GLOBAL RAINBOW DOMINATION IN GRAPHS

J. AMJADI, S.M. SHEIKHOESLAMI, AND L. VOLKMANN

Received 27 May, 2014

Abstract. For a positive integer \( k \), a \( k \)-rainbow dominating function (\( k \)-RDF) of a graph \( G \) is a function \( f \) from the vertex set \( V(G) \) to the set of all subsets of the set \( \{1, 2, \ldots, k\} \) such that for any vertex \( v \in V(G) \) with \( f(v) = \emptyset \), the condition \( \bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\} \) is fulfilled, where \( N(v) \) is the neighborhood of \( v \). The weight of a \( k \)-RDF \( f \) is the value \( \omega(f) = \sum_{v \in V} |f(v)| \). A \( k \)-RDF \( f \) is called a global \( k \)-rainbow dominating function (\( G_k \)-RDF) if \( f \) is also a \( k \)-RDF of the complement \( \overline{G} \) of \( G \). The global \( k \)-rainbow domination number of \( G \), denoted by \( \gamma_{GR}(G) \), is the minimum weight of a \( G_k \)-RDF on \( G \). In this paper, we initiate the study of the global \( k \)-rainbow domination number and we establish some sharp bounds for it.

2010 Mathematics Subject Classification: 05C69

Keywords: \( k \)-rainbow dominating function, \( k \)-rainbow domination number, global \( k \)-rainbow dominating function, global \( k \)-rainbow domination number

1. INTRODUCTION

In this paper, \( G \) is a simple graph with vertex set \( V(G) \) and edge set \( E(G) \) (briefly \( V, E \)). The order \( |V| \) of \( G \) is denoted by \( n = n(G) \). Denote by \( K_n \) the complete graph, by \( C_n \) the cycle and by \( P_n \) the path of order \( n \), respectively. For every vertex \( v \in V(G) \), the open neighborhood \( N_G(v) = N(v) \) is the set \( \{u \in V(G) \mid uv \in E(G)\} \) and its closed neighborhood is the set \( N_G[v] = N[v] = N(v) \cup \{v\} \). The degree of a vertex \( v \in V \) is \( \deg_G(v) = \deg(v) = |N(v)| \). The open neighborhood of a set \( S \subseteq V \) is the set \( N_G(S) = N(S) = \bigcup_{v \in S} N(v) \), and the closed neighborhood of \( S \) is the set \( N_G[S] = N[S] = N(S) \cup S \). The minimum and maximum degrees of \( G \) are respectively denoted by \( \delta(G) = \delta \) and \( \Delta(G) = \Delta \). A leaf of a graph is a vertex of degree 1, a support vertex is a vertex adjacent to a leaf and a strong support vertex is a vertex adjacent to at least two leaves. For a vertex \( v \) in a rooted tree \( T \), let \( C(v) \) denote the set of children of \( v \). Let \( D(v) \) denote the set of descendants of \( v \) and \( D[v] = D(v) \cup \{v\} \). The maximal subtree at \( v \) is the subtree of \( T \) induced by \( D[v] \), and is denoted by \( T_v \). We use [12, 19] for terminology and notation which are not defined here.

A subset \( S \) of vertices of \( G \) is a dominating set if \( N[S] = V \). The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of \( G \). A dominating set \( S \) of
G is the global dominating set of G if S is a dominating set both of G and \(G^c\). The global domination number \(\gamma_r(G)\) of G is the minimum cardinality of a global dominating set. The global domination number was introduced independently by Brigham and Dutton [7] (the term factor domination number was used) and Sampathkumar [15] and has been studied by several authors (see for example [3, 20]). Since then some variants of the global domination parameter, such as connected (total) global domination, global minus domination, and global Roman domination, have been studied [4, 5, 10, 13].

For a positive integer \(k\), a \(k\)-rainbow dominating function (kRDF) of a graph G is a function \(f \colon V(G) \to P\{1, 2, \ldots, k\}\) such that for any vertex \(v \in V(G)\) with \(f(v) = \emptyset\), the condition \(\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}\) is fulfilled. The weight of a kRDF \(f\) is the value \(\omega(f) = \sum_{v \in V} |f(v)|\). The \(k\)-rainbow domination number of a graph G, denoted by \(\gamma_{rk}(G)\), is the minimum weight of a kRDF of G. A \(\gamma_r(G)\)-function is a kRDF dominating function of G with weight \(\gamma_{rk}(G)\). Note that \(\gamma_{r1}(G)\) is the classical domination number \(\gamma(G)\).

The \(k\)-rainbow domination number was introduced by Brešar, Henning, and Rall [6] and has been studied by several authors (see for example [1, 2, 8, 9, 11, 14, 16–18]). A 2-rainbow dominating function (briefly, rainbow dominating function) \(f : V \mapsto \mathcal{P}\{1, 2\}\) can be represented by the ordered partition \((V_0, V_1, V_2, V_{1,2})\) (or \((V_0^f, V_1^f, V_2^f, V_{1,2}^f)\) to refer \(f\)) of V, where \(V_0 = \{v \in V \mid f(v) = \emptyset\}\), \(V_1 = \{v \in V \mid f(v) = \{1\}\}\), \(V_2 = \{v \in V \mid f(v) = \{2\}\}\), and \(V_{1,2} = \{v \in V \mid f(v) = \{1, 2\}\}\). In this representation, its weight is \(\omega(f) = |V_1| + |V_2| + 2|V_{1,2}|\).

