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RECURSION FORMULAS FOR G_1 AND G_2 HORN HYPERGEOMETRIC FUNCTIONS

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Abstract. The aim of this paper is to present various recursion formulas for Horn hypergeometric functions by the contiguous relations of hypergeometric series.These recursion formulas allow us to state the functions G_1 and G_2 Horn hypergeometric functions as a combination of themselves.

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1. Introduction

The first G_1 and G_2 Horn hypergeometric functions were defined by the series [3–5]

$$G_1\left(\alpha,\beta,\beta';x,y\right)=\sum_{m,p=0}^{\infty}(\alpha)_{m+p}(\beta)_{p-m}\left(\beta'\right)_{m-p}\frac{x^m}{m!}\frac{y^p}{p!}$$

$$|x|<r \;\;,\;\; |y|<s \;\;,\;\; r+s=1$$

and

$$G_2\left(\alpha,\alpha',\beta,\beta';x,y\right)=\sum_{m,p=0}^{\infty}(\alpha)_m\left(\alpha'\right)_p(\beta)_{p-m}\left(\beta'\right)_{m-p}\frac{x^m}{m!}\frac{y^p}{p!}$$

$$|x|<1 \;\;,\;\; |y|<1,$$

respectively. Recently, Opps *et al*. [1] have obtained some recursion formulas for the function F_2 by the contiguous relation of the Gauss hypergeometric function ${}_2F_1$. In [6], Wang gave some recursion formulas for Appell hypergeometric functions. The aim of our present investigation is to construct various recursion formulas for each of Horn hypergeometric functions G_1 and G_2.

Recall that gamma function is defined in [2,3] by

$$\Gamma(n)=\int_0^{\infty}t^{n-1}e^{-t}dt \;\;,\;\; \operatorname{Re}(n)>0.$$

The Pochhammer symbol $(\lambda)_n$ is denoted by

$$(\lambda)_n := \lambda(\lambda+1)\ldots(\lambda+n-1)\ ,\ (n\in\mathbb{N}:=\{1,2,3,\ldots\})\ \text{and}\ (\lambda)_0 := 1$$

and its well known form is also given in [1] as

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n}\ ,\ (n\in\mathbb{N}:=\{1,2,3,\ldots\}). \tag{1.1}$$

It easily follows from (1.1) that

$$(\lambda)_{m-n} = (\lambda)_m(\lambda+m)_{-n}, \tag{1.2}$$

for $m,n\in\mathbb{N}$.

2. Recursion formulas of G_1

In this section, we give some recursion formulas for the function G_1. We start the following theorem.

Theorem 1. *Recursion formulas for the function G_1 are as follows* :

$$\begin{aligned} G_1(\alpha+n,\beta,\beta';x,y) &= G_1(\alpha,\beta,\beta';x,y) \\ &+ (\beta)_{-1}\beta' x\sum_{k=1}^{n} G_1(\alpha+k,\beta-1,\beta'+1;x,y) \\ &+ \beta(\beta')_{-1}\, y\sum_{k=1}^{n} G_1(\alpha+k,\beta+1,\beta'-1;x,y) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} G_1(\alpha-n,\beta,\beta';x,y) &= G_1(\alpha,\beta,\beta';x,y) \\ &- (\beta)_{-1}\beta' x\sum_{k=0}^{n-1} G_1(\alpha-k,\beta-1,\beta'+1;x,y) \\ &- \beta(\beta')_{-1}\, y\sum_{k=0}^{n-1} G_1(\alpha-k,\beta+1,\beta'-1;x,y) \end{aligned} \tag{2.2}$$

Proof. From the definition of the function G_1 and transformation

$$(\alpha+1)_{m+p} = (\alpha)_{m+p}\left(1+\frac{m}{\alpha}+\frac{p}{\alpha}\right)$$

we can get the following relation:

$$\begin{aligned} G_1(\alpha+1,\beta,\beta';x,y) &= G_1(\alpha,\beta,\beta';x,y) \\ &+ (\beta)_{-1}\beta' x G_1(\alpha+1,\beta-1,\beta'+1;x,y) \\ &+ \beta(\beta')_{-1}\, y G_1(\alpha+1,\beta+1,\beta'-1;x,y). \end{aligned} \tag{2.3}$$

