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SECTIONAL SWITCHING MAPPINGS IN SEMILATTICES

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ABSTRACT. Consider a $\vee$-lattice $\mathcal{S} = (S, \vee)$ with a greatest element 1. An interval $[a, 1]$ for $a \in S$ is called a section. A mapping f of $[a, 1]$ onto itself is called a switching mapping if $f(a) = 1$, $f(1) = a$ and for $x \in [a, 1]$, $a \neq x \neq 1$ we have $a \neq f(x) \neq 1$. We study $\vee$-lattices with switching mappings on all the sections. If for $p, q \in S$, $p \leq q$ the mapping on the section $[q, 1]$ is determined by that of $[p, 1]$, we say that the compatibility condition is satisfied. We will get conditions for antitony of switching mappings and a connection with complementation in sections will be shown.

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1. BASIC CONCEPTS

A mapping $f : A \to A$ is called an *involution* if $f(f(x)) = x$ for each $x \in A$. Let $(A, \leq)$ be an ordered set. A mapping $f : A \to A$ is *antitone* if $x \leq y$ implies $f(y) \leq f(x)$ for all $x, y \in A$. Let $\mathcal{S} = (S, \vee, 1)$ be a join-semilattice with the greatest element 1. For $a \in S$, the interval $[a, 1]$ will be called a *section* (of $\mathcal{S}$). We will study semilattices with 1 where for each $a \in S$ there is a mapping on the section $[a, 1]$; such a structure will be called a *semilattice with sectional mappings*. To distinguish among these mappings, we introduce the following notation:

for each $a \in S$ and $x \in [a, 1]$, denote by x^a the image of x in this sectional mapping on $[a, 1]$. Thus $x \mapsto x^a$ is a symbol for the corresponding sectional mapping on the section $[a, 1]$.

Hence, semilattices with sectional mappings can be considered as algebras with partial unary operations $x \mapsto x^a$ whose number is equal to the cardinality of S. To avoid the difficulty with the types of these partical algebras and to transform them into total algebras, let us introduce the following:

Let $\mathcal{S} = (S, \vee, 1)$ be a semilattice with sectional mappings. Define the so-called *induced operation* on S by the rule $x \cdot y = (x \vee y)^y$.

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Evidently, "$\cdot$" is everywhere defined binary operation on S since $x \vee y \in [y, 1]$ for any $x, y \in S$. Also conversely, if "$\cdot$" is induced on S then for each $a \in S$ and $x \in [a, 1]$ we have $x \cdot a = (x \vee a)^a = x^a$.

Hence, the induced operation determines all the sectional mappings. Due to this, semilattices with sectional mappings will be considered as algebras of type $(2, 2, 0)$ in the signature $\langle \vee, \cdot, 1 \rangle$. For certain properties of sectional mappings we will specify the corresponding properties of the induced operation and vice versa.

Lemma 1. *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional involutions. The following conditions are equivalent for $a \in S$:*

(a) *$x \mapsto x^a$ is antitone,*

(b) *the section $[a, 1]$ is a lattice where*

$$x \wedge_a y = (x^a \vee y^a)^a \qquad \textit{(De Morgan Law).}$$

PROOF. (a) $\Rightarrow$ (b): Since the sectional mapping on $[a, 1]$ is an antitone involution, it is a bijection and $x, y \leq x \vee y$ implies $x^a, y^a \geq (x \vee y)^a$ and the existence of supremum for $x, y \in [a, 1]$ yields existence of the infimum $x \wedge_a y$. Hence, $x^a \wedge_a y^a \geq (x \vee y)^a$. However, $x^a, y^a \geq x^a \wedge_a y^a$ thus, due to $x = x^{aa}$, $y = y^{aa}$, we obtain $x, y \leq (x^a \wedge_a y^a)^a$ whence $x \vee y \leq (x^a \wedge_a y^a)^a$, i.e. $(x \vee y)^a \geq x^a \wedge_a y^a$. Altogether, we obtain (b).

(b) $\Rightarrow$ (a): Let $x, y \in [a, 1]$ and suppose $x \leq y$. Then $x \vee y = y$ and, by (b), $y^a = (x \vee y)^a = x^a \wedge_a y^a$ thus $y^a \leq x^a$, i.e. the sectional mapping on $[a, 1]$ is antitone. □

2. SWITCHING MAPPINGS

We say that a mapping $x \mapsto x^a$ on the section $[a, 1]$ is *weakly switching* if $a^a = 1$ and $1^a = a$. In other words, a weakly switching mapping "switches" the bound elements of the section.

