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# ON TAUBERIAN REMAINDER THEOREMS FOR CESÀRO SUMMABILITY METHOD OF NONINTEGER ORDER

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*Abstract.* In this paper, we prove some Tauberian remainder theorems for Cesàro summability method of noninteger order $\alpha > -1$.

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## 1. INTRODUCTION

Let $A_n^\alpha$ be defined by the generating function $(1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A_n^\alpha x^n$, $(|x|<1)$, where $\alpha > -1$. For a real sequence $u=(u_n)$, the Cesàro means of the sequence $(u_n)$ of noninteger order $\alpha$ are defined by

$$\sigma_n^{(\alpha)}(u) = \frac{1}{A_n^\alpha}\sum_{j=0}^{n} A_{n-j}^{\alpha-1}u_j .$$

We say that a sequence $(u_n)$ is $(C,\alpha)$ summable to a finite number $s$, where $\alpha > -1$ if

$$\lim_{n\to\infty}\sigma_n^{(\alpha)}(u) = s, \tag{1.1}$$

and we write $u_n \to s\ (C,\alpha)$. We denote the backward difference of $(u_n)$, by $\Delta u_n = u_n - u_{n-1}$, with $\Delta u_0 = u_0$. We define $\tau_n(u) = n\Delta u_n \quad (n=0,1,2,..)$ and indicate $\tau_n^{(\alpha)}(u)$ as $(C,\alpha)$ mean of $(\tau_n(u))$.

Note that if taking $\alpha = k$ where $k$ is a nonnegative integer, then we obtain the $(C,k)$ summability method and for $\alpha = 0$, the $(C,0)$ summability is ordinary convergence.

The $(C,\alpha)$ summability method is regular, more generally, if a sequence $(u_n)$ is $(C,\alpha)$ summable to $s$, where $\alpha > -1$ and $\beta \geq \alpha$ for $\alpha,\beta$, then $(u_n)$ is also $(C,\beta)$ summable to $s$. However, the converse is not always true. The converse of this statement is valid under some conditions called Tauberian conditions. Any theorem which states that convergence of a sequence follows from a summability method and some Tauberian condition(s) is said to be a Tauberian theorem. Recently, a number of authors such as Estrada and Vindas [4, 5], Natarajan [15], Çanak et al. [2], Erdem and Çanak [3], Çanak and Erdem [1] have investigated Tauberian theorems for several summability methods.

For a sequence $(u_n)$ and for each integer $m \geq 1$,

$$(n\Delta)_m u_n = n\Delta((n\Delta)_{m-1}u_n), \tag{1.2}$$

where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n$.

For $\alpha > -1$, the identity

$$\tau_n^{(\alpha)}(u) = n\Delta\sigma_n^{(\alpha)}(u) \tag{1.3}$$

was proved by Kogbetliantz [9]. Note that $\tau_n^{(0)}(u) = \tau_n(u)$.

The identity

$$\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u) = \frac{1}{\alpha+1}\tau_n^{(\alpha+1)}(u) \tag{1.4}$$

is used in the various steps of proofs (see [10]).

Çanak et al. [2] represent the identity

$$n\Delta\tau_n^{\alpha+1} = (\alpha+1)(\tau_n^{\alpha} - \tau_n^{\alpha+1}), \tag{1.5}$$

for $\alpha > -1$.

Erdem and Çanak [3] prove that for $\alpha > -1$ and any integer $k \geq 1$

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = \sum_{j=1}^{k}(-1)^{j+1}A_k^{(j)}(\alpha)n\Delta\tau_n^{(\alpha+j)}(u), \tag{1.6}$$

where $A_k^{(j)}(\alpha) = a_k^{(j-1)}(\alpha) + a_k^{(j)}(\alpha)$, $\quad a_k^{(0)}(\alpha) = 0$ and

$$a_k^{(j)}(\alpha) = \Pi_{i=j+1}^{k}(\alpha+i) \sum_{\substack{j+1 \leq t_1, t_2, \ldots, t_{j-1} \leq k \\ r<s \Rightarrow t_r \leq t_s}} (\alpha+t_1)(\alpha+t_2)\ldots(\alpha+t_{j-1}).$$