By applying this contiguous relation to function G_1 with the parameter $\alpha+2$, we have

$$\begin{aligned}
&G_1\left(\alpha+2,\beta,\beta';x,y\right)\\
&=G_1\left(\alpha+1,\beta,\beta';x,y\right)+(\beta)_{-1}\beta^{'}xG_1\left(\alpha+2,\beta-1,\beta'+1;x,y\right)\\
&\quad+\beta\left(\beta^{'}\right)_{-1}yG_1\left(\alpha+2,\beta+1,\beta'-1;x,y\right)\\
&=G_1\left(\alpha,\beta,\beta';x,y\right)\\
&\quad+(\beta)_{-1}\beta^{'}x\left[G_1\left(\alpha+1,\beta-1,\beta'+1;x,y\right)+G_1\left(\alpha+2,\beta-1,\beta'+1;x,y\right)\right]\\
&\quad+\beta\left(\beta^{'}\right)_{-1}y[G_1\left(\alpha+1,\beta+1,\beta'-1;x,y\right)+G_1\left(\alpha+2,\beta+1,\beta'-1;x,y\right)]
\end{aligned}$$

$$\begin{aligned}
G_1\left(\alpha+n,\beta,\beta';x,y\right)=G_1\left(\alpha,\beta,\beta';x,y\right)&\\
+(\beta)_{-1}\beta'x\sum_{k=1}^{n}G_1\left(\alpha+k,\beta-1,\beta'+1;x,y\right)&\\
+\beta\left(\beta'\right)_{-1}y\sum_{k=1}^{n}G_1\left(\alpha+k,\beta+1,\beta'-1;x,y\right)&
\end{aligned}$$

If we compute the function G_1 with the parameter $\alpha+n$ by relation (2.3) for n times, we find the formula given by (2.1). Replacing α by $\alpha-1$ in the contiguous relation (2.3), we get

$$\begin{aligned}
G_1\left(\alpha-1,\beta,\beta';x,y\right)=G_1\left(\alpha,\beta,\beta';x,y\right)-(\beta)_{-1}\beta'xG_1\left(\alpha,\beta-1,\beta'+1;x,y\right)&\\
-\beta\left(\beta'\right)_{-1}yG_1\left(\alpha,\beta+1,\beta'-1;x,y\right).&
\end{aligned}$$

If we apply this relation to the function G_1 with the parameter $\alpha-n$ for n times, we obtain the recursion formula (2.2) similar to the proof of formula (2.1). □

By the same contiguous relations (2.1) and (2.2) we can express the functions G_1 in the above theorem in other forms.

Theorem 2. *The function G_1 satisfies the recursion formulas* :

$$\begin{aligned}
G_1\left(\alpha+n,\beta,\beta';x,y\right)=\sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n}{i}\binom{n-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}&\\
\times x^k y^i G_1\left(\alpha+i+k,\beta+i-k,\beta'+k-i;x,y\right)&
\end{aligned}\tag{2.4}$$

$$\begin{aligned}
G_1\left(\alpha-n,\beta,\beta';x,y\right)=\sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n}{i}\binom{n-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}&\\
\times(-x)^k(-y)^i G_1\left(\alpha+i+k,\beta+i-k,\beta'+k-i;x,y\right).&
\end{aligned}\tag{2.5}$$

Proof. By the induction method, we only prove the recursion formula given by (2.4). For $n=1$, formula (2.4) is satisfied. Assume that the result (2.4) is true for $n=t$. Then, we need to show that the relation (2.4) is satisfied for $n=t+1$. Setting $n=t$ in (2.4), we have

$$G_1\left(\alpha+t,\beta,\beta';x,y\right)=\sum_{i=0}^{t}\sum_{k=0}^{t-i}\tbinom{t}{i}\tbinom{t-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^k y^i \\ G_1\left(\alpha+i+k,\beta+i-k,\beta'+k-i;x,y\right).$$

