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DEGREE SUM CONDITION FOR FRACTIONAL ID-k-FACTOR-CRITICAL GRAPHS

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Abstract. A graph G is called a fractional ID-k-factor-critical graph if after deleting any independent set of G the resulting graph admits a fractional k-factor. In this paper, we prove that for $k \geq 2$, G is a fractional ID-k-factor-critical graph if $\delta(G) \geq \frac{n}{3} + k$, $\sigma_2(G) \geq \frac{4n}{3}$, $n \geq 6k-8$. The result is best possible in some sense.

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1. INTRODUCTION

All graphs considered in this paper are finite, loopless, and without multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. We use $N_G[x]$ to denote $N_G(x) \cup \{x\}$. We denote the minimum degree and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively.

Let $k \geq 1$ be an integer. A spanning subgraph F of G is called a *k-factor* if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \to [0,1]$ be a function. If $\sum_{x \in e} h(e) = k$ for any $x \in V(G)$, then we call $G[F_h]$ a *fractional k-factor* of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. The following result on degree condition for fractional k-factor is known.

Theorem 1 (Yu et al. [10]). *Let k be an integer with $k \geq 2$, and let G be a graph of order n with $n \geq 4k-3$, $\delta(G) \geq k$. If*

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{2}$$

for each pair of non-adjacent vertices u and v of G, then G has a fractional k-factor.

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In what follows, we always assume that n is order of G, i.e., $n = |V(G)|$, and G is not complete. Chang et al. [1] introduced the concept of fractional independent-set-deletable k-factor critical (shortly, ID-k-factor critical) graph, that is, if removing any independent I from G, the resulting graph has a fractional k-factor. Also, Chang et al. [1] proved that if $n \geq 6k-8$ and $\delta(G) \geq \frac{2n}{3}$, then G is fractional ID-k-factor-critical. More results on fractional ID-k-factor-critical graphs can be found in Gao and Wang [2–6] and Jin [8].

In this paper, we focus on the degree sum condition for fractional ID-k-factor-critical graph. Let $\sigma_2(G) = \min\{d_G(u)+d_G(v)\}$ for each pair of non-adjacent vertices u and v of G. Niessen [9] researched the degree sum condition for a graph which exists regular factor. Iida and Nishimura [7] studied the existence of factor by virtue of $\sigma_2(G)$, and proved that if $n \geq 4k-5$, kn is even, $\delta(G) \geq k$, and $\sigma_2(G) \geq n$, then G has a k-factor. The main result in our paper study the degree sum condition for fractional ID-k-factor-critical graphs and give as follows:

Theorem 2. *Let $k \geq 2$ be an integer, and let G be a graph of order n with $n \geq 6k-8$. If $\delta(G) \geq \frac{n}{3}+k$ and $\sigma_2(G) \geq \frac{4n}{3}$, then G is a fractional ID-k-factor-critical graph.*

Also, we will show that Theorem 2 is sharp in some sense.

In order to prove our main result, we need the following lemma which is the necessary and sufficient condition for the existence of a fractional k-factor in a graph.

Lemma 1 (L. Zhang and G. Liu [11]). *Let $k \geq 1$ be an integer, and let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S,T) = k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \geq 0$$

where $T = \{x : x \in V(G)-S, d_{G-S}(x) \leq k-1\}$.

2. Proof of Theorem 2

Suppose that G satisfies the conditions of Theorem 2, but is not a fractional ID-k-factor-critical graph. Then there exist an independent set I such that $G' = G-I$ has no fractional k-factor. By the argument of Lemma 1, there exists a subset S of $V(G')$ such that

$$\delta_{G'}(S,T) = k|S| + \sum_{x\in T} d_{G'-S}(x) - k|T| \leq -1. \tag{2.1}$$

Here, $T = \{x : x \in V(G')-S, d_{G'-S}(x) \leq k-1\}$.

If G' is a completed graph, then G' has fractional k-factor from the degree sum condition, the bound of n and the definition of fractional k-factor. This is a contradiction.

If $|I| = 1$, then $n' \geq 6k-9$. It is easy to verify that $\delta(G') \geq k$ and $\max\{d_{G'}(u), d_{G'}(v)\} \geq \frac{n'}{2} = \frac{n-1}{2}$ for each pair of non-adjacent vertices u and v of G'. Thus, the results holds from Theorem 1.