Lemma 2. *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional mappings.*

(a) *If for each $p \in S$ the sectional mapping $x \mapsto x^p$ is an involution, then the induced operation satisfies the identity*

$$(x \cdot y) \cdot y = (y \cdot x) \cdot x = x \vee y. \tag{I1}$$

(b) *If for each $p \in S$ the sectional mapping $x \mapsto x^p$ is weakly switching and the induced operation satisfies* (I1)*, then every sectional mapping is an involution.*

PROOF. (a) Since $x \vee y \in [y, 1]$ we have $x \cdot y = (x \vee y)^y \geq y$. Thus, if the sectional mapping is an involution, we infer

$$(x \cdot y) \cdot y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y,$$

whence (I1) is evident.

(b) Let each sectional mapping be weakly switching, let $p \in S$ and $x \in [p, 1]$. Then $x \vee p = x$ and, by (I1),

$$x^{pp} = (x \cdot p) \cdot p = (p \cdot x) \cdot x = ((p \vee x)^x \vee x)^x = (x^x \vee x)^x = (1 \vee x)^x = 1^x = x$$

and thus $x \mapsto x^p$ is an involution. □

Remark 1. Identity (I1) is called *quasi-commutativity* in [1, 2].

A weakly switching mapping $x \mapsto x^p$ will be called a *switching mapping* if $a \neq x^a \neq 1$ for each $x \in [a, 1]$ with $a \neq x \neq 1$.

Remark 2. Every join-semilattice $\mathcal{S} = (S, \vee, 1)$ with a greatest element can be considered as a semilattice with sectional switching mappings. One can take for each $a \in S$ and every $x \in [a, 1]$ $a^a = 1$, $1^a = a$ and $x^a = x$ for $a \neq x \neq 1$. Hence, our concept is really universal and very natural for semilattices.

Lemma 3. *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional switching mappings, let $\leq$ be its induced order. Then $x \leq y$ if and only if $x \cdot y = 1$.*

PROOF. If $x \leq y$, then $x \cdot y = (x \vee y)^y = y^y = 1$. Conversely, if $x \cdot y = 1$, then $(x \vee y)^y = 1$ thus, since it is a switching mapping, $x \vee y = y$, whence $x \leq y$. □

The following lemma is almost evident.

Lemma 4. *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional weakly switching mappings. Then $\mathcal{S}$ satisfies the identities*

$$x \cdot x = 1, \qquad 1 \cdot x = x, \qquad x \cdot 1 = 1. \tag{I2}$$

Theorem 1. *The class of all semilattices with sectional switching mappings is congruence distributive and weakly regular.**

PROOF. Consider the binary terms $r_1(x, y) = x \cdot y$ and $r_2(x, y) = y \cdot x$. By Lemma 4, $r_1(x, x) = r_2(x, x) = 1$. Converesely, let $r_1(x, y) = r_2(x, y) = 1$. By Lemma 3, it yields $x \leq y$ and $y \leq x$ thus $x = y$. Applying Theorem 6.4.3 of [4] (the Csákány Theorem), we conclude that the class $\mathcal{W}$ of all semilattices with sectional switching mappings is weakly regular.

Moreover, for $t(x, y) = y \cdot x$ we have $t(x, x) = 1$ and $t(1, x) = 1$ (by Lemma 4); thus, by Theorem 8.3.2 of [4], the class $\mathcal{W}$ is arithmetical in 1 and hence distributive in 1, i. e., $[1]_{\Theta \wedge (\Phi \vee \Psi)} = [1]_{(\Theta \wedge \Phi) \vee (\Theta \wedge \Psi)}$ for any $\mathcal{A} \in \mathcal{W}$ and $\Theta, \Phi, \Psi \in \operatorname{Con} \mathcal{A}$. Since $\mathcal{W}$ is weakly regular, this yields the congruence distributivity of $\mathcal{W}$. □

We are interesting in the question when sectional switching mappings are antitone. For this, we use the identity involved in [3] (see also [5, 6]).

