ON TAUBERIAN REMAINDER THEOREMS FOR CESÀRO SUMMABILITY METHOD OF NONINTEGER ORDER

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Abstract. In this paper, we prove some Tauberian remainder theorems for Cesàro summability method of noninteger order $\alpha > -1$.

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1. INTRODUCTION

Let $A^\alpha_n$ be defined by the generating function $(1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A^\alpha_n x^n$, $(|x| < 1)$, where $\alpha > -1$. For a real sequence $u = (u_n)$, the Cesàro means of the sequence $(u_n)$ of noninteger order $\alpha$ are defined by

$$
\sigma_n^{(\alpha)}(u) = \frac{1}{A^\alpha_n} \sum_{j=0}^{n} A^\alpha_{n-j} u_j.
$$

We say that a sequence $(u_n)$ is $(C, \alpha)$ summable to a finite number $s$, where $\alpha > -1$ if

$$
\lim_{n \to \infty} \sigma_n^{(\alpha)}(u) = s,
$$

and we write $u_n \to s (C, \alpha)$. We denote the backward difference of $(u_n)$, by $\Delta u_n = u_n - u_{n-1}$, with $\Delta u_0 = u_0$. We define $\tau_n(u) = n \Delta u_n \ (n = 0, 1, 2, \ldots)$ and indicate $\tau_n^{(\alpha)}(u)$ as $(C, \alpha)$ mean of $(\tau_n(u))$.

Note that if taking $\alpha = k$ where $k$ is a nonnegative integer, then we obtain the $(C, k)$ summability method and for $\alpha = 0$, the $(C, 0)$ summability is ordinary convergence.

The $(C, \alpha)$ summability method is regular, more generally, if a sequence $(u_n)$ is $(C, \alpha)$ summable to $s$, where $\alpha > -1$ and $\beta \geq \alpha$ for $\alpha, \beta$, then $(u_n)$ is also $(C, \beta)$ summable to $s$. However, the converse is not always true. The converse of this statement is valid under some conditions called Tauberian conditions. Any theorem which states that convergence of a sequence follows from a summability method and some Tauberian condition(s) is said to be a Tauberian theorem. Recently, a number
of authors such as Estrada and Vindas [4, 5], Natarajan [15], Çanak et al. [2], Erdem and Çanak [3], Çanak and Erdem [1] have investigated Tauberian theorems for several summability methods.

For a sequence \( \{u_n\} \) and for each integer \( m \geq 1 \),
\[
(n \Delta)_m u_n = n \Delta ((n \Delta)_{m-1} u_n),
\]
(1.2)
where \((n \Delta)_0 u_n = u_n\) and \((n \Delta)_1 u_n = n \Delta u_n\).

For \( \alpha > -1 \), the identity
\[
\tau_n^{(\alpha)}(u) = n \Delta \sigma_n^{(\alpha)}(u)
\]
(1.3)
was proved by Kogbetliantz [9]. Note that \( \tau_n^{(0)}(u) = \tau_n(u) \).

The identity
\[
\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u) = \frac{1}{\alpha + 1} \tau_n^{(\alpha+1)}(u)
\]
(1.4)
is used in the various steps of proofs (see [10]).

Çanak et al. [2] represent the identity
\[
(n \Delta)^{\alpha+1} = (\alpha + 1)(\tau_n^\alpha - \tau_n^{\alpha+1}),
\]
(1.5)
for \( \alpha > -1 \).

Erdem and Çanak [3] prove that for \( \alpha > -1 \) and any integer \( k \geq 1 \)
\[
(n \Delta)_k \tau_n^{(\alpha+k)}(u) = \sum_{j=1}^{k} (-1)^{j+1} A_k^{(j)}(\alpha) n \Delta \tau_n^{(\alpha+j)}(u),
\]
(1.6)
where \( A_k^{(j)}(\alpha) = a_k^{(j)}(\alpha) + a_k^{(j)}(\alpha) \), \( a_k^{(0)}(\alpha) = 0 \) and
\[
a_k^{(j)}(\alpha) = \prod_{i=j+1}^{k} (\alpha + i) \sum_{j+1 \leq t_1, t_2, \ldots, t_{j-1} \leq k} \frac{(\alpha + t_1)(\alpha + t_2) \cdots (\alpha + t_{j-1})}{\alpha + 1 < r \Rightarrow t_r \leq t_s}
\]

2. TAUBERIAN REMAINDER THEOREMS

Let \( \lambda = (\lambda_n) \) be a nondecreasing sequence of positive numbers such that \( \lambda_n \to \infty \).

