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# INVERSION FORMULAS FOR GRAPHS

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*Abstract.* In this note we derive some combinatorial identities from inversion formulas in the completion of the dual of graph Hopf algebra. As a consequence some identities involving Stirling and Bell numbers are obtained.

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## 1. INTRODUCTION

The graph Hopf algebra $\mathcal{G}$, introduced by Schmitt [3] as the incidence Hopf algebra of graphs, is a well known example of a combinatorial Hopf algebra. It is also called chromatic Hopf algebra in [1], because of its relation with chromatic polynomials of graphs.

The graded Hopf algebra dual $\mathcal{G}^*$ of the graph Hopf algebra $\mathcal{G}$ is isomorphic to the polynomial Hopf algebra $k[\mathbb{G}_{\mathrm{conn}}]$ generated by connected simple graphs. This determines the completion $\widehat{\mathcal{G}}^*$ of the graded dual of the graph Hopf algebra as the Hopf algebra of formal power series $k[[\mathbb{G}_{\mathrm{conn}}]]$ in variables corresponding to connected graphs. For a class of graphs $\mathcal{C}$ is defined the characteristic element $c(\mathcal{C}) \in \widehat{\mathcal{G}}^*$. We define elements which count the numbers of ordered and unordered decompositions of graphs onto subgraphs from the class $\mathcal{C}$. By using inversion formulas in the algebra $\widehat{\mathcal{G}}^*$ we obtain some combinatorial identities for graphs which are satisfied by these numbers. As a consequence, some numerical identities involving the Stirling numbers of the second kind and the ordered Bell numbers are obtained.

## 2. HOPF ALGEBRA OF GRAPHS $\mathcal{G}$

For a graph $\Gamma$ denote by $V(\Gamma)$ and $E(\Gamma)$ the sets of vertices and edges. All graphs considered are simple, i.e. without multiple edges and loops. By $|\Gamma|$ we denote the number of vertices of a graph $\Gamma$. Let $\Gamma|_I$ be the induced subgraph of a graph $\Gamma$ on the set of vertices $I \subset V(\Gamma)$.

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For graphs $\Gamma$ and $\Gamma_1, \ldots, \Gamma_k$, define $\binom{\Gamma}{\Gamma_1,\ldots,\Gamma_k}$ to be the number of ordered set partitions $I : I_1 \sqcup \ldots \sqcup I_k = V(\Gamma)$ such that $\Gamma|_{I_j}$ is isomorphic to $\Gamma_j$ for all $j = 1, \ldots, k$.

2.1. *Hopf algebras*

For a detailed exposition of Hopf algebras see [5]. Fix a field $k$. A bialgebra $\mathcal{H}$ is a vector space over $k$ equipped with linear maps

$$m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \text{ and } \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H},$$

respectively the multiplication and comultiplication, such that the following properties are satisfied:

(1) $(\mathcal{H}, m, u)$ is an associative algebra with the unit $u : k \to \mathcal{H}$

(2) $(\mathcal{H}, \Delta, \epsilon)$ is a coassociative coalgebra with the counit $\epsilon : \mathcal{H} \to k$

(3) $\Delta$ and $\epsilon$ are multiplicative morphisms (equivalently, $m$ and $u$ are comultiplicative morphisms). If there exists a bialgebra automorphism $S : \mathcal{H} \to \mathcal{H}$ such that

$$m \circ (S \otimes I) \circ \Delta = m \circ (I \otimes S) \circ \Delta = u \circ \epsilon,$$

where $I : \mathcal{H} \to \mathcal{H}$ is the identity map, then $\mathcal{H}$ is a Hopf algebra and $S$ is its antipode. A Hopf algebra $\mathcal{H}$ is graded if $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ and the multiplication and comultiplication respect this decomposition

$$m(\mathcal{H}_i \otimes \mathcal{H}_j) \subset \mathcal{H}_{i+j}, \ \Delta(\mathcal{H}_k) \subset \sum_{i+j=k} \mathcal{H}_i \otimes \mathcal{H}_j.$$

$\mathcal{H}$ is connected if $dim(\mathcal{H}_0) = 1$. A graded connected bialgebra $\mathcal{H}$ posses the antipode $S$ determined recursively as follows: $S(h) = h$ for $h \in \mathcal{H}_0$, and $(m \circ (S \otimes I) \circ \Delta)(h) = 0$ for $h \in \mathcal{H}_i, i > 0$.