A kRDF \(f\) is called a global \(k\)-rainbow dominating function (GkRDF) if \(f\) is also a kRDF of the complement \(G^c\) of G. The global \(k\)-rainbow domination number of G, denoted by \(\gamma_{grk}(G)\), is the minimum weight of a GkRDF on G. A \(\gamma_{grk}(G)\)-function is a GkRDF of G with weight \(\gamma_{grk}(G)\). Since every global \(k\)-rainbow dominating function \(f\) of G is a kRDF of G and \(G^c\), and assigning 1 to the vertices with nonempty label under \(f\) is a global dominating set of G, and since assigning \(\{1, 2, \ldots, k\}\) to the vertices of a global dominating set yields a GkRDF, we deduce that

\[
\max\{\gamma_g(G), \gamma_{rk}(G), \gamma_{grk}(G^c)\} \leq \gamma_{grk}(G) \leq k\gamma_g(G).
\]

We note that the global \(k\)-rainbow domination number can differ significantly from the \(k\)-rainbow domination number. For example, for \(n \geq k + 1\), \(\gamma_{rk}(K_n) = k\) and \(\gamma_{grk}(K_n) = n\).

Our purpose in this paper is to initiate the study of the global \(k\)-rainbow domination number in graphs. We study basic properties of the global \(k\)-rainbow domination number and we establish some bounds for it.

We make use of the following results in this paper.

**Theorem A ([14]).** For any graph G of order n and maximum degree \(\Delta(G) \geq 1\),
\[
\gamma_{rk}(G) \geq \frac{k n}{\Delta(G) + k}.
\]
Theorem B ([6]). For \( n \geq 1 \),
\[
\gamma_{r2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.
\]

Theorem C ([6]). For \( n \geq 3 \),
\[
\gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.
\]

Theorem D ([1]). If \( G \) is a graph of order \( n \), then
\[
\gamma_{rk}(G) \leq n - \Delta(G) + k - 1.
\]

Theorem E ([9]). Let \( G \) be a connected graph. If there is a path \( v_3v_2v_1 \) in \( G \) with \( \deg(v_2) = 2 \) and \( \deg(v_1) = 1 \), then \( G \) has a \( \gamma_{r2}(G) \)-function \( f \) such that \( f(v_1) = \{1\} \), and \( 2 \in f(v_3) \).

Since the function \( f \) defined by \( f(v) = \{1\} \) for each \( v \in V(G) \) is a GkRDF of a graph \( G \), we have the first part of the following observation. The second part is easy to see and therefore its proof is omitted.

Observation 1. If \( G \) is a graph of order \( n \), then \( \gamma_{rk}(G) \leq n \). Furthermore, if \( 1 \leq n \leq 4 \), then \( \gamma_{rk}(G) = n \).

2. Graphs with \( \gamma_{rk}(G) = \gamma_{grk}(G) \)

In this section we provide some sufficient conditions for a graph to satisfy \( \gamma_{rk}(G) = \gamma_{grk}(G) \).

Proposition 1. If \( G \) is a disconnected graph with at least two components of order at least \( k \), then
\[
\gamma_{grk}(G) = \gamma_{rk}(G).
\]

 Proof. Let \( G_1, G_2, \ldots, G_k \) be the components of \( G \). Assume, without loss of generality, that \( |V(G_i)| \geq k \) for \( i = 1, 2 \). Let \( f \) be a \( \gamma_{rk}(G) \)-function. Obviously, \( \sum_{v \in V(G_i)} |f(v)| \geq k \) for \( i = 1, 2 \). If \( f(x) = \emptyset \) for some \( x \in V(G_i) \), then clearly \( \bigcup_{v \in V(G_i)} f(v) = \{1, 2, \ldots, k\} \), otherwise we may assume \( \bigcup_{v \in V(G_i)} f(v) = \{1, 2, \ldots, k\} \) for \( i = 1, 2 \) because \( |V(G_1)| \geq k \) and \( |V(G_2)| \geq k \). Then \( f \) is a GkRDF of \( G \) and hence \( \gamma_{grk}(G) \leq \gamma_{rk}(G) \). Now the result follows from (1.1). \( \square \)

According to Proposition 1, if \( G \) is the disjoint union of two copies of the complete graph \( K_n \) \((n \geq k)\), then \( \gamma_{grk}(G) = \gamma_{rk}(G) \).