A sequence \( \{u_n\} \) is called bounded with the rapidity \( (\lambda_n) \) (in short \( \lambda \)-bounded) if
\[
\lambda_n(u_n - s) = O(1),
\]
with \( \lim_{n \to \infty} u_n = s \). Let
\[
m^\lambda = \{ u = (u_n) | \lim_{n \to \infty} u_n = s \text{ and } \lambda_n(u_n - s) = O(1) \}.
\]
(2.1)
A sequence \( \{u_n\} \) is called \( \lambda \)-bounded by the \((C, \alpha)\) method of summability if
\[
\lambda_n(\sigma_n^{(\alpha)}(u) - s) = O(1),
\]
(2.2)
with \( \lim_{n \to \infty} \sigma_n^{(\alpha)}(u) = s \). Shortly, we write \( u \in ((C, \alpha), m^\lambda) \).
G. Kangro [7] introduced the concepts of Tauberian remainder theorems using summability with given rapidity λ. G. Kangro [8] and Tammeraid [16, 17] proved some Tauberian remainder theorems for several summability methods, such as Riesz, Cesàro, Hölder and Euler-Knopp methods. Recently, various authors have represented some Tauberian remainder theorems (see [12, 13]). In [18], Tammeraid proved some Tauberian remainder theorems in which the \((C, \alpha)\) summability method is used. Tauberian remainder theorems have also been studied by many authors via the Fourier integral method [6, 11].

Meronen and Tammeraid [14] proved the following Tauberian remainder theorems:

**Theorem 1.** Let the condition

\[ \lambda_n \tau_n^{(1)}(u) = O(1) \]

be satisfied. If \( u \in ((C, 1), m^\lambda) \), then \( u \in m^\lambda \).

**Theorem 2.** Let the conditions

\[
\begin{align*}
\lambda_n \tau_n(u) &= O(1), \\
\lambda_n \Delta \tau_n^{(1)}(u) &= O(1)
\end{align*}
\]

be satisfied. If \( u \in ((C, 1), m^\lambda) \), then \( u \in m^\lambda \).

The main purpose of this paper is to prove several Tauberian remainder theorems for Cesàro summability method of noninteger order \( \alpha > -1 \). Our main theorems improve Theorem 1 and Theorem 2 given by Meronen and Tammeraid [14].

3. **A Lemma**

We require the following lemma to be used in the proofs of main theorems.

**Lemma 1.** Let \( \alpha > -1 \). For any integer \( k \geq 2 \),

\[
(n \Delta)_{k-1} \tau_n^{(\alpha+k)}(u) = B_{1,1}^{\alpha} \tau_n^{(\alpha)}(u) - B_{1,1}^{\alpha} \sigma_n^{(\alpha)}(u) + B_{1,1}^{\alpha} \sigma_n^{(\alpha+1)}(u) \\
+ \sum_{j=2}^{k} \left( B_{j-1,j} \sigma_n^{(\alpha+j-2)}(u) - 2B_{j-1,j} \sigma_n^{(\alpha+j-1)}(u) + B_{j,j} \sigma_n^{(\alpha+j)}(u) \right),
\]

where \( B_{m,l} = (\alpha + m)(\alpha + l)(-1)^{m+1} A_k^{(m)}(\alpha) \) and \( A_k^{(j)}(\alpha) = a_k^{(j-1)}(\alpha) + a_k^{(j)}(\alpha) \), \( a_k^{(0)}(\alpha) = 0 \) and

\[
a_k^{(j)}(\alpha) = \Pi_{i=j+1}^{k}(\alpha + i) \sum_{j+1 \leq t_1, t_2, \ldots, t_{j-1} \leq k} (\alpha + t_1)(\alpha + t_2)\ldots(\alpha + t_{j-1}).
\]
Proof. From identity (1.6), we have
\[ (n \Delta)k r_n^{(\alpha+k)}(u) = \sum_{j=1}^{k} (-1)^j A_k^{(j)}(\alpha)n \Delta r_n^{(\alpha+j)}(u) \]
\[ = A_k^{(1)}(\alpha)n \Delta r_n^{(\alpha+1)}(u) + \sum_{j=2}^{k} (-1)^j A_k^{(j)}(\alpha)n \Delta r_n^{(\alpha+j)}(u). \]

