω satisfies (2.4), we say $\omega \in A_\infty(\mathbb{R}^n)$. It is well known that

$$A_\infty(\mathbb{R}^n) = \bigcup_{1\le p<\infty} A_p(\mathbb{R}^n).$$

Let $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for any $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, where χ_E is the characteristic function of the set E. In [9], Lu and Yang introduced weighted Herz spaces.

Definition 2. Let $\alpha \in \mathbb{R}, 0 < p, q < \infty$ and ω_1, ω_2 be two weight functions on $\mathbb{R}^n$. The homogenous weighted Herz space $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) = \left\{ f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}, \omega_2) : \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \left(\sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\alpha p/n} \|f\chi_k\|_{L^q(\omega_2)}^p \right)^{1/p}.$$

For any $k \in \mathbb{Z}, \lambda > 0$ and any measurable function f on $\mathbb{R}^n$, we write $E_k(\lambda, f) = \{x \in C_k : |f(x)| > \lambda\}$.

Definition 3 ([13]). Let $\alpha \in \mathbb{R}, 0 < p, q < \infty$ and ω_1, ω_2 be two weight function on $\mathbb{R}^n$. A measurable function f on $\mathbb{R}^n$ is said to belong to the homogeneous weighted weak Herz space $W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ if

$$\|f\|_{W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \sup_{\lambda>0} \lambda \left(\sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\alpha p/n} [\omega_2(E_k(\lambda, f))]^{p/q} \right)^{1/p} < \infty.$$

Obviously, if $\alpha = 0$, then $\dot{K}_p^{0,p}(\omega_1,\omega_2) = L^p(\omega_2)$ and $W\dot{K}_p^{0,p}(\omega_1,\omega_2)$ $= WL^p(\omega_2)$ for any $0 < p < \infty$. Thus, weighted Herz spaces are generalizations of the weighted Lebesgue spaces.

Definition 4. A locally integrable function b is said to be in $BMO(\mathbb{R}^n)$ if

$$\sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx = \|b\|_* < \infty,$$

where $b_B = \frac{1}{|B|} \int_B b(y) dy$.

Lemma 2. *(John-Nirenberg inequality, see [7]) Let $b \in BMO(\mathbb{R}^n)$. Then for any ball $B \subset \mathbb{R}^n$, there exist constant C_1, C_2 such that for all $\lambda > 0$,*

$$|\{x \in B : |b(x) - b_B| > \lambda\}| \le C_1 |B| exp(-C_2 \lambda / \|b\|_*).$$

Lemma 3 ([11]). *Let $\omega \in A_\infty$. Then the norm of $BMO(\omega, \mathbb{R}^n)$ is equivalent to the norm of $BMO(\mathbb{R}^n)$, where*

$$BMO(\omega, \mathbb{R}^n) = \left\{ b : \|b\|_{*,\omega} = \sup_{B \subset \mathbb{R}^n} \frac{1}{\omega(B)} \int_B |b(x) - b_{B,\omega}| \omega(x) dx \right\},$$

and

$$b_{B,\omega} = \frac{1}{\omega(B)} \int_B b(z) \omega(z) dz.$$

Lemma 4. *Suppose $\omega \in A_\infty(\mathbb{R}^n), b \in BMO(\mathbb{R}^n)$. Then for any $p \ge 1$ we have*

$$\left(\frac{1}{\omega(B)} \int_B |b(x) - b_{B,\omega}|^p \omega(x) dx \right)^{1/p} \le C \|b\|_*.$$

Proof. Since $\omega(x) \in A_\infty(\mathbb{R}^n)$, then (iii) of Lemma 1 and Lemma 2 imply

$$\omega(\{x \in B : |b(x)-b_B| > \lambda\}) \leq C\omega(B)exp(-C_2\delta\lambda/\|b\|_*).$$