Proposition 2. If \( G \) is a disconnected graph with \( r \geq 2 \) components \( G_1, G_2, \ldots, G_r \) of order at most \( k - 1 \) such that \( \sum_{i=1}^{r} |V(G_i)| \geq k \), then
\[
\gamma_{grk}(G) = \gamma_{rk}(G).
\]

 Proof. Assume that \( \bigcup_{i=1}^{r} V(G_i) = \{v_1, v_2, \ldots, v_s\} \), and let \( f \) be a \( \gamma_{rk}(G) \)-function. Then clearly \( f(v_i) \neq \emptyset \) for each \( i \). Define \( g : V(G) \rightarrow \mathcal{P}\{\{1, 2, \ldots, k\}\} \) by \( g(v_i) = \{k - i - 1\} \) for \( 1 \leq i \leq k - 1 \), \( g(v_i) = \{1\} \) for \( i = k, k + 1, \ldots, s \) and \( g(x) = f(x) \)
for \( x \in V(G) - \{v_1, v_2, \ldots, v_s\} \). Then obviously \( g \) is a GkRDF of \( G \) of weight \( \omega(g) = \gamma_{grk}(G) \) and the proof is complete. \( \Box \)

According to Proposition 2, if \( G \) is the disjoint union of \( k \) copies of \( K_1 \) and a copy of the complete graph \( K_n (n \geq k) \), then \( \gamma_{grk}(G) = \gamma_{r_k}(G) \).

**Theorem 1.** For any connected graph \( G \) with radius \( \text{rad}(G) \geq 4 \), \( \gamma_{gr2}(G) = \gamma_{r2}(G) \).

**Proof.** Let \( f = (V_0, V_1, V_2, V_{1,2}) \) be a \( \gamma_{r2}(G) \)-function such that \( |V_{1,2}| \) is maximum. We show that \( f \) is a G2RDF of \( G \). Suppose to the contrary that \( f \) is not a 2RDF of \( G \). Then there exists a vertex \( v \in V_0 \) such that \( V_{1,2} \subseteq N(v) \) and either \( V_1 \subseteq N(v) \) or \( V_2 \subseteq N(v) \). Assume, without loss of generality, that \( V_1 \subseteq N(v) \). Let \( u \) be an arbitrary vertex in \( V(G) \). If \( u \in V_1 \cup V_{1,2} \), then \( d(u, v) = 1 \). If \( u \in V_0 \), then \( u \) and \( v \) have a common neighbor in \( V_1 \) or \( V_{1,2} \) implying that \( d(u, v) \leq 2 \). Let \( u \in V_2 \). If \( u \) has a neighbor in \( V_1 \cup V_{1,2} \), then \( d(u, v) \leq 2 \) as above. If \( u \) has a neighbor \( w \) in \( V_0 \), then \( d(u, v) \leq d(u, w) + d(w, v) \leq 3 \). Otherwise, since \( G \) is connected, \( u \) has a neighbor \( x \) in \( V_2 \). Then the function \( g \) defined by \( g(u) = \emptyset, g(x) = \{1, 2\} \) and \( g(y) = f(y) \) for \( y \in V(G) - \{u, x\} \), is a \( \gamma_{r2}(G) \)-function which contradicts the choice of \( f \). Thus \( f \) is a G2RDF of \( G \) and the proof is complete. \( \Box \)

**Corollary 1.** Let \( G \) be a connected graph of diameter \( \text{diam}(G) \geq 7 \). Then

\[ \gamma_{gr2}(G) = \gamma_{r2}(G). \]

The next results is an immediate consequence of Theorems B, C and 1.

**Corollary 2.** For \( n \geq 8 \),

\[ \gamma_{gr2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1. \]

**Corollary 3.** For \( n \geq 8 \),

\[ \gamma_{gr2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor. \]

### 3. Bounds on the Global \( k \)-Rainbow Domination Number

In this section we present some sharp lower and upper bounds on \( \gamma_{grk}(G) \).

**Proposition 3.** For any integer \( k \geq 2 \) and any graph \( G \) of order \( n \geq 2k \),

\[ \gamma_{grk}(G) \geq 2k. \]

**Proof.** Let \( f \) be a \( \gamma_{grk}(G) \)-function, and let \( V_0 = \{v \in V(G) \mid f(v) = \emptyset\} \). If \( V_0 = \emptyset \), then \( \gamma_{grk}(G) = n \geq 2k \). Let \( V_0 \neq \emptyset \) and \( v \in V_0 \). Then \( \bigcup_{x \in N_G(v)} f(x) = \{1, 2, \ldots, k\} \) and \( \bigcup_{x \in N_G(v)} f(x) = \{1, 2, \ldots, k\} \). Since \( N_G(v) \cap N_G(v) = \emptyset \), we obtain \( \gamma_{grk}(G) = \omega(f) \geq 2k \), as desired. \( \Box \)
This bound is sharp for the disjoint union of two copies of the complete graph \( K_n \) \((n \geq k + 1)\).

**Proposition 4.** For any graph \( G \) of order \( n \geq 4 \), \( \gamma_{gr2}(G) = 4 \) if and only if \( G \) satisfies one of the following properties.

(i) \( n = 4 \),

(ii) there exist two vertices \( u \) and \( v \) in \( G \) such that \( N(u) \cap N(v) = \emptyset \) and \( N[u] \cup N[v] = V \).

(iii) there exist three distinct vertices \( u, v, w \) in \( G \) such that \( N(u) \cap (N(v) \cup N(w)) = \emptyset \) and \( N(u) \cup (N(v) \cap N(w)) = V - \{u, v, w\} \).