It follows from identity (1.5) that
\[ (n \Delta)k r_n^{(\alpha+k)}(u) = (\alpha + 1)A_k^{(1)}(\alpha)(r_n^{(\alpha)}(u) - r_n^{(\alpha+1)}(u)) \]
\[ + \sum_{j=2}^{k} (\alpha + j)(-1)^j A_k^{(j)}(\alpha)(r_n^{(\alpha+j-1)}(u) - r_n^{(\alpha+j)}(u)). \]

By identity (1.4), we can write the above equation as
\[ (n \Delta)k r_n^{(\alpha+k)}(u) = (\alpha + 1)A_k^{(1)}(\alpha)r_n^{(\alpha)}(u) - (\alpha + 1)^2 A_k^{(1)}(\alpha) \left( \sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u) \right) \]
\[ + \sum_{j=2}^{k} (\alpha + j)(-1)^j A_k^{(j)}(\alpha) \left( (\alpha + j - 1)(\sigma_n^{(\alpha+j-2)}(u) - \sigma_n^{(\alpha+j-1)}(u)) \right) \]
\[ - (\alpha + j)(\sigma_n^{(\alpha+j-1)}(u) - \sigma_n^{(\alpha+j)}(u)). \]

Therefore,
\[ (n \Delta)k r_n^{(\alpha+k)}(u) \]
\[ = (\alpha + 1)A_k^{(1)}(\alpha)r_n^{(\alpha)}(u) - (\alpha + 1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha)}(u) + (\alpha + 1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha+1)}(u) \]
\[ + \sum_{j=2}^{k} \left( (\alpha + j)(\alpha + j - 1)(-1)^j A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-2)}(u) \right) \]
\[ - (\alpha + j)(\alpha + j - 1)(-1)^j A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-1)}(u) \]
\[ - (\alpha + j)^2(-1)^j A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-1)}(u) + (\alpha + j)^2(-1)^j A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j)}(u) \].

Hence, we have
\[ (n \Delta)k r_n^{(\alpha+k)}(u) \]
\[ = (\alpha + 1)A_k^{(1)}(\alpha)r_n^{(\alpha)}(u) - (\alpha + 1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha)}(u) + (\alpha + 1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha+1)}(u) \]
\[ + \sum_{j=2}^{k} \left( (\alpha + j)(\alpha + j - 1)(-1)^j A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-2)}(u) \right) \]
\[ - (\alpha + j)(2\alpha + 2j - 1)(-1)^j A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-1)}(u). \]
\[ + (\alpha + j)^2 (-1)^{j+1} A_k^{(j)}(\alpha) \sigma_n^{(\alpha+j)}(u). \]

Taking \((\alpha + m)(\alpha + l)(-1)^{m+1} A_k^{(m)}(\alpha) = B_{m,l},\) we obtain

\[ (n \Delta) k \tau_n^{(\alpha+k)}(u) = B_{1,1} \tau_n^{(\alpha)}(u) - B_{1,1} \sigma_n^{(\alpha)}(u) + B_{1,1} \sigma_n^{(\alpha+1)}(u) \]

\[ + \sum_{j=2}^{k} \left( B_{j-1,j-1} \sigma_n^{(\alpha+j-2)}(u) - 2B_{j-1,j-2} \sigma_n^{(\alpha+j-1)}(u) + B_{j-1,j} \sigma_n^{(\alpha+j)}(u) \right). \]

Thus, we conclude that Lemma 1 is true for each integer \(k \geq 2.\)

4. MAIN RESULTS

In the main theorems, we prove some Tauberian remainder theorems to recover \(\lambda\)-bounded by the \((C, \alpha)\) summability of a sequence out of \(\lambda\)-bounded by the \((C, \alpha + j)\) summability for \(j = 1, 2\) and any integer \(j = k\), and some suitable conditions. In special cases of main theorems, we obtain some classical type Tauberian remainder theorems for the \((C, 1)\) summability method.