*Example* 1. Let $X = \{x_1, x_2, \ldots\}$ be a countable set with the rank function $rk : X \to \mathbb{N}$ and $k[X]$ be the polynomial algebra over a field $k$ generated by $X$. Define the comultiplication $\Delta : k[X] \to k[X] \otimes k[X]$ on variables by $\Delta(x_n) = 1 \otimes x_n + x_n \otimes 1, n \geq 1$ and extend it algebraically to $k[X]$. It turns $k[X]$ into a graded, connected Hopf algebra over $k$. The antipode $S : k[X] \to k[X]$ is uniquely determined by $S(x_n) = -x_n$ for all $n$. The Hopf algebra $k[X]$ is called the *polynomial Hopf algebra* generated by the set $X$.

Recall the definition of the graph Hopf algebra $\mathcal{G}$, introduced in [3]. It is linearly spanned by all isomorphism classes of finite simple graphs. A graduation $\mathcal{G} = \bigoplus_{n\geq 0} \mathcal{G}_n$ is given by the number of vertices. The space $\mathcal{G}$ is a Hopf algebra with the multiplication defined by disjoint union of graphs $\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \sqcup \Gamma_2$ and the comultiplication

$$\Delta(\Gamma) = \sum_{I \subset V(\Gamma)} \Gamma|_I \otimes \Gamma|_{V(\Gamma)\setminus I}. \tag{2.1}$$

The Hopf algebra $\mathcal{G}$ is graded, connected, commutative and cocommutative. The antipode $S : \mathcal{G} \to \mathcal{G}$ is determined by Takeuchi's general formula

$$S(\Gamma) = \sum_{k \geq 1} (-1)^k \sum_{J_1 \sqcup \ldots \sqcup J_k = V(\Gamma)} \prod_{j=1}^{k} \Gamma|_{J_j},$$

where the inner sum is over all ordered set partitions $(J_1, \ldots, J_k)$ of the set of vertices $V(\Gamma)$. A more combinatorial formula involving acyclic orientations on graphs is obtained recently in [2].

$\mathcal{G}$ is algebraically isomorphic to the polynomial algebra $k[\mathbb{G}_{\text{conn}}]$ generated by the family of all isomorphism classes of connected simple graphs.

Let $\mathcal{G}^* = \bigoplus_{n \geq 0} \mathcal{G}_n^*$ be the graded dual of the Hopf algebra of graphs $\mathcal{G}$. If we denote the value of $\delta \in \mathcal{G}^*$ on $\Gamma \in \mathcal{G}$ by $\langle \delta, \Gamma \rangle$, the multiplication and comultiplication on $\mathcal{G}^*$ are determined by the identities

$$\langle \delta\eta, \Gamma \rangle = m \circ (\delta \otimes \eta) \circ \Delta(\Gamma),$$
$$\langle \Delta(\delta), \Gamma_1 \otimes \Gamma_2 \rangle = \delta(\Gamma_1 \Gamma_2).$$

The Hopf algebra $\mathcal{G}^*$ is commutative and cocommutative with the set of linear generators formed by all $\delta_\Gamma$, where

$$\langle \delta_\Gamma, \Gamma' \rangle = \begin{cases} 1, & \Gamma = \Gamma' \\ 0, & \text{otherwise} \end{cases}.$$

The following product formula holds

$$\delta_{\Gamma_1} \cdots \delta_{\Gamma_k} = \sum_{|\Gamma| = n} \binom{\Gamma}{\Gamma_1, \ldots, \Gamma_k} \delta_\Gamma, \tag{2.2}$$

where $n = |\Gamma_1| + \cdots + |\Gamma_k|$. Note that taking duals is not an algebra morphism since $\delta_{\Gamma_1}\delta_{\Gamma_2} \neq \delta_{\Gamma_1\Gamma_2}$, but the restriction to connected graphs generates the algebra morphism $\Phi : k[\mathbb{G}_{\text{conn}}] \to \mathcal{G}^*$. It is actually an isomorphism of Hopf algebras which is a consequence of Schmitt's work on Whitney systems [4].