(iv) there exist four distinct vertices \( u, v, w, x \) in \( G \) such that \( (N(u) \cap N(v)) \setminus \{w, x\} = \emptyset, (N(w) \cap N(x)) \setminus \{u, v\} = \emptyset, (N[u] \cup N[v]) \setminus \{w, x\} = V - \{w, x\} \) and \( (N[w] \cup N[x]) \setminus \{u, v\} = V - \{u, v\} \).

**Proof.** If \( n = 4 \), then it is clear that \( \gamma_{gr2}(G) = 4 \). Let \( n \geq 5 \). If (ii) holds, then the function \( f : V \rightarrow \mathcal{P}(\{1, 2\}) \) defined by \( f(u) = f(v) = \{1, 2\} \) and \( f(z) = \emptyset \) for \( z \in V(G) - \{u, v\} \), is a 2RDF of \( G \) and \( \overline{G} \) which yields \( \gamma_{gr2}(G) = 4 \) by Proposition 3.

If (iii) holds, then the function \( f : V \rightarrow \mathcal{P}(\{1, 2\}) \) defined by \( f(u) = \{1\}, f(w) = \{2\} \) and \( f(z) = \emptyset \) for \( z \in V(G) - \{u, v, w\} \), is a 2RDF of \( G \) and \( \overline{G} \) which yields \( \gamma_{gr2}(G) = 4 \) again. Let (iv) hold. Then the function \( f : V \rightarrow \mathcal{P}(\{1, 2\}) \) defined by \( f(u) = \{1\}, f(w) = \{2\} \) and \( f(z) = \emptyset \) for \( z \in V(G) - \{u, v, w, x\} \), is a 2RDF of \( G \) and \( \overline{G} \). This implies that \( \gamma_{gr2}(G) = 4 \).

Conversely, Let \( \gamma_{gr2}(G) = 4 \) and let \( f = (V_0, V_1, V_2, V_{1,2}) \) be a \( \gamma_{gr2}(G) \)-function such that \( |V_{1,2}| \) is maximum. We consider three cases.

**Case 1.** \( |V_{1,2}| = 2 \).

Let \( V_{1,2} = \{u, v\} \). Then \( V_0 = V(G) - \{u, v\} \). Since \( f \) is a G2RDF, each vertex in \( w \in V(G) - \{u, v\} \) must be adjacent to a vertex in \( \{u, v\} \) in both \( G \) and \( \overline{G} \). It follows that \( N[u] \cup N[v] = V \) and \( N(u) \cap N(v) = \emptyset \), i.e. \( G \) satisfies (ii).

**Case 2.** \( |V_{1,2}| = 1 \).

Then \( |V_1| = |V_2| = 1 \). Let \( V_{1,2} = \{u\}, V_1 = \{v\} \) and \( V_2 = \{w\} \). Hence \( V_0 = V(G) - \{u, v, w\} \). Every vertex of \( w \in V(G) - \{u, v, w\} \) must be adjacent to \( u \) or both of \( u, v \) in \( G \) and \( \overline{G} \) because \( f \) is a 2RDF of \( G \) and \( \overline{G} \). This yields \( N(u) \cap (N(v) \cup N(w)) = \emptyset \) and \( N(u) \cup (N(v) \cup N(w)) = V - \{u, v, w\} \). Thus \( G \) satisfies (iii) in this case.

**Case 3.** \( |V_{1,2}| = 0 \).

If \( V_0 = \emptyset \), then \( V_1 \cup V_2 = V(G) \) which implies that \( 4 = \gamma_{gr2}(G) = |V_1 \cup V_2| = n \), i.e. \( G \) satisfies (i). Now assume that \( V_0 \neq \emptyset \) and let \( z \in V_0 \). Since \( f \) is a 2RDF of \( G \) and \( \overline{G} \), \( \bigcup_{v \in \mathbb{N}_G(z)} f(v) = \{1, 2\} \) and \( \bigcup_{v \in \mathbb{N}_{\overline{G}}(z)} f(v) = \{1, 2\} \). Assume that \( u, w \in \mathbb{N}_G(z) \) and \( x, v \in \mathbb{N}_{\overline{G}}(z) \) such that \( f(u) = f(v) = \{1\} \) and \( f(w) = f(x) = \{2\} \). Since \( f \) is a G2RDF, each vertex in \( V(G) - \{u, v, w, x\} \) must be adjacent to a vertex in \( \{u, v\} \) and a vertex in \( \{w, x\} \) in \( G \) and \( \overline{G} \). It follows that \( (N(u) \cap N(v)) \setminus \{w, x\} = \emptyset, (N(w) \cap N(x)) \setminus \{u, v\} = \emptyset, (N[u] \cup N[v]) \setminus \{w, x\} = V - \{w, x\} \) and \( (N[w] \cup N[x]) \setminus \{u, v\} = V - \{u, v\} \). Thus \( G \) satisfies (iv). This completes the proof. \( \square \)
Proposition 5. Let $k \geq 2$ be an integer. If the graph $G$ has $r \geq 1$ components $G_1, G_2, \ldots, G_r$ with $\sum_{i=1}^{r} |V(G_i)| \leq k - 1$ then
\[ \gamma_{grk}(G) \leq \gamma_{rk}(G) + k - \sum_{i=1}^{r} |V(G_i)|. \]