**Theorem 3.** Let the conditions

\[ \lambda_n n \Delta \tau_n^{(\alpha+1)}(u) = O(1), \quad (4.1) \]

and

\[ \lambda_n \tau_n^{(\alpha)}(u) = O(1) \quad (4.2) \]

be satisfied for \(\alpha > -1.\) If \(u \in ((C, \alpha + 1), m^\lambda),\) then \(u \in ((C, \alpha), m^\lambda).\)

**Proof.** From identity (1.5), we have

\[ \lambda_n n \Delta \tau_n^{(\alpha+1)}(u) = \lambda_n (\alpha + 1)(\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u)) \]

\[ = \lambda_n (\alpha + 1) \tau_n^{(\alpha)}(u) - \lambda_n (\alpha + 1) \tau_n^{(\alpha+1)}(u). \]

From identity (1.4), we obtain

\[ \lambda_n n \Delta \tau_n^{(\alpha+1)}(u) = \lambda_n (\alpha + 1) \tau_n^{(\alpha)}(u) - \lambda_n (\alpha + 1)^2 (\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u)). \]

Rewritten the above equation, we have

\[ \lambda_n (\alpha + 1)^2 (\sigma_n^{(\alpha)}(u) - s) = \lambda_n (\alpha + 1)^2 (\sigma_n^{(\alpha+1)}(u) - s) \]

\[ + \lambda_n (\alpha + 1) \tau_n^{(\alpha)}(u) - \lambda_n n \Delta \tau_n^{(\alpha+1)}(u). \]

Using (4.1) and (4.2), we get

\[ \lambda_n (\alpha + 1)^2 (\sigma_n^{(\alpha)}(u) - s) = O(1) + O(1) + O(1) = O(1). \]

Therefore, \(\lambda_n (\sigma_n^{(\alpha)}(u) - s) = O(1).\) That means \(u \in ((C, \alpha), m^\lambda).\)

Notice that taking \(\alpha = 0,\) we obtain Theorem 2.
Proposition 1. Let the conditions
\[
\lambda_n(n \Delta)^2 \tau_n^{(\alpha^2)}(u) = O(1),
\]
(4.3)
\[
\lambda_n \tau_n^{(\alpha)}(u) = O(1),
\]
(4.4)
and
\[
\lambda_n(\sigma_n^{(\alpha^2)}(u) - s) = O(1)
\]
(4.5)
be satisfied for \(\alpha > -1\). If \(u \in ((C, \alpha + 1), m^2)\), then \(u \in ((C, \alpha), m^2)\).

Proof. Taking \(k = 2\) in Lemma 1, we have
\[
\lambda_n(n \Delta)^2 \tau_n^{(\alpha^2)}(u) = \lambda_n(\alpha + 2)(\alpha + 1) \left( \tau_n^{(\alpha)}(u) - \tau_n^{(\alpha + 1)}(u) \right)
\]
\[
\lambda_n(\alpha + 2)^2 \left( \tau_n^{(\alpha + 1)}(u) - \tau_n^{(\alpha + 2)}(u) \right)
\]
\[
= \lambda_n(\alpha + 2)(\alpha + 1) \tau_n^{(\alpha)}(u) - \lambda_n(\alpha + 2)(\alpha + 1) \tau_n^{(\alpha + 1)}(u)
\]
\[
- \lambda_n(\alpha + 2)^2 \left( \tau_n^{(\alpha)}(u) - \tau_n^{(\alpha + 1)}(u) \right)
\]
\[
+ \lambda_n(\alpha + 2)^2 \left( \tau_n^{(\alpha + 1)}(u) - \tau_n^{(\alpha + 2)}(u) \right)
\]
From identity (1.4), we get
\[
\lambda_n(n \Delta)^2 \tau_n^{(\alpha^2)}(u)
\]
\[
= \lambda_n(\alpha + 2)(\alpha + 1) \tau_n^{(\alpha)}(u) - \lambda_n(\alpha + 2)(\alpha + 1) \left( (\alpha + 1)(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha + 1)}(u)) \right)
\]
\[
- \lambda_n(\alpha + 2)^2 \left( (\alpha + 1)(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha + 1)}(u)) \right)
\]
\[
+ \lambda_n(\alpha + 2)^2 \left( (\alpha + 1)(\sigma_n^{(\alpha + 1)}(u) - \sigma_n^{(\alpha + 2)}(u)) \right)
\]
\[
= \lambda_n(\alpha + 2)(\alpha + 1) \tau_n^{(\alpha)}(u) - \lambda_n(\alpha + 2)(\alpha + 1) \tau_n^{(\alpha + 1)}(u)
\]
\[
+ \lambda_n(\alpha + 1)^2(\alpha + 2)\sigma_n^{(\alpha^2)}(u)
\]
\[
+ \lambda_n(\alpha + 2)^2(\alpha + 1)\sigma_n^{(\alpha+1)}(u) + \lambda_n(\alpha + 2)^2(\alpha + 1)\sigma_n^{(\alpha)}(u)
\]
\[
- \lambda_n(\alpha + 2)^2(\alpha + 1)(\sigma_n^{(\alpha+1)}(u) - \sigma_n^{(\alpha+2)}(u))
\]
Rewriting the above equation, we have
\[
\lambda_n(\alpha + 1)(\alpha + 2)(2\alpha + 3)(\sigma_n^{(\alpha)}(u) - s)
\]
\[
= -\lambda_n \Delta \tau_n^{(\alpha^2)}(u) + \lambda_n(\alpha + 2)(\alpha + 1) \tau_n^{(\alpha)}(u) + \lambda_n((\alpha + 2)^3 + (\alpha + 2)^2(\alpha + 1)
\]
\[
+ (\alpha + 1)^2(\alpha + 2) - s)(\sigma_n^{(\alpha+1)}(u) - \sigma_n^{(\alpha^2)}(u))
\]
Using (4.3), (4.4) and (4.5), we get
\[
\lambda_n(\alpha + 1)(\alpha + 2)(2\alpha + 3)(\sigma_n^{(\alpha)}(u) - s) = O(1) + O(1) + O(1) = O(1).
Therefore, \( \lambda_n(\sigma_n^{(\alpha)}(u) - s) = O(1) \). That means \( u \in ((C, \alpha), m^\lambda) \). □