Replacing α by $\alpha+1$ in the above relation, we obtain

$$G_1\left(\alpha+t+1,\beta,\beta';x,y\right)=\sum_{i=0}^{t}\sum_{k=0}^{t-i}\tbinom{t}{i}\tbinom{t-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^k y^i \\ G_1\left(\alpha+i+k+1,\beta+i-k,\beta'+k-i;x,y\right).$$

In the above equality, we apply the contiguous relation (2.3) with the transformations $\alpha\to\alpha+i+k$, $\beta\to\beta+i-k$, $\beta'\to\beta'+k-i$. Using the relations

$$\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k}$$

and

$$\binom{n}{k}=0\ ,\ \ when\ \ k>n\ \ or\ \ k<0$$

and with some simplifications, we deduce

$$\begin{aligned}
&G_1\left(\alpha+t+1,\beta,\beta';x,y\right)\\
&=\sum_{i=0}^{t}\sum_{k=0}^{t-i}\tbinom{t}{i}\tbinom{t-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^k y^i G_1\left(\alpha+i+k,\beta+i-k,\beta'+k-i;x,y\right)\\
&\quad+\sum_{i=0}^{t}\sum_{k=0}^{t-i}\tbinom{t}{i}\tbinom{t-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^{k+1}y^i(\beta+i-k)_{-1}\left(\beta'+k-i\right)_1\\
&\quad G_1\left(\alpha+1+i+k,\beta-1+i-k,\beta'+1+k-i;x,y\right)\\
&\quad+\sum_{i=0}^{t}\sum_{k=0}^{t-i}\tbinom{t}{i}\tbinom{t-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^k y^{i+1}(\beta+i-k)_1\left(\beta'+k-i\right)_{-1}\\
&\quad G_1\left(\alpha+1+i+k,\beta+1+i-k,\beta'-1+k-i;x,y\right)\\
&=\sum_{i=0}^{t+1}\sum_{k=0}^{t+1-i}\tbinom{t+1}{i}\tbinom{t-i+1}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^k y^i
\end{aligned}$$

$$\times G_1\left(\alpha+i+k,\beta+i-k,\beta'+k-i;x,y\right),$$

where we replace k by $k-1$ in the second summation term and i by $i-1$ in the third summation term in the first equality. So, we obtain the recursion formula (2.4). In a similar manner, the relation (2.5) can be easily proved. □

Theorem 3. *For the function G_1, we have*

$$\begin{aligned}G_1\left(\alpha,\beta+n,\beta';x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right) \\ &\quad +\alpha\left(\beta'\right)_{-1} y\sum_{k=1}^{n} G_1\left(\alpha+1,\beta+k,\beta'-1;x,y\right) \\ &\quad -\alpha\beta' x\sum_{k=1}^{n}\frac{(\beta+k-1)_{-1}}{(\beta+k-1)} G_1\left(\alpha+1,\beta+k-2,\beta'+1;x,y\right)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}G_1\left(\alpha,\beta-n,\beta';x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right) \\ &\quad -\alpha\left(\beta'\right)_{-1} y\sum_{k=1}^{n} G_1\left(\alpha+1,\beta-k+1,\beta'-1;x,y\right) \\ &\quad -\alpha\beta' x\sum_{k=1}^{n}\frac{(\beta-k)_{-1}}{(\beta-k)} G_1\left(\alpha+1,\beta-k-1,\beta'+1;x,y\right).\end{aligned} \tag{2.7}$$

Proof. Using the definition of the function G_1 and the equality

$$(\beta+1)_{p-m} = (\beta)_{p-m}\left(1+\frac{p}{\beta}-\frac{m}{\beta}\right)$$

we can easily obtain the contiguous function

$$\begin{aligned}G_1\left(\alpha,\beta+1,\beta';x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right) \\ &\quad +\alpha\left(\beta'\right)_{-1} yG_1\left(\alpha+1,\beta+1,\beta'-1;x,y\right) \\ &\quad -\alpha\frac{(\beta)_{-1}}{\beta}\beta' xG_1\left(\alpha+1,\beta-1,\beta'+1;x,y\right).\end{aligned} \tag{2.8}$$