## 2. Tauberian remainder theorems

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_n \to \infty$. A sequence $(u_n)$ is called bounded with the rapidity $(\lambda_n)$ (in short $\lambda$-bounded) if

$$\lambda_n(u_n - s) = O(1),$$

with $\lim_{n\to\infty} u_n = s$. Let

$$m^{\lambda} = \{u = (u_n) | \lim_{n\to\infty} u_n = s \quad \text{and} \quad \lambda_n(u_n - s) = O(1)\}. \tag{2.1}$$

A sequence $(u_n)$ is called $\lambda$-bounded by the $(C,\alpha)$ method of summability if

$$\lambda_n(\sigma_n^{(\alpha)}(u) - s) = O(1), \tag{2.2}$$

with $\lim_{n\to\infty}\sigma_n^{(\alpha)}(u) = s$. Shortly, we write $u \in ((C,\alpha), m^{\lambda})$.

G. Kangro [7] introduced the concepts of Tauberian remainder theorems using summability with given rapidity $\lambda$. G. Kangro [8] and Tammeraid [16, 17] proved some Tauberian remainder theorems for several summability method, such as Riesz, Cesàro, Hölder and Euler-Knopp methods. Recently, various authors have represented some Tauberian remainder theorems (see [12, 13]). In [18], Tammeraid proved some Tauberian remainder theorems in which the $(C,\alpha)$ summability method is used. Tauberian remainder theorems have also been studied by many authors via the Fourier integral method. [6, 11]

Meronen and Tammeraid [14] proved the following Tauberian remainder theorems:

**Theorem 1.** *Let the condition*

$$\lambda_n \tau_n^{(1)}(u) = O(1)$$

*be satisfied. If* $u \in ((C,1), m^{\lambda})$, *then* $u \in m^{\lambda}$.

**Theorem 2.** *Let the conditions*

$$\lambda_n \tau_n(u) = O(1),$$
$$\lambda_n n \Delta \tau_n^{(1)}(u) = O(1)$$

*be satisfied. If* $u \in ((C,1), m^{\lambda})$, *then* $u \in m^{\lambda}$.

The main purpose of this paper is to prove several Tauberian remainder theorems for Cesàro summability method of noninteger order $\alpha > -1$. Our main theorems improve Theorem 1 and Theorem 2 given by Meronen and Tammeraid [14].

## 3. A LEMMA

We require the following lemma to be used in the proofs of main theorems.

**Lemma 1.** *Let* $\alpha > -1$. *For any integer* $k \geq 2$,

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = B_{1,1-\alpha}\tau_n^{(\alpha)}(u) - B_{1,1}\sigma_n^{(\alpha)}(u) + B_{1,1}\sigma_n^{(\alpha+1)}(u)$$
$$+ \sum_{j=2}^{k}\left(B_{j,j-1}\sigma_n^{(\alpha+j-2)}(u) - 2B_{j,j-\frac{1}{2}}\sigma_n^{(\alpha+j-1)}(u) + B_{j,j}\sigma_n^{(\alpha+j)}(u)\right),$$

*where* $B_{m,l} = (\alpha+m)(\alpha+l)(-1)^{m+1}A_k^{(m)}(\alpha)$ *and* $A_k^{(j)}(\alpha) = a_k^{(j-1)}(\alpha) + a_k^{(j)}(\alpha)$, $a_k^{(0)}(\alpha) = 0$ *and*

$$a_k^{(j)}(\alpha) = \Pi_{i=j+1}^{k}(\alpha+i) \sum_{\substack{j+1 \leq t_1, t_2, \ldots, t_{j-1} \leq k \\ r<s \Rightarrow t_r \leq t_s}} (\alpha+t_1)(\alpha+t_2)\ldots(\alpha+t_{j-1}).$$

*Proof.* From identity (1.6), we have

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = \sum_{j=1}^{k}(-1)^{j+1}A_k^{(j)}(\alpha)n\Delta\tau_n^{(\alpha+j)}(u)$$
$$= A_k^{(1)}(\alpha)n\Delta\tau_n^{(\alpha+1)}(u) + \sum_{j=2}^{k}(-1)^{j+1}A_k^{(j)}(\alpha)n\Delta\tau_n^{(\alpha+j)}(u).$$

It follows from identity (1.5) that

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = (\alpha+1)A_k^{(1)}(\alpha)(\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u))$$
$$+ \sum_{j=2}^{k}(\alpha+j)(-1)^{j+1}A_k^{(j)}(\alpha)(\tau_n^{(\alpha+j-1)}(u) - \tau_n^{(\alpha+j)}(u)).$$