Now, we represent a Tauberian remainder theorem which generalizes Theorem 3 and Proposition 1.

**Theorem 4.** Let the conditions

\[
\lambda_n(n\Delta_k \tau_n^{(\alpha+k)}(u) = O(1),
\]

\[
\lambda_n \tau_n^{(\alpha)}(u) = O(1),
\]

and

\[
\lambda_n(\sigma_n^{(\alpha+j)}(u) - s) = O(1) \quad \text{for} \quad 2 \leq j \leq k
\]

be satisfied for \( \alpha > -1 \). If \( u \in ((C, \alpha + 1), m^\lambda) \), then \( u \in ((C, \alpha), m^\lambda) \).

**Proof.** From Lemma 1 we have

\[
\lambda_n(n\Delta_k \tau_n^{(\alpha+k)}(u) - B_{1,1-a} \lambda_n \tau_n^{(\alpha)}(u) - B_{1,1} \lambda_n \sigma_n^{(\alpha+1)}(u) + B_{1,1} \lambda_n \sigma_n^{(\alpha+1)}(u)
\]

\[
+ \lambda_n \sum_{j=2}^{k} \left( B_{j,j-1} \sigma_n^{(\alpha+j-2)}(u) - 2B_{j-2,j-1} \sigma_n^{(\alpha+j-1)}(u) + B_{j,j} \sigma_n^{(\alpha+j)}(u) \right).
\]

Rewritten the above equation, we have

\[
B_{1,1} \lambda_n(\sigma_n^{(\alpha)}(u) - s)
\]

\[
= B_{1,1-a} \lambda_n \tau_n^{(\alpha)}(u) - \lambda_n(n\Delta_k \tau_n^{(\alpha+k)}(u) + B_{1,1} \lambda_n \sigma_n^{(\alpha+1)}(u) - s)
\]

\[
+ \lambda_n \sum_{j=2}^{k} B_{j,j-1}(\sigma_n^{(\alpha+j-2)}(u) - s) - \lambda_n \sum_{j=2}^{k} 2B_{j-2,j-1}(\sigma_n^{(\alpha+j-1)}(u) - s)
\]

\[
+ \lambda_n \sum_{j=2}^{k} B_{j,j}(\sigma_n^{(\alpha+j)}(u) - s).
\]

Using (4.6), (4.7) and (4.8), we get

\[
B_{1,1} \lambda_n(\sigma_n^{(\alpha)}(u) - s) = O(1) + O(1) + O(1) + O(1) + O(1) + O(1) = O(1).
\]

Therefore, \( \lambda_n(\sigma_n^{(\alpha)}(u) - s) = O(1) \). That means \( u \in ((C, \alpha), m^\lambda) \). □

**Theorem 5.** Let the condition

\[
\lambda_n \tau_n^{(\alpha+j+1)}(u) = O(1) \quad \text{for} \quad 0 \leq j \leq k - 1
\]

be satisfied for \( \alpha > -1 \). If \( u \in ((C, \alpha + k), m^\lambda) \), then \( u \in ((C, \alpha), m^\lambda) \).