If we apply this contiguous relation for two times for the function G_1 with the parameter $\beta+2$, we get

$$\begin{aligned}G_1\left(\alpha,\beta+2,\beta';x,y\right) &= G_1\left(\alpha,\beta+1,\beta';x,y\right) \\ &\quad +\alpha\left(\beta'\right)_{-1} yG_1\left(\alpha+1,\beta+2,\beta'-1;x,y\right) \\ &\quad -\alpha\frac{(\beta+1)_{-1}}{(\beta+1)}\left(\beta'\right)_{1} xG_1\left(\alpha+1,\beta,\beta'+1;x,y\right) \\ &= G_1\left(\alpha,\beta,\beta';x,y\right)\end{aligned}$$

$$+\alpha\left(\beta'\right)_{-1}yG_1\left(\alpha+1,\beta+1,\beta'-1;x,y\right)$$
$$-\alpha\frac{(\beta)_{-1}}{\beta}\beta' xG_1\left(\alpha+1,\beta-1,\beta'+1;x,y\right)$$
$$+\alpha\left(\beta'\right)_{-1}yG_1\left(\alpha+1,\beta+2,\beta'-1;x,y\right)$$
$$-\alpha\frac{(\beta+1)_{-1}}{(\beta+1)}\beta' xG_1\left(\alpha+1,\beta,\beta'+1;x,y\right).$$

By iterating this method on G_1 with $\beta+n$ for n times, we find

$$\begin{aligned}G_1\left(\alpha,\beta+n,\beta';x,y\right) &= G_1\left(\alpha,\beta+1,\beta';x,y\right)\\ &+\alpha\left(\beta'\right)_{-1}y\sum_{k=1}^{n}G_1\left(\alpha+1,\beta+k,\beta'-1;x,y\right)\\ &-\alpha\beta' x\sum_{k=1}^{n}\frac{(\beta+k-1)_{-1}}{(\beta+k-1)}G_1\left(\alpha+1,\beta+k-2,\beta'+1;x,y\right).\end{aligned}$$

Replacing β by $\beta-1$ in contiguous relation (2.8), we obtain

$$\begin{aligned}G_1\left(\alpha,\beta-1,\beta';x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right)-\alpha\left(\beta'\right)_{-1}yG_1\left(\alpha+1,\beta,\beta'-1;x,y\right)\\ &+\alpha\frac{(\beta-1)_{-1}}{(\beta-1)}\beta' xG_1\left(\alpha+1,\beta-2,\beta'+1;x,y\right).\end{aligned}$$

If we apply this relation to the function G_1 with the parameter $\beta-n$ for n times, we obtain the recursion formulas (2.7). □

Theorem 4. *For the function G_1, the equalities*

$$\begin{aligned}G_1\left(\alpha,\beta,\beta'+n;x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right)\\ &+\alpha(\beta)_{-1}x\sum_{k=1}^{n}G_1\left(\alpha+1,\beta-1,\beta'+k;x,y\right)\\ &-\alpha\beta y\sum_{k=1}^{n}\frac{(\beta'+k-1)_{-1}}{(\beta'+k-1)}G_1\left(\alpha+1,\beta+1,\beta'+k-2;x,y\right)\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}G_1\left(\alpha,\beta,\beta'-n;x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right)\\ &-\alpha(\beta)_{-1}x\sum_{k=1}^{n}G_1\left(\alpha+1,\beta-1,\beta'-k+1;x,y\right)\\ &+\alpha\beta y\sum_{k=1}^{n}\frac{(\beta'-k)_{-1}}{(\beta'-k)}G_1\left(\alpha+1,\beta+1,\beta'-k-1;x,y\right)\end{aligned}\tag{2.10}$$

hold.

Proof. If we use following equalities

$$\begin{aligned} G_1\left(\alpha,\beta,\beta'+1;x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right) \\ &\quad +\alpha\left(\beta\right)_{-1} x G_1\left(\alpha+1,\beta-1,\beta'+1;x,y\right) \\ &\quad -\alpha\beta\frac{(\beta')_{-1}}{\beta'} y G_1\left(\alpha+1,\beta+1,\beta'-1;x,y\right) \\ G_1\left(\alpha,\beta,\beta'-1;x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right) \\ &\quad -\alpha\left(\beta\right)_{-1} x G_1\left(\alpha+1,\beta-1,\beta';x,y\right) \\ &\quad +\alpha\beta\frac{(\beta'-1)_{-1}}{(\beta'-1)} y G_1\left(\alpha+1,\beta+1,\beta'-2;x,y\right) \end{aligned}$$

we obtain recursion formulas given by (2.9) and (2.10). □

3. Recursion formulas of G_2

In this section, we give some recursion formulas for the function G_2. We first present the recursion formulas for the function G_2 about the parameter α and α'.