**Theorem 1.** *The graded dual $\mathcal{G}^*$ of the graph Hopf algebra $\mathcal{G}$ is isomorphic to the polynomial Hopf algebra $k[\mathbb{G}_{\text{conn}}]$ generated by connected simple graphs.*

## 3. Inversion formulas

Let $\widehat{\mathcal{G}}^*$ be the completion of the graded dual $\mathcal{G}^*$ of the graph Hopf algebra $\mathcal{G}$. Theorem 1 implies that $\widehat{\mathcal{G}}^*$ is isomorphic to the Hopf algebra of formal power series $k[[\mathbb{G}_{\text{conn}}]]$. Given an element $x \in \widehat{\mathcal{G}}^*$, by $[n]x$ we denote its $n$-th homogeneous summand.

For a class of graphs $\mathcal{C}$ let $c(\mathcal{C}) = \sum_{\Gamma \in \mathcal{C}} \delta_\Gamma$ be its characteristic element

$$c(\mathcal{C})(\Gamma) = \begin{cases} 1, & \Gamma \in \mathcal{C} \\ 0, & \text{otherwise} \end{cases}.$$

Define

$$u(\mathcal{C}) = \sum_{k\geq 0} c(\mathcal{C})^k = \frac{\epsilon}{\epsilon - c(\mathcal{C})} \quad \text{and} \quad \exp(c(\mathcal{C})) = \sum_{k\geq 0} \frac{1}{k!} c(\mathcal{C})^k.$$

These elements have the following combinatorial meanings.

**Lemma 1.** *The values $u(\mathcal{C})(\Gamma)$ and $\exp(c(\mathcal{C}))(\Gamma)$ are the numbers of all ordered and unordered decompositions of a graph $\Gamma$ onto subgraphs from the class $\mathcal{C}$.*

*Proof.* The $n$-th homogeneous summand of the $k$-power of the characteristic element $c(\mathcal{C})$ is given by

$$[n]c(\mathcal{C})^k = \sum_{\substack{\Gamma_1,\dots,\Gamma_k \in \mathcal{C} \\ |\Gamma_1|+\cdots+|\Gamma_k| = n}} \delta_{\Gamma_1}\cdots\delta_{\Gamma_k}.$$

Therefore, for a graph $\Gamma$ on $n$ vertices we have by the product formula (2.2)

$$c(\mathcal{C})^k(\Gamma) = \sum_{\substack{\Gamma_1,\dots,\Gamma_k \in \mathcal{C} \\ |\Gamma_1|+\cdots+|\Gamma_k| = n}} \binom{\Gamma}{\Gamma_1\cdots\Gamma_k}, \tag{3.1}$$

which is precisely the number of all ordered decompositions of the graph $\Gamma$ onto $k$ subgraphs from the class $\mathcal{C}$. The lemma follows from definitions of $u(\mathcal{C})$ and $\exp(c(\mathcal{C}))$. □

**Theorem 2.** *Given a class $\mathcal{C}$ of simple graphs and a graph $\Gamma$ on the vertex set $V \neq \varnothing$, then*

$$c(\mathcal{C})(\Gamma) = \sum_{I_1\sqcup\ldots\sqcup I_k = V} (-1)^{k-1} \prod_{j=1}^{k} u(\mathcal{C})(\Gamma|_{I_j}), \tag{3.2}$$

$$c(\mathcal{C})(\Gamma) = \sum_{I_1\sqcup\ldots\sqcup I_k = V} \frac{(-1)^{k-1}}{k} \prod_{j=1}^{k} \exp(c(\mathcal{C}))(\Gamma|_{I_j}), \tag{3.3}$$

*where the sums are over all ordered decompositions of $V$.*

*Proof.* The statement follows from the inversion formulas

$$c(\mathcal{C}) = \frac{u(\mathcal{C}) - \epsilon}{u(\mathcal{C})} = \sum_{k>0} (-1)^{k-1}(u(\mathcal{C}) - \epsilon)^k,$$

$$c(\mathcal{C}) = \log(\epsilon + (\exp(c(\mathcal{C})) - \epsilon)) = \sum_{k>0} \frac{(-1)^{k-1}}{k} (\exp(c(\mathcal{C})) - \epsilon)^k,$$

by calculating at the graph $\Gamma$. □

For an integer $n$ let $\alpha = (i_1, \dots, i_k) \models n$ be a composition, i.e. $i_j, j = 1, \dots, k$ are positive integers and $i_1 + \cdots + i_k = n$. Denote by $\alpha(j) = i_j$ the $j$-th part of $\alpha$ and by $k(\alpha) = k$ the number of its parts. Let $\binom{n}{\alpha} = \frac{n!}{i_1! i_2! \cdots i_k!}$ be the multinomial coefficient. It counts the number of all ordered set partitions $(I_1, \dots, I_k)$ of the finite set $V$ with prescribed sizes of parts $|I_1| = i_1, \dots, |I_k| = i_k$. Recall that a *proper coloring* of a graph $\Gamma$ with at most $k$ colors is the map $f : V(\Gamma) \to [k]$ such that there are no monochromatic edges, i.e. $f(v_1) \neq f(v_2)$ if $\{v_1, v_2\}$ is an edge of $\Gamma$. The chromatic polynomial $\chi(\Gamma, k)$ counts the numbers of proper colorings of a graph $\Gamma$.