**Proof.** Suppose that \( u \in ((C, \alpha + k), m^\lambda) \). Taking \( j = k - 1 \) in (4.9), it follows from the identity

\[
\tau_n^{(\alpha+k)}(u) = (\alpha+k)(\sigma_n^{(\alpha+k-1)}(u) - \sigma_n^{(\alpha+k)}(u))
\]
that we obtain
\[
\lambda_n(\alpha + k)(\sigma_n^{\alpha+k-1}(u) - s) = \lambda_n\tau_n^{\alpha+k}(u) + \lambda_n(\alpha + k)(\sigma_n^{\alpha+k}(u) - s)
\]
= \(O(1) + O(1) = O(1)
\)
then we obtain \(\lambda_n(\sigma_n^{\alpha+k-1} - s) = O(1).\) Hence, that means
\[
u \in ((C, \alpha + k - 1), m^\lambda).
\]
From identity (1.4), we have
\[
\tau_n^{\alpha+k-1}(u) = (\alpha + k - 1)(\sigma_n^{\alpha+k-2}(u) - \sigma_n^{\alpha+k-1}(u)).
\]
Taking \(j = k - 2\) in (4.9), we obtain
\[
\lambda_n(\alpha + k - 1)(\sigma_n^{\alpha+k-2}(u) - s)
\]
\[
= \lambda_n\tau_n^{\alpha+k-1}(u) + \lambda_n(\alpha + k - 1)(\sigma_n^{\alpha+k-1}(u) - s) = O(1) + O(1) = O(1)
\]
Therefore we have
\[
u \in ((C, \alpha + k - 2), m^\lambda).
\]
Continuing in this way, we obtain that
\[
u \in ((C, \alpha + 1), m^\lambda).
\]
Taking \(j = 0\) in (4.9), we obtain \(\lambda_n\tau_n^{\alpha+1} = O(1).\) From identity (1.4), we have
\[
\lambda_n(\alpha + 1)(\sigma_n^\alpha(u) - s) = \lambda_n\tau_n^{\alpha+1}(u) + \lambda_n(\alpha + 1)(\sigma_n^{\alpha+1}(u) - s)
\]
= \(O(1) + O(1) = O(1)
\]
This completes the proof. \(\square\)

**Theorem 6.** Let the condition
\[
\lambda_n(n\Delta)_{j}\tau_n^{\alpha+j}(u) = O(1) \quad \text{for} \quad 0 \leq j \leq k,
\]
be satisfied for \(\alpha > -1.\) If \(u \in ((C, \alpha + k), m^\lambda),\) then \(u \in ((C, \alpha), m^\lambda).\)

**Proof.** By identity (1.6) for \(k = 1,\) it follows
\[
\lambda_n n \Delta \tau_n^{\alpha+1}(u) = \lambda_n(\alpha + 1)(\tau_n^\alpha(u) - \tau_n^{\alpha+1}(u))
\]
\[
= \lambda_n(\alpha + 1)\tau_n^{\alpha}(u) - \lambda_n(\alpha + 1)\tau_n^{\alpha+1}(u).
\]
Taking \(j = 0\) and \(j = 1\) in (4.10), we obtain
\[
\lambda_n\tau_n^{\alpha+1}(u) = O(1)
\]
From identity (1.6) for \(k = 2,\) we get
\[
\lambda_n(n\Delta)_{2}\tau_n^{\alpha+2}(u) = \lambda_n(\alpha + 2)(\alpha + 1)\left(\tau_n^\alpha(u) - \tau_n^{\alpha+1}(u)\right)
\]
\[
- \lambda_n(\alpha + 2)^2\left(\tau_n^{\alpha+1}(u) - \tau_n^{\alpha+2}(u)\right).
\]
Taking $j = 0$ and $j = 2$ in (4.10), we obtain

$$\lambda_n \tau_n^{(\alpha+2)}(u) = O(1)$$

Continuing in this way, by Lemma 1, we obtain

$$\lambda_n (n \Delta_k^{(\alpha+k)}(u) = (\alpha + 1) A_k^{(1)}(\alpha) \lambda_n (\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u))$$

$$\quad + \lambda_n \sum_{j=2}^{k} (\alpha + j)(-1)^{j+1} A_k^{(j)}(\alpha)(\tau_n^{(\alpha+j-1)}(u) - \tau_n^{(\alpha+j)}(u)).$$

Taking $j = 0$ and $j = k$ in (4.10), we obtain

$$\lambda_n \tau_n^{(\alpha+k)}(u) = O(1).$$

The conditions in Theorem 5 hold, the proof is completed. □

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