We now consider $|I| \geq 2$ and G' is not complete. Obviously, $T \neq \varnothing$ and $S \neq \varnothing$ by $|I| \geq 2$ and $\delta(G) \geq \frac{n}{3}+k$. Let $d_1 = \min\{d_{G'-S}(x) : x \in T\}$ and choose $x_1 \in T$ such that $d_{G'-S}(x_1) = d_1$. If $T - N_T[x_1] \neq \varnothing$, let $d_2 = \min\{d_{G'-S}(x) : x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{G'-S}(x_2) = d_2$. So, $d_1 \leq d_2$. Let $|S| = s$, $|T| = t$, $|N_T[x_1]| = p$. Then we have $p \leq d_1+1$, $d_{G'-S}(T) \geq d_1 p + d_2(t-p)$ and $ks - kt + d_1 p + d_2(t-p) \leq k|S| - k|T| + d_{G'-S}(T) < 0$. Thus,

$$|S| \leq \frac{k|T| - d_{G'-S}(T) - 1}{k} \leq \frac{k|T|-1}{k},$$

i.e., $1 \leq s \leq t - \frac{1}{k}$.

Let $|V(G')| = n'$. We obtain $2n' \geq \sigma_2(G) \geq \frac{4n}{3} \geq \frac{4}{3}(6k-8)$. Since n' is an integer, we get $n' \geq 4k-5$. If $\sigma_2(G') < n'$, then $\frac{4(n'+|I|)}{3} \leq \sigma_2(G) < n' + 2|I|$, i.e., $n' < 2|I| \leq \frac{2n}{3}$. This contradicts to $\sigma_2(G) \geq \frac{4n}{3}$ and $|I| \geq 2$. Therefore, $\sigma_2(G') \geq n'$. Furthermore, we obtain $\delta(G') \geq k$ by $|I| \geq 2$ and $\delta(G) \geq \frac{n}{3}+k$.

We consider following two cases:

Case 1. $T = N_T[x_1]$**.** In this case, $t = p \leq d_1+1$ and $d_2 = 0$. If $d_1 = k-1$, then $t \leq k$, $k|S| - k|T| + d_{G'-S}(T) \geq ks - kt + d_1 p = ks - kt + (k-1)t \geq ks - t \geq 0$, which contradicts (2.1). If $0 \leq d_1 \leq k-2$, then $t \leq d_1+1 \leq k-1$. By $\delta(G') \geq k$ and $d_{G'}(x_1) \leq s + d_1$, we have $s \geq k - d_1$. Thus, $k|S| - k|T| + d_{G'-S}(T) \geq ks - kt + d_1 p \geq k(k-d_1) + (d_1-k)t = (k-d_1)(k-t) > 0$, which contradicts (2.1).

Case 2. $T - N_T[x_1] \neq \varnothing$**.** We consider following three subcases.

Case 2.1. $d_1 = d_2 = k-1$**.** In this subcase, $k|S| - k|T| + d_{G'-S}(T) \geq ks - kt + d_1 p + d_2(t-p) = ks - kt + (k-1)p + (k-1)(t-p) = ks - t \geq 0$, which contradicts (2.1). In fact, if $ks \leq t-1$, then $s + ks + 1 \leq s + t \leq n'$. Note that x_1, x_2 are not adjacent in G'. Thus, $2(s+k-1) \geq \sigma_2(G') \geq n' \geq s + sk + 1$. We get $s = 1$. Thus, $2(1+k-1) \geq 2(s+k-1) \geq \sigma_2(G') \geq n' \geq 4k-5$, i.e., $k = 2$. In this case, $d_1 = d_2 = 1$, $s = 1$, $t \geq 3$, $n' \geq 4$. We have $4 \leq n' \leq \sigma_2(G') \leq 2s+2 = 4$, i.e., $t = 3$, $n' = 4$. Thus, the vertex $T - \{x_1, x_2\}$ has degree 2 in T, and we can check that $k|S| - k|T| + d_{G'-S}(T) = 0$. This is a contradiction.

Case 2.2. $0 \leq d_1 \leq k-2$ and $d_2 = k-1$. In this subcase, $p \leq d_1+1 \leq k-1$. Since x_1 and x_2 are not adjacent in G', we have $(s+k-1) + (s+d_1) \geq \sigma_2(G') \geq n' \geq 4k-5$, i.e., $n' \leq 2s + k - 1 + d_1$ and $s \geq \frac{3k-d_1-4}{2}$. Thus,

$$k|S| - k|T| + d_{G'-S}(T)$$
$$\geq ks - kt + d_1 p + d_2(t-p)$$

$$\geq ks - k(n'-s) + (d_1-k+1)(d_1+1) + (k-1)(n'-s)$$
$$= (k+1)s - n' - k + 1 + d_1^2 + (2-k)d_1$$

$$\geq (k+1)s-(2s+k-1+d_1)-k+1+d_1^2+(2-k)d_1$$
$$=(k-1)s-2k+2+d_1^2+(1-k)d_1$$
$$\geq (k-1)\frac{3k-d_1-4}{2}-2k+2+d_1^2+(1-k)d_1$$
$$=d_1^2+\frac{3}{2}(1-k)d_1+(k-1)\frac{3k-4}{2}-2k+2.$$