So

$$\begin{aligned}\int_B |b(x)-b_B|^p\omega(x)dx &\leq \int_0^\infty p\lambda^{p-1}\omega(\{x \in B : |b(x)-b_B| > \lambda\})\,d\lambda \\ &\leq C\omega(B)\int_0^\infty p\lambda^{p-1}exp(-C_2\delta\lambda\|b\|_*)d\lambda \\ &\leq C\omega(B)\|b\|_*^p.\end{aligned}$$

Thus

$$\frac{1}{\omega(B)}\int_B |b(x)-b_{B,\omega}|^p\omega(x)dx \leq \frac{C}{\omega(B)}\int_B |b(x)-b_B|^p\omega(x)dx \leq C\|b\|_*^p.$$

□

3. Proof of Theorem 1

Let $f \in \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$. Then

$$\begin{aligned}&\left\|\mu_\Omega^\rho(f)\right\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \\ \leq\, & C\sum_{j=-\infty}^{\infty}\left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}\left\|(\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)}\right)^p \\ &+ C\sum_{j=-\infty}^{\infty}\left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\left\|(\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)}\right)^p \\ &+ C\sum_{j=-\infty}^{\infty}\left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}}\left(\sum_{k=j+2}^{\infty}\left\|(\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)}\right)^p \\ =\, & E_1+E_2+E_3.\end{aligned}$$

By the fact that μ_Ω^ρ is a bounded operator on $L^q(\omega_2)$, we get

$$\begin{aligned}E_2 =& C\sum_{j=-\infty}^{\infty}\left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\left\|(\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)}\right)^p \\ &\leq C\sum_{j=-\infty}^{\infty}\left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\left\|f\chi_k\right\|_{L^q(\omega_2)}\right)^p \\ &\leq C\|f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.\end{aligned}$$

For any $x \in C_j$ and $y \in C_k \cap \{y : |x-y| < t\}$ with $j \geq k+2$, we have $t > |x-y| \geq |x| - |y| \geq |x|/2 \geq C2^{jn}$. So

$$\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right| = \left(\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f_k(y)dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2}$$
$$\leq C2^{-j(n-\rho)}\|f\chi_k\|_{L^1}\left(\int_{C2^{jn}}^\infty \frac{dt}{t^{2\rho+1}}\right)^{1/2}.$$

Then

$$\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right| = C2^{-jn}\|f\chi_k\|_{L^1}, \tag{3.1}$$

and

$$\left\|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \leq C2^{-jn}\omega_2(B_j)^{1/q}\|f\chi_k\|_{L^1}.$$

By Hölder's inequality,

$$\|f\chi_k\|_{L^1} \leq C\|f\chi_k\|_{L^q(\omega_2)}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}. \tag{3.2}$$

Since $\omega_2 \in A_{q_2}(\mathbb{R}^n) \subset A_q(\mathbb{R}^n)$, by (2.2) and (2.3) we get,

$$(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}(\int_{B_j}\omega_2(x)dx)^{1/q}$$
$$= (\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}(\int_{B_k}\omega_2(x)dx)^{1/q}\left(\frac{\omega_2(B_j)}{\omega_2(B_k)}\right)^{1/q}$$
$$\leq C2^{kn+(j-k)nq_2/q}. \tag{3.3}$$

Then, for $j \geq k+2$ we have

$$\left\|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \leq C\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)n(q_2/q-1)}.$$

Thus

$$E_1 = C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}\|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\|_{L^q(\omega_2)}\right)^p$$
$$\leq C\|b\|_*^p\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{j-2}(\omega_1(B_j))^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)n(q_2/q-1)}\right)^p$$
$$\leq C\|b\|_*^p\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{j-2}(\omega_1(B_k))^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)(\alpha q_1+q_2n/q-n)}\right)^p.$$

When $0 < p \le 1$, we get

$$
\begin{aligned}
E_1 &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \sum_{j=-\infty}^{k+2} 2^{(j-k)p(\alpha q_1+q_2 n/q-n)} \\
&\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \\
&\le C\|b\|_*^p \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)}^p.
\end{aligned}
$$