By identity (1.4), we can write the above equation as

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = (\alpha+1)A_k^{(1)}(\alpha)\tau_n^{(\alpha)}(u) - (\alpha+1)^2 A_k^{(1)}(\alpha)\left(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u)\right)$$
$$+ \sum_{j=2}^{k}(\alpha+j)(-1)^{j+1}A_k^{(j)}(\alpha)\Big((\alpha+j-1)(\sigma_n^{(\alpha+j-2)}(u) - \sigma_n^{(\alpha+j-1)}(u))$$
$$-(\alpha+j)(\sigma_n^{(\alpha+j-1)}(u) - \sigma_n^{(\alpha+j)}(u))\Big).$$

Therefore,

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u)$$
$$= (\alpha+1)A_k^{(1)}(\alpha)\tau_n^{(\alpha)}(u) - (\alpha+1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha)}(u) + (\alpha+1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha+1)}(u)$$
$$+ \sum_{j=2}^{k}\Big((\alpha+j)(\alpha+j-1)(-1)^{j+1}A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-2)}(u)$$
$$- (\alpha+j)(\alpha+j-1)(-1)^{j+1}A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-1)}(u)$$
$$- (\alpha+j)^2(-1)^{j+1}A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-1)}(u) + (\alpha+j)^2(-1)^{j+1}A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j)}(u)\Big).$$

Hence, we have

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u)$$
$$= (\alpha+1)A_k^{(1)}(\alpha)\tau_n^{(\alpha)}(u) - (\alpha+1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha)}(u) + (\alpha+1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha+1)}(u)$$
$$+ \sum_{j=2}^{k}\Big((\alpha+j)(\alpha+j-1)(-1)^{j+1}A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-2)}(u)$$
$$- (\alpha+j)(2\alpha+2j-1)(-1)^{j+1}A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-1)}(u)$$

$$+\,(\alpha+j)^2(-1)^{j+1}A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j)}(u)\Big).$$

Taking $(\alpha+m)(\alpha+l)(-1)^{m+1}A_k^{(m)}(\alpha)=B_{m,l}$, we obtain

$$(n\Delta)_k\tau_n^{(\alpha+k)}(u)=B_{1,1-\alpha}\tau_n^{(\alpha)}(u)-B_{1,1}\sigma_n^{(\alpha)}(u)+B_{1,1}\sigma_n^{(\alpha+1)}(u)$$
$$+\sum_{j=2}^{k}\Big(B_{j,j-1}\sigma_n^{(\alpha+j-2)}(u)-2B_{j,j-\frac{1}{2}}\sigma_n^{(\alpha+j-1)}(u)+B_{j,j}\sigma_n^{(\alpha+j)}(u)\Big).$$

Thus, we conclude that Lemma 1 is true for each integer $k\geq 2$. □

## 4. MAIN RESULTS

In the main theorems, we prove some Tauberian remainder theorems to recover $\lambda$-bounded by the $(C,\alpha)$ summability of a sequence out of $\lambda$-bounded by the $(C,\alpha+j)$ summability for $j=1,2$ and any integer $j=k$, and some suitable conditions. In special cases of main theorems, we obtain some classical type Tauberian remainder theorems for the $(C,1)$ summability method.

**Theorem 3.** *Let the conditions*

$$\lambda_n n\Delta\tau_n^{(\alpha+1)}(u)=O(1), \tag{4.1}$$

*and*

$$\lambda_n\tau_n^{(\alpha)}(u)=O(1) \tag{4.2}$$

*be satisfied for $\alpha>-1$. If $u\in((C,\alpha+1),m^\lambda)$, then $u\in((C,\alpha),m^\lambda)$.*

*Proof.* From identity (1.5), we have

$$\lambda_n n\Delta\tau_n^{(\alpha+1)}(u)=\lambda_n(\alpha+1)(\tau_n^{(\alpha)}(u)-\tau_n^{(\alpha+1)}(u))$$
$$=\lambda_n(\alpha+1)\tau_n^{(\alpha)}(u)-\lambda_n(\alpha+1)\tau_n^{(\alpha+1)}(u).$$

From identity (1.4), we obtain

$$\lambda_n n\Delta\tau_n^{(\alpha+1)}(u)=\lambda_n(\alpha+1)\tau_n^{(\alpha)}(u)-\lambda_n(\alpha+1)^2(\sigma_n^{(\alpha)}(u)-\sigma_n^{(\alpha+1)}(u)).$$