If $k\geq 5$, then $\frac{3}{4}(k-1)\leq k-2$ and d_1 can reach to $\frac{3}{4}(k-1)$. We get

$$d_1^2+\frac{3}{2}(1-k)d_1+(k-1)\frac{3k-4}{2}-2k+2$$
$$\geq \frac{9}{16}(k-1)^2-\frac{9}{8}(k-1)^2+(k-1)\frac{3k-4}{2}-2k+2$$
$$=\frac{15}{16}k^2-\frac{35}{8}k+\frac{55}{16}$$
$$\geq \frac{15}{16}k^2-\frac{35}{8}k+\frac{55}{16}>0,$$

which contradicts (2.1).

If $k=2,3,4$, then

$$d_1^2+\frac{3}{2}(1-k)d_1+(k-1)\frac{3k-4}{2}-2k+2$$
$$\geq (k-2)^2+\frac{3}{2}(1-k)(k-2)+(k-1)\frac{3k-4}{2}-2k+2$$
$$=k^2-5k+5.$$

If $k=4$, then $k^2-5k+5\geq 0$, which contradicts (2.1).

If $k=3$, then $d_2=2$, $d_1=0$ or 1. If $d_1=0$, then $s\geq\frac{n'}{2}-1$ and $t\leq\frac{n'}{2}+1$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(\frac{n'}{2}-1)-k(\frac{n'}{2}+1)+2(\frac{n'}{2}+1-1)\geq 2k-5>0$, which contradicts (2.1). Assume $d_1=1$. If $n'\geq 8=4k-4$, then we get contradiction similarly as what we discuss above. If $n'=7$, then $n\leq 10$ since $n'\geq\frac{2n}{3}$. And, if $s\geq 3$, we obtain $k|S|-k|T|+d_{G'-S}(T)\geq 0$. The last situation is $k=3$, $n'=7$, $s=2$. Thus, $\sigma_2(G)\leq 13$ which contradicts $\sigma_2(G)\geq\frac{4n}{3}$.

Assume $k=2$. Then $d_1=0$ and $d_2=1$. If $G'-S-T\neq\varnothing$, then $t\leq n'-s-1$ and $k|S|-k|T|+d_{G'-S}(T)\geq 2s-2(n'-s-1)+(n'-s-p-1)\geq 3s-n'-p+1\geq 3s-n\geq 3s-(2s+1)=s-1\geq 0$, which contradicts (2.1). Suppose $G'-S-T=\varnothing$. If $n'\geq 4k-3=5$, then $s\geq\frac{n'-1}{2}$ and $t\leq\frac{n'+1}{2}$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k\frac{n'-1}{2}-k\frac{n'+1}{2}+(\frac{n'+1}{2}-1)\geq 0$, which contradicts (2.1). If $n'=4k-4=4$, then $s\geq 2$ and $t\leq 2$ by s is an integer. Thus, $k|S|-k|T|+d_{G'-S}(T)>0$, which contradicts (2.1). If $n=3=4k-5$, then $s\geq 1$ and $t\leq 2$. If $t\leq 1$, then $s\geq 2$ and we have $k|S|-k|T|+d_{G'-S}(T)\geq 0$, which contradicts (2.1). The last case is $s=1$ and $t=2$. Then at least one vertex in T is of degree at least 2 in $G'-S$. Thus, $k|S|-$

$k|T|+d_{G'-S}(T)\geq ks-kt+d_1p+d_2(t-p)\geq k-2k+(2-1)+1=0$, which contradicts (2.1).