Theorem 2. *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional switching mappings.*

*That is, if $\Theta, \Phi \in \operatorname{Con} \mathcal{S}$ and $[1]_\Theta = [1]_\Phi$ then $\Theta = \Phi$.

(a) *If $\mathcal{S}$ satisfies the identity*

$$(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1 \tag{I3}$$

then every switching mapping on $\mathcal{S}$ is antitone.

(b) *If every sectional switching mapping on $\mathcal{S}$ is an involution then it is antitone if and only if $\mathcal{S}$ satisfies* (I3).

PROOF. (a) Suppose $z \in S$, $x, y \in [z, 1]$ and $x \leq y$. By Lemma 3 we have $x \cdot y = 1$ and, by Lemma 4 and (I3) we infer

$$(y \cdot z) \cdot (x \cdot z) = ((1 \cdot y) \cdot z) \cdot (x \cdot z) = (((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1.$$

By Lemma 3 we have $y \cdot z \leq x \cdot z$ and thus $y^z = y \cdot z \leq x \cdot z = x^z$.

(b) Let the sectional switching mappings on $\mathcal{S}$ are antitone involutions. By Lemma 2 we have $(x \cdot y) \cdot y = x \vee y$.

Since $x \vee y \vee z \geq x \vee z$ and $x \vee y \vee z, x \vee z \in [z, 1]$, we obtain $((x \cdot y) \cdot y) \cdot z = (x \vee y \vee z)^z \leq (x \vee z)^z = x \cdot z$. By Lemma 3 we infer $(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1$. The converse is given by (a). □

3. THE COMPATIBILITY CONDITION

We will consider a semilattice with sectional mappings where the mapping in a smaller section is determined by that of a bigger one. More precisely, we say that $\mathcal{S} = (S, \vee, \cdot, 1)$ satisfies the *compatibility condition* if

$$p \leq q \leq x \text{ implies that } x^q = x^p \vee q. \tag{CC}$$

It is easy to verify that (CC) can be equivalently expressed as the following identity

$$(y \vee z) \cdot (x \vee y) = ((y \vee z) \cdot x) \vee (x \vee y) \tag{CCI}$$

since $x \leq x \vee y \leq x \vee y \vee z$ and $(y \vee z) \cdot (x \vee y) = (x \vee y \vee z)^{(x \vee y)}$, $(y \vee z) \cdot x = (x \vee y \vee z)^x$.

Let us also note that the compatibility condition is satisfied for complementation in any Boolean semilattice (see [1]) and in any orthomodular semilattice [2], its modification holds also for semilattices with sectionally antitone involutions which are implication algebras for MV-algebras [6].

We are going to show that (CC) does not imply either antitone or involuton of switching mappings.

Example 1. Let $\mathcal{S} = (\{p, x, y, z, 1\}, \vee, \cdot, 1)$ be a semilattice depicted on Figure 1, where the sectional mappings are given as follows:

$$\begin{aligned}
\text{on } [p, 1]: &\quad x^p = y^p = z,\ z^p = y,\ p^p = 1,\ 1^p = p,\\
\text{on } [x, 1]: &\quad x^x = 1,\ 1^x = x,\\
\text{on } [y, 1]: &\quad y^y = 1,\ 1^y = y,\\
\text{on } [z, 1]: &\quad z^z = 1,\ 1^z = z,
\end{aligned}$$

on $[1,1]: \quad 1^1 = 1.$

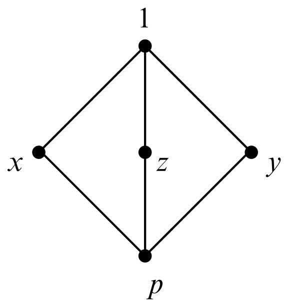

FIGURE 1.

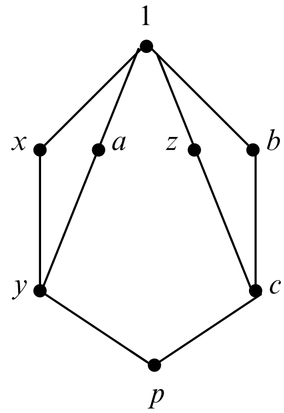

FIGURE 2.

One can easily verify that (CC) is satisfied by $\mathcal{S}$ and all the sectional mappings are antitone switching mappings. However, $a \mapsto a^p$ is not a bijection since $x \neq y$ but $x^p = y^p$.