When $p > 1$, by Hölder's inequality we get

$$
\begin{aligned}
E_1 &\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j-2} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p 2^{(j-k)(\alpha q_1+q_2 n/q-n)} \right) \\
&\quad \left(\sum_{k=-\infty}^{j-2} 2^{(j-k)(\alpha q_1+q_2 n/q-n)} \right)^{p/p'} \\
&\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \\
&\le C\|b\|_*^p \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)}^p.
\end{aligned}
$$

Let us now turn to estimate the last term E_3. In the case $k \ge j+2$, for any $x \in C_j$ and $y \in C_k \cap \{y : |x-y| < t\}$, we have $t > |x-y| \ge |y|-|x| \ge |y|/2 \ge C2^{kn}$. Then, similar to the estimates of (3.1),

$$
\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right| \le C2^{-kn}\|f\chi_k\|_{L^1}. \tag{3.4}
$$

So,

$$
\left\|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \le C2^{-kn}\omega_2(B_j)^{1/q}\|f\chi_k\|_{L^1}. \tag{3.5}
$$

By (2.2) and (2.4),

$$
\Big(\int_{B_k} \omega_2(x)^{1-q'}dx\Big)^{1/q'}\Big(\int_{B_j} \omega_2(x)dx\Big)^{1/q} \le C2^{kn+(j-k)\delta_2 n/q}. \tag{3.6}
$$

Combining with (3.2), (3.5) and (3.6), we have

$$
\left\|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \le C\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)\delta_2 n/q}
$$

for $j \le k-2$.

When $0 < p \le 1$, we get

$$E_3 \le C\|b\|_*^p \sum_{j=-\infty}^{\infty} \omega_1(B_j)^{\frac{\alpha p}{n}} \sum_{k=j+2}^{\infty} \|f\chi_k\|_{L^q(\omega_2)}^p 2^{(j-k)p\delta_2 n/q}$$
$$\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \sum_{j=-\infty}^{k-2} \left(\frac{\omega_1(B_j)}{\omega_1(B_k)}\right)^{\frac{\alpha p}{n}} 2^{(j-k)p\delta_2 n/q}$$
$$\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \sum_{j=-\infty}^{k-2} 2^{(j-k)p(\delta_1\alpha+\delta_2 n/q)}$$
$$\le C\|b\|_*^p \|f\|_{\dot K_q^{\alpha,p}(\omega_1,\omega_2)}^p.$$

When $q > 1$, by Hölder's inequality we get

$$E_3 \le C\|b\|_*^p \sum_{j=-\infty}^{\infty} \left(\sum_{k=j+2}^{\infty} (\omega_1(B_k))^{\frac{\alpha}{n}} \|f\chi_k\|_{L^q(\omega_2)} 2^{(j-k)(\delta_1\alpha+\delta_2 n/q)}\right)^p$$
$$\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} \left(\sum_{k=j+2}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p 2^{(j-k)(\delta_1\alpha+\delta_2 n/q)}\right)$$
$$\left(\sum_{k=j+2}^{\infty} 2^{(j-k)(\delta_1\alpha+\delta_2 n/q)}\right)^{p/p\prime}$$
$$\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \sum_{j=-\infty}^{k-2} 2^{(j-k)(\delta_1\alpha+\delta_2 n/q)}$$
$$\le C\|b\|_*^p \|f\|_{\dot K_q^{\alpha,p}(\omega_1,\omega_2)}^p.$$

Combining the above estimates for E_1, E_2 and E_3, the proof of Theorem 1 is completed.