Theorem 5. *The function G_2 satisfies the recursion formulas :*

$$\begin{aligned} G_2\left(\alpha+n,\alpha',\beta,\beta';x,y\right) &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) \\ &\quad +(\beta)_{-1}\beta' x\sum_{k=1}^{n} G_2\left(\alpha+k,\alpha',\beta-1,\beta'+1;x,y\right) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} G_2\left(\alpha-n,\alpha',\beta,\beta';x,y\right) &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) \\ &\quad -(\beta)_{-1}\beta' x\sum_{k=0}^{n-1} G_2\left(\alpha-k,\alpha',\beta-1,\beta'+1;x,y\right). \end{aligned} \tag{3.2}$$

Proof. By the definition of the function G_2, we get

$$\begin{aligned} G_2\left(\alpha+1,\alpha',\beta,\beta';x,y\right) &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) \\ &\quad +(\beta)_{-1}\beta' x G_2\left(\alpha+1,\alpha',\beta-1,\beta'+1;x,y\right). \end{aligned}$$

Applying this relation n times recursively, as the same as we have done in the proof of Theorem 1, we immediately have complete the proof (3.1). □

The function G_2 in the above theorem can be expressed in other forms as follows:

Theorem 6. *For the function G_2, the equalities*

$$G_2\left(\alpha+n,\alpha',\beta,\beta';x,y\right)$$

$$=\sum_{k=0}^{n}\binom{n}{k}(\beta)_{-k}\left(\beta'\right)_{k}x^{k}G_2\left(\alpha+k,\alpha',\beta-k,\beta'+k;x,y\right) \tag{3.3}$$

and

$$G_2\left(\alpha-n,\alpha',\beta,\beta';x,y\right) =\sum_{k=0}^{n}\binom{n}{k}(\beta)_{-k}\left(\beta'\right)_{k}(-x)^{k}G_2\left(\alpha,\alpha',\beta-k,\beta'+k;x,y\right), \tag{3.4}$$

hold.

Proof. By the inductive method as we have done in the proof of Theorem 2, the proof can be easily seen. □

Theorem 7. *The function G_2 satisfies the following recursion formulas*

$$\begin{aligned} G_2\left(\alpha,\alpha'+n,\beta,\beta';x,y\right) &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) \\ &\quad +\beta\left(\beta'\right)_{-1}y\sum_{k=1}^{n}G_2\left(\alpha,\alpha'+k,\beta+1,\beta'-1;x,y\right) \\ &=\sum_{k=0}^{n}\binom{n}{k}(\beta)_{k}\left(\beta'\right)_{-k}y^{k}G_2\left(\alpha,\alpha'+k,\beta+k,\beta'-k;x,y\right) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} G_2\left(\alpha,\alpha'-n,\beta,\beta';x,y\right) &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) \\ &\quad -\beta\left(\beta'\right)_{-1}y\sum_{k=0}^{n-1}G_2\left(\alpha,\alpha'-k,\beta+1,\beta'-1;x,y\right) \\ &=\sum_{k=0}^{n}\binom{n}{k}(\beta)_{k}\left(\beta'\right)_{-k}(-y)^{k}G_2\left(\alpha,\alpha',\beta+k,\beta'-k;x,y\right) \end{aligned} \tag{3.6}$$

Proof. By the definition of the function G_2, we get

$$\begin{aligned} G_2\left(\alpha,\alpha'+1,\beta,\beta';x,y\right) &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) \\ &\quad +\beta\left(\beta'\right)_{-1}yG_2\left(\alpha,\alpha'+1,\beta+1,\beta'-1;x,y\right) \end{aligned}$$

and

$$\begin{aligned} G_2\left(\alpha,\alpha'-1,\beta,\beta';x,y\right) &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) \\ &\quad -\beta\left(\beta'\right)_{-1}yG_2\left(\alpha,\alpha',\beta+1,\beta'-1;x,y\right). \end{aligned}$$

By applying this relation for n times, as the same as we have done in the proof of Theorem 1, we complete the proof. □

Now, we present the recursion formulas of the function G_2 about the parameter β and β'.