Example 2. Let $\mathcal{S} = (\{p, a, b, c, x, y, z, 1\}, \vee, \cdot, 1)$ be the semilattice shown on Figure 2.

The sectional mappings are defined as follows:

$$\begin{aligned}
\text{on } [p,1]: &\quad x^p = a,\ y^p = c,\ z^p = b, a^p = x,\ c^p = y,\ b^p = z,\ p^p = 1,\\
&\quad 1^p = p,\\
\text{on } [y,1]: &\quad x^y = a,\ a^y = x,\ y^y = 1,\ 1^y = y,\\
\text{on } [c,1]: &\quad z^c = b,\ b^c = z,\ c^c = 1,\ 1^c = c,\\
\text{on } [x,1]: &\quad x^x = 1,\ 1^x = x,\\
\text{on } [a,1]: &\quad a^a = 1,\ 1^a = a,\\
\text{on } [z,1]: &\quad z^z = 1,\ 1^z = z,\\
\text{on } [b,1]: &\quad b^b = 1,\ 1^b = b,\\
\text{on } [1,1]: &\quad 1^1 = 1.
\end{aligned}$$

It is easy to verify that all of them are switching mappings satisfying (CC) and, moreover, they are involutions. However, the mapping $v \mapsto v^p$ is not antitone, since $y \leq x$ but $x^p = a$, $y^p = c$ are incomparable.

Lemma 5. *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional mappings satisfying the compatibility condition. Then*

(a) $x \vee x^p = 1$ *for each* $p \in S$ *and each* $x \in [p, 1]$*;*

(b) *If* $z \mapsto z^p$ *is a switching mapping for* $p \neq 1$*, then* $x^p \neq x$ *and if* $x < y$ *then* $x^p \neq y^p$ *for each* $x, y \in [p, 1]$*;*

(c) *If all the sectional mappings are switching, then no section of* $\mathcal{S}$ *can be a chain with more than two elements.*

PROOF. (a) Since $p \leq x \leq x$, we infer directly by (CC) $1 = x^x = x^p \vee x$.

(b) If $z \mapsto z^p$ is a switching mapping on $[p, 1]$ and $x, y \in [p, 1]$, then if $x^p = x$, by (a), we obtain $1 = x^p \vee x = x$ and, hence, $1 = x^p = 1^p = p$, a contradiction.

If $x < y$ and $x^p = y^p$, then, by (CC) and (a), $y^x = y^p \vee x = x^p \vee x = 1$. Since the sectional mapping is switching, it yields $y = x$, a contradiction.

(c) Suppose that $[p, 1]$ is a chain with more than two elements. Then there exists $x \in [p, 1]$, $p \neq x \neq 1$. We have $x^p \neq p$, $x^p \neq 1$ and, by (a), $1 = x^p \vee x = \max(x, x^p)$, a contradiction. □

Let us recall that a join semilattice $\mathcal{S} = (S, \vee, 1)$ with 1 where for $p \in S$ the section $[p, 1]$ is a lattice $([p, 1], \vee, \wedge_p)$ is called a *nearlattice* (the concept was introduced by M. Scholander [9] in 1950s).

We are interested in the case where the sectional switching mappings are sectional complementations.

Theorem 3. *Let* $\mathcal{S} = (S, \vee, \cdot, 1)$ *be a nearlattice with sectional switching mappings satisfying the compatibility condition. If* $x \mapsto x^p$ *is antitone on* $[p, 1]$*, then* x^p *is a complement of* x *for each* $x \in [p, 1]$*.*

PROOF. Assume that the sectional switching mapping on $[p, 1]$ is antitone. By Lemma 5, we have $x \vee x^p = 1$ and $x^p \vee x^{pp} = 1$ for each $x \in [p, 1]$. Take $z = x \wedge_p x^p$. Then $z \leq x$, $z \leq x^p$ and, due to the antitone property of this mapping, also $z^p \geq x^p$, $z^p \geq x^{pp}$. Thus, $z^p \geq x^p \vee x^{pp} = 1$. Therefore, it follows that $z^p = 1$, i. e., $z = p$ and x^p is a complement of x in the lattice $([p, 1], \vee, \wedge_p)$. □

Remark 3. The complement x^p of x in $[p, 1]$ need not be an orthocomplement although the mapping is antitone. We can see in Example 1 that this mapping need not be an involution: we have $x^{pp} = z^p = y \neq x$.