Proof. Let $\bigcup_{i=1}^{r} V(G_i) = \{v_1, v_2, \ldots, v_s\}$, and let $f$ be a $\gamma_{rk}(G)$-function. Clearly, $f(v_i) \neq \emptyset$ for each $i$. Define $g : V(G) \to \mathcal{P}\{1, 2, \ldots, k\}$ by $g(v_i) = \{s, s + 1, \ldots, k\}$, $g(v_i) = \{i\}$ for $i = 1, 2, \ldots, s - 1$ and $g(x) = f(x)$ for $x \in V(G) - \{v_1, v_2, \ldots, v_s\}$. Then obviously $g$ is a $GkRDGF$ of $G$ with weight $\omega(g) = \gamma_{rk}(G) + k - s$ and so $\gamma_{grk}(G) \leq \gamma_{rk}(G) + k - \sum_{i=1}^{r} |V(G_i)|$. □

Let $H$ be the disjoint union of $r \leq k - 1$ isolated vertices and a star $K_{1,s}$ with $s \geq k$. Then $\gamma_{rk}(H) = r + k$ and $\gamma_{grk}(H) = 2k$. This example demonstrates that Proposition 5 is tight.

Proposition 6. Let $G$ be a graph of order $n \geq 4$ and $u, v \in V(G)$. If $uv \not\in E(G)$, then
\[ \gamma_{grk}(G) \leq n - \deg(u) - \deg(v) + 2|N(u) \cap N(v)| + 2k - 2, \]
and if $uv \in E(G)$, then
\[ \gamma_{grk}(G) \leq n - \deg(u) - \deg(v) + 2|N(u) \cap N(v)| + 2k. \]

Proof. Define $f : V(G) \to \mathcal{P}\{1, 2, \ldots, k\}$ as follows
\[ f(z) = \begin{cases} 
\{1, 2, \ldots, k\} & \text{if } z \in \{u, v\} \\
\emptyset & \text{if } z \in ((N(u) \cup N(v)) - \{u, v\}) \setminus (N(u) \cap N(v)) \\
\{1\} & \text{otherwise.} 
\end{cases} \]

It is easy to see that $f$ is a $GkRDGF$ of $G$ which attains the bound. This completes the proof. □

Corollary 4. If $G$ is a connected triangle-free graph of order $n \geq 4$, then
\[ \gamma_{grk}(G) \leq \min\{n - \Delta(G) - \delta(G) + 2k, \gamma_{rk}(G) + 2k - 1\}. \]

Proof. By considering a vertex of maximum degree and one of its neighbors, it follows from Proposition 6 that $\gamma_{grk}(G) \leq n - \Delta(G) - \delta(G) + 2k$. Hence, it is sufficient to show that $\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k - 1$. If $n \leq \gamma_{rk}(G) + 2k - 1$, the result is immediate. Let $n > \gamma_{rk}(G) + 2k - 1$ and let $f$ be a $\gamma_{rk}(G)$-function. Then there exists a vertex $u$ such that $f(u) = \emptyset$. Then $u$ has a neighbor $v$ such that $|f(v)| \geq 1$. Define $g : V(G) \to \mathcal{P}\{1, 2, \ldots, k\}$ by $g(u) = g(v) = \{1, 2, \ldots, k\}$ and $g(x) = f(x)$ otherwise. Clearly, $g$ is a $GkRDGF$ of $G$ and hence $\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k - 1$. This completes the proof. □
Proposition 7. Let \( k \geq 2 \) be an integer, and let \( G \) be a graph of diameter \( \text{diam}(G) \geq 5 \). Then

\[
\gamma_{g\text{-rk}}(G) \leq \gamma_{rk}(G) + 2k - 2.
\]

Proof. If \( G \) is disconnected, then the result follows from Propositions 1 and 5. Henceforth, we assume that \( G \) is connected. Let \( f \) be a \( \gamma_{rk}(G) \)-function. Let \( v_1v_2\ldots v_d \) be a diametral path in \( G \). If \( f(v_1) = f(v_d) = \emptyset \), then we have \( \bigcup_{x \in N(v_1)} f(x) = \{1, 2, \ldots, k\} \) and \( \bigcup_{x \in N(v_d)} f(x) = \{1, 2, \ldots, k\} \). Since \( \text{diam}(G) \geq 5 \), we have \( N(v_1) \cap N(v_d) = \emptyset \). It follows that \( f \) is a GkRDF of \( G \) and hence \( \gamma_{g\text{-rk}}(G) = \gamma_{rk}(G) \). If \( f(v_1) \neq \emptyset \) and \( f(v_d) \neq \emptyset \), then the function \( g : V \to \mathcal{P}\{1, 2, \ldots, k\} \) defined by \( g(v_1) = g(v_d) = \{1, 2, \ldots, k\} \) and \( g(x) = f(x) \) for \( x \in V(G) - \{v_1, v_d\} \), is a GkRDF of \( G \) of weight at most \( \omega(f) + 2k - 2 \) and so \( \gamma_{g\text{-rk}}(G) \leq \gamma_{rk}(G) + 2k - 2 \). Now let \( f(v_1) = \emptyset \) and \( f(v_d) \neq \emptyset \) (the case \( f(v_1) \neq \emptyset \) and \( f(v_d) = \emptyset \) is similar). Define \( g : V \to \mathcal{P}\{1, 2, \ldots, k\} \) by \( g(v_d) = \{1, 2, \ldots, k\} \) and \( g(x) = f(x) \) for \( x \in V(G) - \{v_d\} \). Obviously, \( g \) is a GkRDF of \( G \) of weight at most \( \omega(f) + k - 1 \) and so \( \gamma_{g\text{-rk}}(G) \leq \gamma_{rk}(G) + k - 1 \). This completes the proof. \( \square \)