4. Proof of Theorem 2

Let $f \in \dot K_q^{\alpha,p}(\omega_1,\omega_2)$. Then

$$\lambda^p \sum_{j=-\infty}^{\infty} \left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}} \left(\omega_2(\{x \in C_j : |\mu_\Omega^\rho(f)(x)| > \lambda\})\right)^{p/q}$$
$$\le \lambda^p \sum_{j=-\infty}^{\infty} \left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}} \left(\omega_2\left(\left\{x \in C_j : \sum_{k=-\infty}^{j-2} |\mu_\Omega^\rho(f_k)(x)| > \lambda/3\right\}\right)\right)^{p/q}$$

$$+\lambda^p \sum_{j=-\infty}^{\infty} \left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}} \left(\omega_2\left(\left\{x \in C_j : \sum_{k=j-1}^{j+1} |\mu_\Omega^\rho(f_k)(x)| > \lambda/3\right\}\right)\right)^{p/q}$$

$$+\lambda^p \sum_{j=-\infty}^{\infty} \left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}} \left(\omega_2\left(\left\{x \in C_j : \sum_{k=j-2}^{\infty} |\mu_\Omega^\rho(f_k)(x)| > \lambda/3\right\}\right)\right)^{p/q}$$

$$= F_1 + F_2 + F_3.$$

Applying Chebyshev's inequality [4] and Theorem A, we obtain

$$\begin{aligned} F_2 \le\ & C\lambda^p \sum_{j=-\infty}^{\infty} \left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}} \left(\frac{1}{\lambda^q} \sum_{k=j-1}^{j+1} \|\mu_\Omega^\rho(f_k)\|_{L^q(\omega_2)}^q\right)^{p/q} \\ & \le C \sum_{j=-\infty}^{\infty} \left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}} \left(\sum_{k=j-1}^{j+1} \|f\chi_k\|_{L^q(\omega_2)}\right)^p \\ & \le C\|f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}. \end{aligned}$$

For any $x \in C_j$ and $y \in C_k \cap \{y : |x-y| < t\}$ with $j \ge k+2$, by the inequalities (3.1), (3.2) and (3.3) we have

$$\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right| \le\ C\left(\omega_2(B_j)\right)^{-1/q} 2^{-jn+kn+(j-k)nq_2/q} \|f\chi_k\|_{L^q(\omega_2)}.$$

Noting the fact $\alpha q_1 = n(1-q_2/q)$, then

$$\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right| \le C\left(\omega_2(B_j)\right)^{-1/q} 2^{(k-j)\alpha q_1} \|f\chi_k\|_{L^q(\omega_2)}.$$

Moreover, since $0 < p \le 1$, then for any $x \in C_j$,

$$\begin{aligned} & \sum_{k=-\infty}^{j-2} \left|\mu_\Omega^\rho(f_k)(x)\right| \\ & \le C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n} \\ & \sum_{k=-\infty}^{j-2} 2^{(k-j)\alpha q_1}\left(\omega_1(B_k)\right)^{\frac{\alpha}{n}} \|f\chi_k\|_{L^q(\omega_2)} \left(\frac{\omega_1(B_j)}{\omega_1(B_k)}\right)^{\frac{\alpha}{n}} \\ & \le C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n} \sum_{k=-\infty}^{j-2} \left(\omega_1(B_k)\right)^{\frac{\alpha}{n}} \|f\chi_k\|_{L^q(\omega_2)} \end{aligned}$$

$$\le C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\left(\sum_{k=-\infty}^{j-2}\left(\omega_1(B_k)\right)^{\frac{\alpha p}{n}}\|f\chi_k\|_{L^q(\omega_2)}^p\right)^{1/p}$$

$$\le C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

If

$$\left\{x\in C_j:\sum_{k=-\infty}^{j-2}|\mu_\Omega^\rho(f_k)(x)|>\lambda/3\right\}=\varnothing,$$

then

$$F_1\le C\|f\|_{\dot{K}_p^{\alpha,p}(\omega_1,\omega_2)}^p$$

holds is trivially. Now we suppose

$$\left\{x\in C_j:\sum_{k=-\infty}^{j-2}|\mu_\Omega^\rho(f_k)(x)|>\lambda/3\right\}\neq\varnothing.$$