Case 2.3. $0\leq d_1\leq d_2\leq k-2$**.** In this subcase, $k-1-d_2\geq 1$ and $n'-s-t\geq 0$. So, $(k-1-d_2)(n'-s-t)>ks-kt+d_1p+d_2(t-p)$. Thus, $(k-d_2)(n'-s)-ks>(d_1-d_2)p+(n'-s-t)\geq(d_1-d_2)(d_1+1)+(n'-s-t)$, i.e.,

$$(k-d_2)(n'-s)-ks\geq(d_1-d_2)(d_1+1)+1. \tag{2.2}$$

In terms of $n'\geq 4k-5$, we obtain

$$d_2\frac{n'}{2}\geq d_2(2k-\frac{5}{2}). \tag{2.3}$$

In view of $s\geq\frac{n'-d_1-d_2}{2}$, we have

$$(s-\frac{n'}{2})(2k-d_2)\geq-\frac{d_1+d_2}{2}(2k-d_2). \tag{2.4}$$

Adding (2.2), (2.3) and (2.4), we get

$$\begin{aligned}0&\geq d_1^2+\frac{d_2^2}{2}-\frac{d_1d_2}{2}+d_1-\frac{7}{2}d_2+1+(d_2-d_1)k\\&\geq d_1^2+\frac{d_2^2}{2}-\frac{d_1d_2}{2}+d_1-\frac{7}{2}d_2+1+(d_2-d_1)(d_2+2)\\&=d_1^2+\frac{3}{2}d_2^2-\frac{3}{2}d_1d_2-\frac{3}{2}d_2-d_1+1.\end{aligned}$$

Equivalent to

$$(d_1-(\frac{3}{4}d_2+\frac{1}{2}))^2+(\frac{\sqrt{15}}{4}d_2-\frac{9}{2\sqrt{15}})^2-\frac{3}{5}\leq 0.$$

We have

$$0\leq d_1\leq d_2\leq 2$$

by $(\frac{\sqrt{15}}{4}d_2-\frac{9}{2\sqrt{15}})^2-\frac{3}{5}\leq 0$.

- If $d_1=d_2=2$. In this case, if $n'\geq 4k-4$, then $s\geq\frac{n'}{2}-2$ and $t\leq n'-s\leq\frac{n'}{2}+2$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(\frac{n'}{2}-2)-k(\frac{n'}{2}+2)+2(\frac{n'}{2}+2)\geq 0$, which contradicts (2.1). If $n'=4k-5$, then $s\geq 2k-4$ and $t\leq n'-s\leq 2k-1$ since s is an integer. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-4)-k(2k-1)+2(2k-1)=k-2\geq 0$, which contradicts (2.1).
- If $d_1=1$ and $d_2=2$. In this case, if $n'\geq 4k-3$, then $s\geq\frac{n'-3}{2}$ and $t\leq n'-s\leq\frac{n'+3}{2}$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k\frac{n'-3}{2}-k\frac{n'+3}{2}+2+2\frac{n'-1}{2}\geq k-2\geq 0$, which contradicts (2.1). If $n'=4k-4$, then $s\geq 2k-3$ and $t\leq n'-s\leq 2k-1$ since s is an integer. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-3)-k(2k-1)+2+2(2k-1-2)=2k-4\geq 0$, which contradicts (2.1). If $n'=4k-5$, then $s\geq 2k-4$ and $t\leq n'-s\leq 2k-1$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-4)-k(2k-1)+2+$

$2(2k-3)=k-4\geq 0$ if $k\geq 4$, which contradicts (2.1). If $k=2$, then $n'=4k-5=3$. In terms of $n'\geq \frac{2}{3}n$, we get n=4, which contradicts $|I|\geq 2$. In particular, for $k=3$. If $n'\geq 8=4k-4$, then we get $k|S|-k|T|+d_{G'-S}(T)\geq 0$. If $n'=7$, then $n\leq 10$ since $n'\geq \frac{2n}{3}$. And, if $s\geq 3$, we get $k|S|-k|T|+d_{G'-S}(T)\geq 0$. The last situation is $k=3$, $n'=7$, $s=2$. Thus, $\sigma_2(G)\leq 13$ which which contradicts $\sigma_2(G)\geq \frac{4n}{3}$.