Proposition 8. If \( G \) is a graph of diameter 3 or 4, then

\[
\gamma_{g\text{-rk}}(G) \leq \gamma_{rk}(G) + 2k.
\]

Proof. Let \( f \) be a \( \gamma_{rk}(G) \)-function, and let \( u \) and \( v \) be two vertices of \( G \) such that \( d(u, v) = \text{diam}(G) \). Then the function \( g : V \to \mathcal{P}\{1, 2, \ldots, k\} \) defined by \( g(u) = g(v) = \{1, 2, \ldots, k\} \) and \( g(x) = f(x) \) for \( x \in V(G) - \{u, v\} \), is a GkRDF of \( G \) and therefore \( \gamma_{g\text{-rk}}(G) \leq \gamma_{rk}(G) + 2k \). \( \square \)

Theorem 2. If \( G \) is a graph of order \( n \geq 4 \) with minimum degree \( \delta(G) \), then

\[
\gamma_{g\text{-rk}}(G) \leq \gamma_{rk}(G) + \delta(G) + k - 1.
\]

This bound is sharp for stars \( K_{1,t} \) (\( t \geq 2k - 1 \)) by Proposition 3.

Proof. If \( G \) is disconnected, then the result follows from Propositions 1 and 5. Therefore we assume that \( G \) is connected. Let \( u \) be a vertex of minimum degree \( \delta(G) \), \( f \) be a \( \gamma_{rk}(G) \)-function and \( B = \{x \in N(u) \mid f(x) = \emptyset\} \).

If \( f(u) = \emptyset \), then \( \bigcup_{v \in N(u) - B} f(v) = \{1, 2, \ldots, k\} \). Then obviously the function \( g : V(G) \to \mathcal{P}\{1, 2, \ldots, k\} \) defined by \( g(u) = \{1, 2, \ldots, k\} \), \( g(x) = \{1\} \) if \( x \in B \) and \( g(z) = f(z) \) otherwise, is a GkRDF of \( G \) with weight at most \( \gamma_{rk}(G) + \delta(G) + k - 1 \) and hence \( \gamma_{g\text{-rk}}(G) \leq \gamma_{rk}(G) + \delta(G) + k - 1 \).

Let \( |f(u)| \geq 1 \). Define \( g : V(G) \to \mathcal{P}\{1, 2, \ldots, k\} \) by \( g(u) = \{1, 2, \ldots, k\} \), \( g(v) = \{1\} \) if \( v \in B \) and \( g(z) = f(z) \) for each \( z \in V(G) - (B \cup \{u\}) \). It is clear that \( g \) is a GkRDF of \( G \) with weight at most \( \gamma_{rk}(G) + \delta(G) + k - 1 \) and hence \( \gamma_{g\text{-rk}}(G) \leq \gamma_{rk}(G) + \delta(G) + k - 1 \). This completes the proof. \( \square \)
4. Global rainbow domination numbers of trees

According to Theorem 2, for any tree $T$ of order $n \geq 4$ we have

$$\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 2. \quad (4.1)$$

In this section we characterize all extremal trees attaining equality in (4.1). We begin with some lemmas giving some sufficient conditions for a tree to have global 2-rainbow domination number less than $\gamma_{r2}(T) + 2$. As a special case, Corollary 1 and Proposition 3 imply the next results.

**Corollary 5.** For any tree $T$ with $\text{diam}(T) \geq 7$, $\gamma_{gr2}(T) = \gamma_{r2}(T)$.

**Corollary 6.** If $T$ is a star of order $n \geq 4$, then $\gamma_{gr2}(T) = \gamma_{r2}(T) + 2$.

**Lemma 1.** Let $T$ be a tree. If $T$ has two strong support vertices, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$.

**Proof.** Let $u$ and $v$ be two strong support vertices of $T$ and let $f$ be a $\gamma_{r2}(T)$-function. Obviously we may assume that $f(u) = f(v) = \{1, 2\}$. Since $T$ is a tree, $u$ and $v$ have at most one common neighbor. If $u$ and $v$ have no common neighbor, then clearly $f$ is a G2RDF of $T$ and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. If $u$ and $v$ have a common neighbor, say $w$, then the function $g$ defined by $g(w) = f(w) \cup \{1\}$ and $g(x) = f(x)$ otherwise, is a G2RDF of $T$ of weight at most $\gamma_{r2}(T) + 1$ and the result follows.