Let

$$S_j=\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}.$$

Since $\alpha>0$, it is easy to see that

$$\lim_{j\to\infty}S_j=0.$$

Then for any $\lambda>0$, we can find a maximal positive integer j_λ such that

$$\lambda/3\le CS_{j_\lambda}\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

So

$$\begin{aligned}
F_1 &\le \lambda^p\sum_{j=-\infty}^{j_\lambda}\left(\omega_2(B_j)\right)^{p/q}\left(\omega_1(B_j)\right)^{\alpha p/n}\\
&\le C\|f\|_{\dot{K}_p^{\alpha,p}(\omega_1,\omega_2)}^p\sum_{j=-\infty}^{j_\lambda}\left(\frac{\omega_1(B_j)}{\omega_1(B_{j_\lambda})}\right)^{\frac{\alpha p}{n}}\left(\frac{\omega_2(B_j)}{\omega_2(B_{j_\lambda})}\right)^{\frac{p}{q}}\\
&\le C\|f\|_{\dot{K}_p^{\alpha,p}(\omega_1,\omega_2)}^p\sum_{j=-\infty}^{j_\lambda}2^{(j-j_\lambda)(\delta_1\alpha p+\delta_2 pn/q)}\\
&\le C\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}^p.
\end{aligned}$$

Let us now estimate F_3. From (3.2), (3.4) and (3.6) we have

$$\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right|\le C\left(\omega_2(B_j)\right)^{-1/q}2^{(j-k)\delta_2 n/q}\|f\chi_k\|_{L^q(\omega_2)}$$

for $j \geq k-2$. So

$$\sum_{k=j-2}^{\infty} |\mu_{\Omega}^{\rho}(f_k)(x)|$$
$$\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}$$
$$\sum_{k=j-2}^{\infty} (\omega_1(B_k))^{\alpha/n} \|f\chi_k\|_{L^q(\omega_2)} 2^{(j-k)\delta_2 n/q}\left(\frac{\omega_1(B_j)}{\omega_1(B_k)}\right)^{\alpha/n}$$
$$\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\sum_{k=j-2}^{\infty}(\omega_1(B_k))^{\alpha/n}\|f\chi_k\|_{L^q} 2^{(j-k)(\delta_2 n/q+\alpha\delta_1)}$$
$$\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\sum_{k=j-2}^{\infty}(\omega_1(B_k))^{\alpha/n}\|f\chi_k\|_{L^q}.$$

Noting that $0 < p \leq 1$, we have

$$\sum_{k=j-2}^{\infty} |\mu_{\Omega}^{\rho}(f_k)(x)|$$
$$\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\left(\sum_{k=j-2}^{\infty} |(\omega_1(B_k))^{\alpha p/n}\|f\chi_k\|_{L^q}^p\right)^{1/p}$$
$$\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

Repeating the arguments used for the term F_1, we can also obtain

$$F_3 \leq C\dot{K}_q^{\alpha,p}(\omega_1,\omega_2).$$

Combining the above estimates for F_1, F_2 and F_3, and taking the supremum for all $\lambda > 0$, the proof of Theorem 2 is finished.

5. Proof of Theorem 3

Let $f \in \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$, then

$$\left\|[b,\mu_{\Omega}^{\rho}](f)\right\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}^p$$
$$\leq C\sum_{j=-\infty}^{\infty}\left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}\left\|([b,\mu_{\Omega}^{\rho}](f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)}\right)^p$$
$$+\ C\sum_{j=-\infty}^{\infty}\left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\left\|([b,\mu_{\Omega}^{\rho}](f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)}\right)^p$$

$$
\begin{aligned}
&+ C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=j+2}^{\infty}\|([b,\mu_\Omega^\rho](f\chi_k)\chi_j(x)\|_{L^q(\omega_2)}\right)^p\\
&= G_1+G_2+G_3.
\end{aligned}
$$