Rewritten the above equation, we have

$$\lambda_n(\alpha+1)^2(\sigma_n^{(\alpha)}(u)-s)=\lambda_n(\alpha+1)^2(\sigma_n^{(\alpha+1)}(u)-s)$$
$$+\lambda_n(\alpha+1)\tau_n^{(\alpha)}(u)-\lambda_n n\Delta\tau_n^{(\alpha+1)}(u).$$

Using (4.1) and (4.2), we get

$$\lambda_n(\alpha+1)^2(\sigma_n^{(\alpha)}(u)-s)=O(1)+O(1)+O(1)=O(1).$$

Therefore, $\lambda_n(\sigma_n^{(\alpha)}(u)-s)=O(1)$. That means $u\in((C,\alpha),m^\lambda)$. □

Notice that taking $\alpha=0$, we obtain Theorem 2.

**Proposition 1.** *Let the conditions*

$$\lambda_n (n\Delta)_2 \tau_n^{(\alpha+2)}(u) = O(1), \tag{4.3}$$

$$\lambda_n \tau_n^{(\alpha)}(u) = O(1), \tag{4.4}$$

*and*

$$\lambda_n (\sigma_n^{(\alpha+2)}(u) - s) = O(1) \tag{4.5}$$

*be satisfied for* $\alpha > -1$. *If* $u \in ((C,\alpha+1), m^{\lambda})$, *then* $u \in ((C,\alpha), m^{\lambda})$.

*Proof.* Taking $k = 2$ in Lemma 1, we have

$$\begin{aligned}
\lambda_n (n\Delta)_2 \tau_n^{(\alpha+2)}(u) &= \lambda_n (\alpha+2)(\alpha+1)\left(\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u)\right) \\
&\quad - \lambda_n (\alpha+2)^2 \left(\tau_n^{(\alpha+1)}(u) - \tau_n^{(\alpha+2)}(u)\right) \\
&= \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha+1)}(u) \\
&\quad - \lambda_n (\alpha+2)^2 \tau_n^{(\alpha+1)}(u) + \lambda_n (\alpha+2)^2 \tau_n^{(\alpha+2)}(u).
\end{aligned}$$

From identity (1.4), we get

$$\begin{aligned}
&\lambda_n (n\Delta)_2 \tau_n^{(\alpha+2)}(u) \\
&= \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n (\alpha+2)(\alpha+1)\left((\alpha+1)(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u))\right) \\
&\quad - \lambda_n (\alpha+2)^2 \left((\alpha+1)(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u))\right) \\
&\quad + \lambda_n (\alpha+2)^2 \left((\alpha+2)(\sigma_n^{(\alpha+1)}(u) - \sigma_n^{(\alpha+2)}(u))\right) \\
&= \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n (\alpha+1)^2(\alpha+2)\sigma_n^{(\alpha)}(u) \\
&\quad + \lambda_n (\alpha+1)^2(\alpha+2)\sigma_n^{(\alpha+1)}(u) - \lambda_n (\alpha+2)^2(\alpha+1)\sigma_n^{(\alpha)}(u) \\
&\quad + \lambda_n (\alpha+2)^2(\alpha+1)\sigma_n^{(\alpha+1)}(u) + \lambda_n (\alpha+2)^3\sigma_n^{(\alpha+1)}(u) - \lambda_n (\alpha+2)^3\sigma_n^{(\alpha+2)}(u) \\
&= \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n (\alpha+1)(\alpha+2)(2\alpha+3)\alpha_n^{(\alpha)}(u) \\
&\quad + \lambda_n ((\alpha+2)^3 + (\alpha+2)^2(\alpha+1) + (\alpha+1)^2(\alpha+2))\sigma_n^{(\alpha+1)}(u) \\
&\quad - \lambda_n (\alpha+2)^3\sigma_n^{(\alpha+2)}(u).
\end{aligned}$$

Rewritten the above equation, we have

$$\begin{aligned}
&\lambda_n (\alpha+1)(\alpha+2)(2\alpha+3)(\sigma_n^{(\alpha)}(u) - s) \\
&= -\lambda_n n\Delta \tau_n^{(\alpha+2)}(u) + \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u) + \lambda_n ((\alpha+2)^3 + (\alpha+2)^2(\alpha+1) \\
&\quad + (\alpha+1)^2(\alpha+2) - s)\sigma_n^{(\alpha+1)}(u) - \lambda_n ((\alpha+2)^3 - s)\sigma_n^{(\alpha+2)}(u).
\end{aligned}$$