**Lemma 2.** Let $T$ be a tree. If $\text{diam}(T) = 6$, then $\gamma_{gr2}(T) = \gamma_{r2}(T)$.

**Proof.** Let $P = v_1v_2 \ldots v_7$ be a diametral path of $T$ and let $f$ be a $\gamma_{r2}(T)$-function. Root $T$ at $v_1$. If $v_2$ and $v_6$ are strong support vertices, then $f$ is a $\gamma_{gr2}(T)$-function since $v_2$ and $v_6$ have no common neighbor. Hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Assume, without loss of generality, that $\text{deg}(v_2) = 2$. By Theorem E, we may assume $f(v_1) = \{1\}$ and $2 \in f(v_3)$. If $v_6$ is a strong support vertex, then we can assume $f(v_6) = \{1, 2\}$ and clearly $f$ is a G2RDF of $T$ implying that $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Henceforth, we assume $\text{deg}(v_6) = 2$. By Theorem E, we may assume $f(v_7) = \{1\}$ and $2 \in f(v_5)$. Define the function $g$ by $g(v) = \{1\}$ if $v \in V(T_{v_9})$ and $f(v) = \{2\}$, $g(v) = \{2\}$ if $v \in V(T_{v_8})$ and $f(v) = \{1\}$ and $g(x) = f(x)$ otherwise. Clearly, $g$ is a G2RDF of $T$ of weight $\gamma_{r2}(T)$ and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. This completes the proof.

**Lemma 3.** Let $T$ be a tree. If $\text{diam}(T) = 5$, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$.

**Proof.** Let $P = v_1v_2 \ldots v_6$ be a diametral path of $T$, and let $f$ be a $\gamma_{r2}(T)$-function. If $v_2$ and $v_5$ are strong support vertices, then $f$ is a $\gamma_{gr2}(T)$-function and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Assume, without loss of generality, that all support vertices adjacent to $v_4$ have degree 2. By Theorem E, we may assume $f(v_6) = \{1\}$ and $2 \in f(v_4)$. Then the function $g$ defined by $g(v_3) = f(v_3) \cup \{1\}$ and $g(x) = f(x)$ otherwise, is a G2RDF of $T$ of weight at most $\gamma_{r2}(T) + 1$ that implies $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$. \hfill \Box
A subdivision of an edge $uv$ is obtained by removing the edge $uv$, adding a new vertex $w$, and adding edges $uw$ and $wv$. The subdivision graph $S(G)$ is the graph obtained from $G$ by subdividing each edge of $G$. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a healthy spider. A wounded spider $S_t$ is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 2$. The center of a spider, is the center of the star whose subdivision produced the spider.

**Definition 1.** For $1 \leq i \leq 2$, let $\mathcal{B}_i$ be the family of trees $T$ defined as follows and let $\mathcal{B} = \bigcup_{i=1}^{2} \mathcal{B}_i$.

$\mathcal{B}_1 : T$ is a spider $S_t$ for some $t \geq 2$ with exception of stars, wounded spiders $S_t$ ($t \geq 3$) with exactly one wounded leg or wounded spiders $S_t$ ($t \geq 3$) with at least four healthy legs.

$\mathcal{B}_2 : T$ is obtained from stars $K_{1,r_1}, K_{1,r_2}, \ldots, K_{1,r_j}$ where $r_k \geq 3$ for $1 \leq k \leq j$, with centers $y_1, y_2, \ldots, y_j$ ($j \geq 2$) by adding a new vertex $x$ and joining $x$ to all vertices $y_j$ and adding at most one pendant edge at $x$.

**Lemma 4.** Let $T$ be a tree. If $\text{diam}(T) = 4$, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$ and equality holds if and only if $T \in \mathcal{B}$.

**Proof.** Let $\text{diam}(T) = 4$ and let $P = v_1v_2v_3v_4v_5$ be a diametral path of $T$. Let $f$ be a $\gamma_{r2}(T)$-function. Consider the following cases.

**Case 1.** $\deg(v_2) = 3$.

Suppose $u, v_1$ are the leaves adjacent to $v_2$. Then we can assume that $f(v_2) = \{1, 2\}$. If $\deg(v_4) \geq 3$, then we may assume $f(v_4) = \{1, 2\}$ and if $\deg(v_4) = 2$ then by Theorem E we can assume $f(v_5) = \{1\}$ and $2 \in f(v_3)$. Define $g : V(T) \rightarrow \mathcal{P}(\{1, 2\})$ by $g(v_1) = \{1\}, g(u) = \{2\}, g(v_2) = \emptyset$ and $g(x) = f(x)$ otherwise. Obviously $g$ is a G2RDF of $T$ of weight $\gamma_{r2}(T)$ and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$.

By Case 1, we may assume that all support vertices adjacent to $v_3$ have degree different from 3.