**Theorem 3.**

$$\sum_{I_1 \sqcup \ldots \sqcup I_k = V(\Gamma)} (-1)^{k-1} \prod_{j=1}^{k} \chi(\Gamma|_{I_j}, |I_j|) = \begin{cases} 1, & \Gamma \text{ is discrete} \\ 0, & \text{otherwise} \end{cases}, \tag{3.4}$$

*where the sum is over all ordered set partitions $(I_1, \dots, I_k)$ of the vertex set $V(\Gamma)$.*

*Proof.* Let $\mathcal{C} = \{D_n\}_{n>0}$ be the class of all discrete graphs. Note that $\binom{\Gamma}{D_{i_1} \cdots D_{i_k}}$ is the number of all proper colorings of $\Gamma$ with exactly $k$ colors such that a color $j$ is taken $i_j$ times, for all $j = 1, \dots, k$. Therefore, from (3.1), we have that $c(\mathcal{C})^k(\Gamma)$ counts the number of all proper colorings with exactly $k$ colors and $(u(\mathcal{C}) - \epsilon)(\Gamma) = \chi(\Gamma, |\Gamma|)$. We apply the inversion formula (3.2) to obtain (3.4). □

Recall that the Stirling number of the second kind

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{\alpha \models n : k(\alpha) = k} \binom{n}{\alpha}$$

is the number of all set partitions of an $n$-element set into $k$ parts. The ordered Bell number

$$F(n) = \sum_{\alpha \models n} \binom{n}{\alpha}$$

counts the total number of ordered set partitions on an $n$-element set. The following corollaries illustrate how Theorems 2 and 3 lead to some identities involving Stirling and Bell numbers.

**Corollary 1.** *The following identity holds for an integer $n$*

$$\sum_{\alpha \models n} \frac{(-1)^{k(\alpha)-1}}{k(\alpha)} \binom{n}{\alpha} = 0. \tag{3.5}$$

*Proof.* Let $\mathcal{C} = \{D_1\}$, where $D_1$ is the graph on the single vertex. Then $c(\mathcal{C}) = \delta_{D_1}$ and $\exp(\delta_{D_1})(\Gamma) = 1$ for any graph $\Gamma$. The identity follows from the inversion formula (3.3) applied on an arbitrary graph $\Gamma$ on $n \neq 1$ vertices. □

Note that the identity (3.5) may be rewritten as

$$\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = 0.$$

**Corollary 2.** *Given an integer n, the following identity holds*

$$\sum_{\alpha \models n}(-1)^{k(\alpha)-1}\binom{n}{\alpha}\prod_{j=1}^{k(\alpha)} F(\alpha(j))) = 1.$$

*Proof.* Set $\Gamma = D_n$ into the formula (3.4) and note that $\chi(D_j, j) = F(j)$. □

## REFERENCES


[1] M. Aguiar, N. Bergeron, and F. Sottile, "Combinatorial hopf algebras and generalized dehn-sommerville relations," *Compositio Mathematica*, vol. 142, pp. 1–30, 2006, doi: 10.1112/S0010437X0500165X.

[2] B. Humpert and J. L. Martin, "The incidence hopf algebra of graphs," *SIAM J. on Discrete Math.*, vol. 26, no. 2, pp. 555–570, 2012, doi: 10.1137/110820075.

[3] W. R. Schmitt, "Incidence hopf algebras," *J. Pure Appl. Algebra*, vol. 96, no. 3, pp. 299–330, 1994, doi: 10.1016/0022-4049(94)90105-8.

[4] W. R. Schmitt, "Hopf algebra methods in graph theory," *J. Pure Appl. Algebra*, vol. 101, no. 1, pp. 77–90, 1995, doi: 10.1016/0022-4049(95)90925-B.

[5] M. E. Sweedler, *Hopf algebras*, ser. Mathematics Lecture Notes Series. New York: Benjamin, 1969.



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