If $\mathcal{S} = (S, \vee, \cdot, 1)$ is a semilattice with sectionally antitone involutions, then we can apply De Morgan law (see Lemma 1) in each section $[p, 1]$ to prove that for $x, y \in [p, 1]$, $(x^p \vee y^p)^p$ is their infimum, i. e., every $[p, 1]$ becomes a lattice where $x \wedge_p y = (x^p \vee y^p)^p$. Hence, $\mathcal{S}$ is in fact a nearlattice. Moreover, if these sectionally antitone involutions satisfy the compatibility condition, we can prove the following.

Theorem 4. *Let* $\mathcal{S} = (S, \vee, \cdot, 1)$ *be a semilattice with sectionally antitone involutions satisfying the compatibility condition. Then for each* $p \in S$ *the section* $[p, 1]$ *is an orthomodular lattice where* x^p *is an orthocomplement of* $x \in [p, 1]$*.*

PROOF. Naturally, sectionally antitone involutions are switching mappings, thus, by Lemma 1 and Theorem 3, $[p, 1]$ is a lattice and x^p is a complement of $x \in [p, 1]$.

Since this sectional mapping is an involution, we have $x^{pp} = x$ and, due to antitony, $x \leq y$ implies $y^p \leq x^p$ for $x, y \in [p, 1]$ thus x^p is an orthocomplement of x in $[p, 1]$. By using the compatibility condition, $p \leq x \leq y$ implies $y^x = y^p \vee x$ and hence

$$y \wedge_p (x \vee y^p) = y \wedge_p y^x = y \wedge_x y^x = x$$

which is the orthomodular law in the lattice $([p, 1], \vee, \wedge_p)$. □

In the remaining part, we will check whether the complement x^p of x in $[p, 1]$ is unique. We will establish a new condition which need not be the compatibility condition.

Theorem 5. *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectionally antitone involutions. If for $p \in S$ and each $x, y \in [p, 1]$ the relation*

$$(x^p \vee y)^p \vee x^p = (y^p \vee x)^p \vee y^p \qquad (*)$$

holds, then $([p, 1], \vee, \wedge_p)$ is a Boolean algebra.

PROOF. Due to Lemma 1, $([p, 1], \vee, \wedge_p)$ is a lattice and we can use De Morgan law for each section. Let $a \in [p, 1]$. Using of the identity $(*)$, we obtain

$$a \vee a^p = a^{pp} \vee a^p = (a^p \vee p)^p \vee a^p = (p^p \vee a)^p \vee p^p = (1 \vee a)^p \vee 1 = 1.$$

Due to the De Morgan law, we have

$$a \wedge_p a^p = a^{pp} \wedge_p a^p = (a^p \vee a)^p = 1^p = p.$$

Hence, a^p is a complement of a in $[p, 1]$.

Suppose now that $u \in [p, 1]$ is a complement of a in $[p, 1]$, i.e. $a \vee u = 1$ and $a \wedge_p u = p$. Using the identity $(*)$ and the De Morgan law again, we derive $a = p \vee a = (a \vee u)^p \vee a = (a^{pp} \vee u)^p \vee a^{pp} = (u^p \vee a^p)^p \vee u^p = (u \wedge_p a) \vee u^p = p \vee u^p = u^p$. Thus, $a^p = u^{pp} = u$, and the complement is unique.

Since the involution is an antitone unique complementation, then, according to [8], $([p, 1], \vee, \wedge_p)$ is distributive. □

Remark 4. Identity $(*)$ is in fact equivalent to the assertion that $([p, 1], \vee, \wedge_p)$ is a Boolean algebra where x^p is a complement of an $x \in [p, 1]$. Indeed, if $([p, 1], \vee, \wedge_p)$ is distributive, then $(x^p \vee y)^p \vee x^p = x^p \vee (x^{pp} \wedge_p y^p) = x^p \vee (x \wedge_p y^p) = (x^p \vee x) \wedge_p (x^p \vee y^p) = 1 \wedge_p (x^p \vee y^p) = x^p \vee y^p$, thus also $(y^p \vee x)^p \vee y^p = y^p \vee x^p = x^p \vee y^p$. Therefore, $(*)$ is satisfied.

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