**Case 2.** $\deg(v_2) > 3$.

Then $f(v_2) = \{1, 2\}$. If $\deg(v_4) = 2$, then by Theorem E we may assume $f(v_5) = \{1\}$ and $2 \in f(v_3)$, and clearly $f$ is a G2RDF of $T$ and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. So we assume that each support vertex adjacent to $v_3$ has degree at least 4. If $v_3$ is a strong support vertex, then $f(v_3) = \{1, 2\}$ and clearly $f$ is a G2RDF of $T$ and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Let $v_3$ be not a strong support vertex. Then $T \in \mathcal{B}_2$ and $T$ has at most two $\gamma_{r2}(T)$-functions which none of them is G2RDF of $T$ and hence $\gamma_{gr2}(T) \geq \gamma_{r2}(T) + 1$. On the other hand, the function $g$ defined by $g(v_3) = \{1\}$ and $g(x) = f(x)$ otherwise is a G2RDF of $T$ of weight $\gamma_{r2}(T) + 1$ implying that $\gamma_{gr2}(T) = \gamma_{r2}(T) + 1$.

By Cases 1 and 2, we may assume that all support vertices adjacent to $v_3$ have degree 2. Thus $T$ is a spider of diameter 4. If $T$ is a wounded spiders $S_t$ ($t \geq 3$) with exactly one wounded leg, then the function $g$ that assigns $\emptyset$ to all support vertices
of $T$ with exception of the center of spider, $\{1\}$ to the center of spider and the leaf adjacent to the center of spider, and $\{2\}$ to the other leaves, is a G2RDF of $T$ of weight $\gamma_{r2}(T)$ implying that $\gamma_{g \cdot r2}(T) = \gamma_{r2}(T)$. Now $T$ is a wounded spider $S_t$ ($t \geq 3$) with at least four healthy legs. Suppose $x$ is the center of $T$ and $u_1, u_2, u_3, u_4$ are leaves at distance two from $x$. Then the function $g$ that assigns $\{1, 2\}$ to $x$, $\emptyset$ to all support vertices of $T$, $\{1\}$ to $u_1, u_2$, and $\{2\}$ to the other leaves, is a G2RDF of $T$ of weight $\gamma_{r2}(T)$ implying that $\gamma_{g \cdot r2}(T) = \gamma_{r2}(T)$. Finally let $T$ be a spider that is not a wounded spider $S_t$ ($t \geq 3$) with exactly one wounded leg or a wounded spider $S_t$ ($t \geq 3$) with at least four healthy legs, that is $T \in \mathcal{B}_1$. It is easy to see that in this case $\gamma_{g \cdot r2}(T) = \gamma_{r2}(T) + 1$ and the proof is complete. □

For $p, q \geq 1$, a double star $DS(p, q)$ is a tree with exactly two vertices that are not leaves, with one adjacent to $p$ leaves and the other to $q$ leaves.

**Lemma 5.** Let $T$ be a tree. If $\text{diam}(T) = 3$, then $\gamma_{g \cdot r2}(T) \leq \gamma_{r2}(T) + 1$ and equality holds if and only if $T = DS(p, q)$ with $q \geq p = 1$.

**Proof.** Let $\text{diam}(T) = 3$. Then $T$ is a double star $DS(p, q)$ with $q \geq p \geq 1$. Let $u, v$ be the vertices of $T$ of degree $p$ and $q$, respectively. If $p \geq 2$, then $u, v$ are strong support vertices with no common neighbor and it follows from the proof of Lemma 1 that $\gamma_{g \cdot r2}(T) = \gamma_{r2}(T)$. Henceforth, assume $p = 1$. If $q = 1$, then $T = P_4$ and clearly $\gamma_{g \cdot r2}(T) = \gamma_{r2}(T) + 1$. Let $q \geq 2$ and $u'$ be the leaf adjacent to $u$. Then $T$ has exactly two $\gamma_{r2}(T)$-functions $f_i$ ($i = 1, 2$) defined by $f_i(v) = \{1, 2\}$, $f_i(u') = \{i\}$ and $f_i(x) = \emptyset$ otherwise. Obviously, none of $f_1$ or $f_2$ is not a G2RDF of $T$ and also the function $g$ defined by $g(u) = \{1\}$ and $g(x) = f_1(x)$ for $x \in V(T) \setminus \{u\}$ is a G2RDF of $T$ that yields $\gamma_{g \cdot r2}(T) \geq \gamma_{r2}(T) + 1$. □

The next theorem is an immediate consequence of (4.1), Corollaries 5, 6 and Lemmas 2, 3, 4, 5.

**Theorem 3.** Let $T$ be a tree of order $n \geq 4$. Then $\gamma_{g \cdot r2}(T) = \gamma_{r2}(T) + 2$ if and only if $T$ is the star $K_{1,t}$ for some $t \geq 3$.

**ACKNOWLEDGEMENT**

The authors would like to thank anonymous referees for their remarks and suggestions that helped improve the manuscript.

**REFERENCES**


GLOBAL RAINBOW DOMINATION IN GRAPHS


Authors’ addresses

J. Amjadi
Azarbaijan Shahid Madani University, Department of Mathematics, Tabriz, I.R. Iran
E-mail address: j-amjadi@azaruniv.edu

S.M. Sheikholeslami
Azarbaijan Shahid Madani University, Department of Mathematics, Tabriz, I.R. Iran
E-mail address: s.m.sheikholeslami@azaruniv.edu

L. Volkmann
RWTH Aachen University, Lehrstuhl II für Matematik, 52056, Aachen, Germany
E-mail address: volkm@math2.rwth-aachen.de