Using (4.3), (4.4) and (4.5), we get

$$\lambda_n (\alpha+1)(\alpha+2)(2\alpha+3)(\sigma_n^{(\alpha)}(u) - s) = O(1) + O(1) + O(1) = O(1).$$

Therefore, $\lambda_n(\sigma_n^{(\alpha)}(u)-s) = O(1)$. That means $u \in ((C,\alpha),m^{\lambda})$. □

Now, we represent a Tauberian remainder theorem which generalizes Theorem 3 and Proposition 1.

**Theorem 4.** *Let the conditions*

$$\lambda_n(n\Delta)_k \tau_n^{(\alpha+k)}(u) = O(1), \tag{4.6}$$

$$\lambda_n \tau_n^{(\alpha)}(u) = O(1), \tag{4.7}$$

*and*

$$\lambda_n(\sigma_n^{(\alpha+j)}(u)-s) = O(1) \quad \textit{for} \quad 2 \le j \le k \tag{4.8}$$

*be satisfied for* $\alpha > -1$. *If* $u \in ((C,\alpha+1),m^{\lambda})$, *then* $u \in ((C,\alpha),m^{\lambda})$.

*Proof.* From Lemma 1 we have

$$\lambda_n(n\Delta)_k \tau_n^{(\alpha+k)}(u) = B_{1,1-\alpha}\lambda_n \tau_n^{(\alpha)}(u) - B_{1,1}\lambda_n \sigma_n^{(\alpha)}(u) + B_{1,1}\lambda_n \sigma_n^{(\alpha+1)}(u)$$
$$+\lambda_n \sum_{j=2}^{k} \left( B_{j,j-1}\sigma_n^{(\alpha+j-2)}(u) - 2B_{j,j-\frac{1}{2}}\sigma_n^{(\alpha+j-1)}(u) + B_{j,j}\sigma_n^{(\alpha+j)}(u) \right),$$

Rewritten the above equation, we have

$$\begin{aligned} &B_{1,1}\lambda_n(\sigma_n^{(\alpha)}(u)-s) \\ &= B_{1,1-\alpha}\lambda_n \tau_n^{(\alpha)}(u) - \lambda_n(n\Delta)_k \tau_n^{(\alpha+k)}(u) + B_{1,1}\lambda_n(\sigma_n^{(\alpha+1)}(u)-s) \\ &\quad + \lambda_n \sum_{j=2}^{k} B_{j,j-1}(\sigma_n^{(\alpha+j-2)}(u)-s) - \lambda_n \sum_{j=2}^{k} 2B_{j,j-\frac{1}{2}}(\sigma_n^{(\alpha+j-1)}(u)-s) \\ &\quad + \lambda_n \sum_{j=2}^{k} B_{j,j}(\sigma_n^{(\alpha+j)}(u)-s). \end{aligned}$$

Using (4.6), (4.7) and (4.8), we get

$$B_{1,1}\lambda_n(\sigma_n^{(\alpha)}(u)-s) = O(1)+O(1)+O(1)+O(1)+O(1)+O(1) = O(1).$$

Therefore, $\lambda_n(\sigma_n^{(\alpha)}(u)-s) = O(1)$. That means $u \in ((C,\alpha),m^{\lambda})$. □

**Theorem 5.** *Let the condition*

$$\lambda_n \tau_n^{(\alpha+j+1)}(u) = O(1) \quad \textit{for} \quad 0 \le j \le k-1, \tag{4.9}$$

*be satisfied for* $\alpha > -1$. *If* $u \in ((C,\alpha+k),m^{\lambda})$, *then* $u \in ((C,\alpha),m^{\lambda})$.