By the fact that $[b,\mu_\Omega^\rho]$ is a bounded operator on $L^q(\omega_2)$, we obtain

$$
G_2\le C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\|f\chi_k\|_{L^q(\omega_2)}\right)^p\le C\|b\|_*^p\|f\|^p_{\dot K_q^{\alpha,p}(\omega_1,\omega_2)}.
$$

Obviously,

$$
\begin{aligned}
&\left\|[b,\mu_\Omega^\rho](f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)}\\
&\le C\left(\int_{C_j}\left|(b(x)-b_{B_k})\mu_\Omega^\rho(f_k)(x)\right|^q\omega_2(x)dx\right)^{1/q}\\
&\quad + C\left(\int_{C_j}\left|\mu_\Omega^\rho((b(\cdot)-b_{B_k})f_k)(x)\right|^q\omega_2(x)dx\right)^{1/q}\\
&= H_1+H_2.
\end{aligned}
$$

For the term G_1, since $j\ge k+2$, by (3.1) we have

$$
\begin{aligned}
H_1 &= C\left(\int_{C_j}\left|(b(x)-b_{B_k})\mu_\Omega^\rho(f_k)(x)\right|^q\omega_2(x)dx\right)^{1/q}\\
&\le C2^{-jn}\|f\chi_k\|_{L^1}\left(\int_{B_j}|b(x)-b_{B_k}|^q\omega_2(x)dx\right)^{1/q}\\
&\le C2^{-jn}\|f\chi_k\|_{L^1}\Big(\Big(\int_{B_j}|b(x)-b_{B_j,\omega_2}|^q\omega_2(x)dx\Big)^{1/q}\\
&\qquad+(|b_{B_j}-b_{B_j,\omega_2}|+|b_{B_j}-b_{B_k}|)\Big(\int_{B_j}\omega_2(x)dx\Big)^{1/q}\Big).
\end{aligned}
$$

By Lemma 4 and the definition of $BMO(\mathbb{R}^n)$, we have

$$
\Big(\int_{B_j}|b(x)-b_{B_j,\omega_2}|^q\omega_2(x)dx\Big)^{1/q}\le C\|b\|_*\Big(\int_{B_j}\omega_2(x)dx\Big)^{1/q}, \tag{5.1}
$$

$$
|b_{B_j}-b_{B_j,\omega_2}|\le C\|b\|_*, \tag{5.2}
$$

and

$$
|b_{B_j}-b_{B_k}|\le C(j-k)\|b\|_*. \tag{5.3}
$$

From (3.2),(3.3) and (5.1)-(5.3), we get

$$H_1 \leq C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{-jn}(\int_{B_j}\omega_2(x)dx)^{1/q}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}$$
$$\leq C\|b\|_\star\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{(j-k)n(q_2/q-1)}$$

for $j \geq k+2$.

Similar to the estimate of (3.1),

$$\left|\mu^\rho_\Omega((b(\cdot)-b_{B_k})f_k)(x)\right| \leq C2^{-jn}\left(\int_{C_k}|b(z)-b_{B_k}||f(z)|dz\right).$$

So

$$\left(\int_{C_j}\left|\mu^\rho_\Omega((b(\cdot)-b_{B_k})f_k)(x)\right|^q\omega_2(x)dx\right)^{1/q}$$
$$\leq C2^{-jn}\left(\omega_2(B_j)\right)^{1/q}\left(\int_{C_k}|b(z)-b_{B_k}||f(z)|dz\right).$$

Since $\omega_2 \in A_{q_2}$, by Lemma 1 we know that $\omega_2^{1-q_2'} \in A_{q_2'}$. Therefore by Lemma 4 we have

$$\left(\int_{B_k}|b(x)-b_{B_k}|^{q'}\omega_2(x)^{1-q'}dx\right)^{1/q'} \leq C\|b\|_*(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}. \quad (5.4)$$