- If $d_1=0$ and $d_2=2$. In this case, if $n'\geq 4k-4$, then $s\geq \frac{n'}{2}-1$ and $t\leq n-s\leq \frac{n'}{2}+1$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(\frac{n'}{2}-1)-k(\frac{n'}{2}+1)+2(\frac{n'}{2}+1-1)=n'-2k\geq 2k-4\geq 0$, which contradicts (2.1). If $n'=4k-5$, then $s\geq 2k-3$ and $t\leq n'-s\leq 2k-2$ since s is an integer. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-3)-k(2k-2)+2(2k-2-1)=3k-6\geq 0$, which contradicts (2.1).
- If $d_1=d_2=1$. In this case, $s\geq \frac{n'}{2}-1$ and $t\leq n'-s\leq \frac{n'}{2}+1$. If $n'\geq 4k-2$, then $k|S|-k|T|+d_{G'-S}(T)\geq k(\frac{n'}{2}-1)-k(\frac{n'}{2}+1)+(\frac{n'}{2}+1)\geq 0$, which contradicts (2.1). If $n'=4k-3$, then $s\geq 2k-2$ and $t\leq n'-s\leq 2k-1$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-2)-k(2k-1)+(2k-1)=k-1>0$, which contradicts (2.1). If $n'=4k-4$, then $s\geq 2k-3$ and $t\leq 2k-1$. If $s\geq 2k-2$ or $t\leq 2k-2$, then we have $k|S|-k|T|+d_{G'-S}(T)\geq 0$. If $s=2k-3$ and $t=2k-1$, then at least one vertex in T is of degree at least 2 in T since t is odd. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-3)-k(2k-1)+(2k-1)+1=0$, which contradicts (2.1). If $n'=4k-5$, then $s\geq 2k-3$ and $t\leq 2k-2$ since s is an integer. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-3)-k(2k-2)+(2k-2)=k-2\geq 0$, which contradicts (2.1).
- If $d_1=0$ and $d_2=1$. In this case, $s\geq \frac{n'-1}{2}$, $t\leq n'-s=\frac{n'+1}{2}$ and $p\leq d_1+1=1$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(\frac{n'-1}{2})-k(\frac{n'+1}{2})+(\frac{n'+1}{2}-1)\geq k-3\geq 0$ if $k\geq 3$, which contradicts (2.1). If $k=2$ and $n'\geq 5=4k-3$, then $k|S|-k|T|+d_{G'-S}(T)\geq k\frac{n'-1}{2}-k\frac{n'+1}{2}+(\frac{n'+1}{2}-1)\geq k-2=0$, which contradicts (2.1). If $n'=4=4k-4$, then $s\geq 2$ and $t\leq 2$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq 2k-2k+(2-1)>0$, which contradicts (2.1). The last situation is $k=2$ and $n'=3=4k-5$. Then $s\geq 1$ and $t\leq 2$. If $s\geq 2$ or $t\leq 1$, then we get $k|S|-k|T|+d_{G'-S}(T)\geq 0$, which contradicts (2.1). Otherwise, $s=1$ and $t=2$. Then at least one vertex in T has degree at least 2 in T since t is even and $d_1=0$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq 2-4+1+1=0$, which contradicts (2.1).
- If $d_1=d_2=0$. In this case, $s\geq \frac{n'}{2}$ and $t\leq \frac{n'}{2}$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq 0$, which contradicts (2.1).

Thus, we complete the proof of Theorem 2. □

Remark 1. We construct some graphs to show that the bounds in the Theorem 2 are best possible.

For $k\geq 3$, let $G=(2k-3)K_1\vee(K_{2k-4}\vee(k-1)K_2)$. Then $n=6k-9$, $\delta(G)=4k-6\geq \frac{n}{3}+k$ and $\sigma_2(G)=8k-12=\frac{4n}{3}$. Let $I=(2k-3)K_1$, $S=K_{2k-4}$. Then $T=(k-1)K_2$ and $k|S|-k|T|+d_{G'-S}(T)=-2<0$. So, G is not a fractional

ID-k-factor-critical graph. For $k = 2$ and $|G| = 3 = 6k-9$, then G is not a fractional ID-k-factor-critical graph. Thus, the bound of n is best possible.

If $k \geq 3$. Let $G = (2k-2)K_1 \vee (K_{2k-3} \vee (2k-2)K_1)$. Then $n = 6k-7$, $\delta(G) = 4k-5 \geq \frac{n}{3}+k$, but $\sigma_2(G) = 8k-10 < \frac{4n}{3}$. Let $I = (2k-2)K_1$, $S = K_{2k-3}$. Then $T = (2k-2)K_1$, $d_{G'-S}(T) = 0$ and $k|S| + \sum_{x\in T} d_{G'-S}(x) - k|T| = -k < 0$. So, G is not a fractional ID-k-factor-critical graph. The condition $\sigma_2(G) \geq \frac{4n}{3}$ is best possible for $k \geq 3$.

At last, the condition that $\delta(G) \geq \frac{n}{3}+k$ cannot be replaced by $\frac{n}{3}+k-1$. We consider a such graph G: n is divided by 3 and $G = \frac{n}{3}K_1 \vee G'$. Let $I = \frac{n}{3}K_1$. Deleting I form G, we have $\delta(G') = k-1$ if $\delta(G) = \frac{n}{3}+k-1$. Therefore, $G-I$ has no fractional k-factor by the definition.

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