*Proof.* Suppose that $u \in ((C,\alpha+k),m^{\lambda})$. Taking $j = k-1$ in (4.9), it follows from the idendity

$$\tau_n^{(\alpha+k)}(u) = (\alpha+k)(\sigma_n^{(\alpha+k-1)}(u) - \sigma_n^{(\alpha+k)}(u))$$

that we obtain

$$\begin{aligned}\lambda_n(\alpha+k)(\sigma_n^{(\alpha+k-1)}(u)-s) &= \lambda_n\tau_n^{(\alpha+k)}(u)+\lambda_n(\alpha+k)(\sigma_n^{\alpha+k}(u)-s)\\ &= O(1)+O(1)=O(1)\end{aligned}$$

then we obtain $\lambda_n(\sigma_n^{(\alpha+k-1)}-s)=O(1)$. Hence, that means

$$u\in((C,\alpha+k-1),m^{\lambda}).$$

From identity (1.4), we have

$$\tau_n^{(\alpha+k-1)}(u)=(\alpha+k-1)(\sigma_n^{(\alpha+k-2)}(u)-\sigma_n^{(\alpha+k-1)}(u)).$$

Taking $j=k-2$ in (4.9), we obtain

$$\begin{aligned}&\lambda_n(\alpha+k-1)(\alpha_n^{(\alpha+k-2)}(u)-s)\\ &=\lambda_n\tau_n^{(\alpha+k-1)}(u)+\lambda_n(\alpha+k-1)(\sigma_n^{(\alpha+k-1)}(u)-s)=O(1)+O(1)=O(1)\end{aligned}$$

Therefore we have

$$u\in((C,\alpha+k-2),m^{\lambda}).$$

Continuing in this way, we obtain that

$$u\in((C,\alpha+1),m^{\lambda}).$$

Taking $j=0$ in (4.9), we obtain $\lambda_n\tau_n^{(\alpha+1)}=O(1)$. From identity (1.4), we have

$$\begin{aligned}\lambda_n(\alpha+1)(\sigma_n^{(\alpha)}(u)-s) &= \lambda_n\tau_n^{(\alpha+1)}(u)+\lambda_n(\alpha+1)(\sigma_n^{(\alpha+1)}(u)-s)\\ &= O(1)+O(1)=O(1)\end{aligned}$$

This completes the proof. □

**Theorem 6.** *Let the condition*

$$\lambda_n(n\Delta)_j\tau_n^{(\alpha+j)}(u)=O(1)\quad \textit{for}\quad 0\le j\le k, \tag{4.10}$$

*be satisfied for* $\alpha>-1$. *If* $u\in((C,\alpha+k),m^{\lambda})$, *then* $u\in((C,\alpha),m^{\lambda})$.

*Proof.* By identity (1.6) for $k=1$, it follows

$$\begin{aligned}\lambda_n n\Delta\tau_n^{(\alpha+1)}(u) &= \lambda_n(\alpha+1)(\tau_n^{(\alpha)}(u)-\tau_n^{(\alpha+1)}(u))\\ &= \lambda_n(\alpha+1)\tau_n^{(\alpha)}(u)-\lambda_n(\alpha+1)\tau_n^{(\alpha+1)}(u).\end{aligned}$$

Taking $j=0$ and $j=1$ in (4.10), we obtain

$$\lambda_n\tau_n^{(\alpha+1)}(u)=O(1)$$

From identity (1.6) for $k=2$, we get

$$\begin{aligned}\lambda_n(n\Delta)_2\tau_n^{(\alpha+2)}(u) &= \lambda_n(\alpha+2)(\alpha+1)\left(\tau_n^{(\alpha)}(u)-\tau_n^{(\alpha+1)}(u)\right)\\ &\quad -\lambda_n(\alpha+2)^2\left(\tau_n^{(\alpha+1)}(u)-\tau_n^{(\alpha+2)}(u)\right).\end{aligned}$$

Taking $j=0$ and $j=2$ in (4.10), we obtain

$$\lambda_n \tau_n^{(\alpha+2)}(u) = O(1)$$

Continuing in this way, by Lemma 1, we obtain

$$\lambda_n (n\Delta)_k \tau_n^{(\alpha+k)}(u) = (\alpha+1) A_k^{(1)}(\alpha)\lambda_n(\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u)) + \lambda_n \sum_{j=2}^{k} (\alpha+j)(-1)^{j+1} A_k^{(j)}(\alpha)(\tau_n^{(\alpha+j-1)}(u) - \tau_n^{(\alpha+j)}(u)).$$

Taking $j=0$ and $j=k$ in (4.10), we obtain

$$\lambda_n \tau_n^{(\alpha+k)}(u) = O(1).$$

The conditions in Theorem 5 hold, the proof is completed. □

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