Using Hölder's inequality , (3.3) and (5.4) we get

$$H_2 \leq C2^{-jn}\int_{B_k}|b(y)-b_{B_k}||f\chi_k(y)|dy(\int_{B_j}\omega_2(x)dx)^{1/q}$$
$$\leq C2^{-jn}\left(\int_{B_k}|b(y)-b_{B_k}|^{q'}\omega_2(y)^{1-q'}dy\right)^{1/q'}\|f\chi_k\|_{L^q(\omega_2)}(\int_{B_j}\omega_2(x)dx)^{1/q}$$
$$\leq C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}2^{-jn}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}(\int_{B_j}\omega_2(x)dx)^{1/q}$$
$$\leq C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)n(q_2/q-1)}.$$

Summarizing the above estimates, we have that for $j \geq k+2$,

$$\left\|[b,\mu^\rho_\Omega](f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \leq C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{(j-k)n(q_2/q-1)}. \quad (5.5)$$

Using (5.5) and repeating the estimation process of E_1, we obtain

$$G_1 \leq C\|b\|_*^p\|f\|^p_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)}$$

for $0 < p < \infty$.

Finally, let us estimate G_3. Since $j \le k-2$, by (3.4) we have

$$H_1 = C\left(\int_{C_j}\left|(b(x)-b_{B_k})\mu_\Omega^\rho(f_k)(x)\right|^q \omega_2(x)dx\right)^{1/q}$$
$$\le C2^{-kn}\|f\chi_k\|_{L^1}\left(\int_{B_j}|b(x)-b_{B_k}|^q\omega_2(x)dx\right)^{1/q}.$$

From (3.2),(5.1)-(5.3) and (3.6), we get

$$H_1 \le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{-jn}(\int_{B_j}\omega_2(x)dx)^{1/q}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}$$
$$\le C\|b\|_\star\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{(j-k)\delta_2 n/q}$$

for $j \le k-2$.

Similar to the estimate of (3.4),

$$\left|\mu_\Omega^\rho((b(\cdot)-b_{B_k})f_k)(x)\right| \le C2^{-kn}\left(\int_{C_k}|b(z)-b_{B_k}||f(z)|dz\right).$$

So

$$\left(\int_{C_j}\left|\mu_\Omega^\rho((b(\cdot)-b_{B_k})f_k)(x)\right|^q\omega_2(x)dx\right)^{1/q}$$
$$\le C2^{-kn}\left(\omega_2(B_j)\right)^{1/q}\left(\int_{C_k}|b(z)-b_{B_k}||f(z)|dz\right).$$

Using Hölder's inequality, (5.4) and (3.6),

$$H_2 \le C2^{-kn}\int_{B_k}|b(y)-b_{B_k}||f\chi_k(y)|dy(\int_{B_j}\omega_2(x)dx)^{1/q}$$
$$\le C2^{-kn}\left(\int_{B_k}|b(y)-b_{B_k}|^{q'}\omega_2(y)^{1-q'}dy\right)^{1/q'}\|f\chi_k\|_{L^q(\omega_2)}(\int_{B_j}\omega_2(x)dx)^{1/q}$$
$$\le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}2^{-kn}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}(\int_{B_j}\omega_2(x)dx)^{1/q}$$
$$\le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)\delta_2 n/q}.$$

Thus

$$\left\|[b,\mu_\Omega^\rho](f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}(k-j)2^{(j-k)\delta_2 n/q}$$

for $j \ge k-2$. Repeating the estimation process of E_3, we obtain

$$G_3 \le C\|b\|_*^p\|f\|^p_{\dot K_q^{\alpha,p}(\omega_1,\omega_2)}$$

for $0 < p < \infty$.

Summing up the estimates of G_1, G_2 and G_3, it completes the proof of Theorem 3.

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