Theorem 8. *Recursion formulas for the function G_2 are as follows*

$$\begin{aligned}&G_2\left(\alpha,\alpha',\beta+n,\beta';x,y\right) \\ &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right)+\alpha'\left(\beta'\right)_{-1}y\sum_{k=1}^{n}G_2\left(\alpha,\alpha'+1,\beta+k,\beta'-1;x,y\right) \\ &\quad -\alpha\beta' x\sum_{k=1}^{n}\frac{(\beta+k-1)_{-1}}{(\beta+k-1)}G_2\left(\alpha+1,\alpha',\beta+k-2,\beta'+1;x,y\right)\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&G_2\left(\alpha,\alpha',\beta-n,\beta';x,y\right) \\ &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right)-\alpha'\left(\beta'\right)_{-1}y\sum_{k=1}^{n}G_2\left(\alpha,\alpha'+1,\beta-k+1,\beta'-1;x,y\right) \\ &\quad +\alpha\beta' x\sum_{k=1}^{n}\frac{(\beta-k)_{-1}}{(\beta-k)}G_2\left(\alpha+1,\alpha',\beta-k-1,\beta'+1;x,y\right).\end{aligned} \tag{3.8}$$

Proof. From the definition of the function G_2, we have the following relation:

$$\begin{aligned}&G_2\left(\alpha,\alpha',\beta+1,\beta';x,y\right) \\ &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right)+\alpha'\left(\beta'\right)_{-1}yG_2\left(\alpha,\alpha'+1,\beta+1,\beta'-1;x,y\right) \\ &\quad -\alpha\beta' x\frac{(\beta)_{-1}}{\beta}G_2\left(\alpha+1,\alpha',\beta-1,\beta'+1;x,y\right).\end{aligned}$$

If we apply this relation for n times, we can easily prove the theorem by the same method as we have done in the proof of Theorem 3. □

Theorem 9. *Recursion formulas for the function G_2 are as follows*

$$\begin{aligned}&G_2\left(\alpha,\alpha',\beta,\beta'+n;x,y\right) \\ &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right)+\alpha\left(\beta\right)_{-1}x\sum_{k=1}^{n}G_2\left(\alpha+1,\alpha',\beta-1,\beta'+k;x,y\right) \\ &\quad -\alpha'\beta y\sum_{k=1}^{n}\frac{(\beta'+k-1)_{-1}}{(\beta'+k-1)}G_2\left(\alpha,\alpha'+1,\beta+1,\beta'+k-2;x,y\right)\end{aligned} \tag{3.9}$$

and

$$G_2\left(\alpha,\alpha',\beta,\beta'-n;x,y\right) \tag{3.10}$$

$$= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) - \alpha\left(\beta\right)_{-1} x \sum_{k=1}^{n} G_2\left(\alpha+1,\alpha',\beta-1,\beta'-k+1;x,y\right)$$
$$+\alpha'\beta y \sum_{k=1}^{n} \frac{(\beta'-k)_{-1}}{(\beta'-k)} G_2\left(\alpha,\alpha'+1,\beta+1,\beta'-k-1;x,y\right).$$

Proof. From the definition of the function G_2, we have the following relation:

$$G_2\left(\alpha,\alpha',\beta,\beta'+n;x,y\right)$$
$$= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) + \alpha\left(\beta\right)_{-1} x G_2\left(\alpha+1,\alpha',\beta-1,\beta'+1;x,y\right)$$
$$-\alpha'\beta y \frac{(\beta')_{-1}}{\beta'} G_2\left(\alpha,\alpha'+1,\beta+1,\beta'-1;x,y\right).$$

If we apply this relation for n times, we can easily prove the theorem by the same method as we have done in the proof of Theorem 3. □

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