Approximately algebraic tensor products

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APPROXIMATELY ALGEBRAIC TENSOR PRODUCTS

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Abstract. Let $X$ and $Y$ be normed spaces over a complete field $F$ with dual spaces $X'$ and $Y'$ respectively. Under certain hypotheses, for given $x \in X$ and $y \in Y$ and a mapping $u$ from $X' \times Y'$ to $F$, we apply Hyers–Ulam approach to find a unique bounded bilinear mapping $v$ near to $u$ such that $||v|| = ||x \otimes y||$.

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1. INTRODUCTION

Let $X$, $Y$, and $Z$ be normed linear spaces over the same field $F$. A mapping $\phi : X \times Y \rightarrow Z$ is said to be bilinear if the mappings $x \mapsto \phi(x, y)$ and $y \mapsto \phi(x, y)$ are linear. A bilinear mapping $\phi : X \times Y \rightarrow Z$ is said to be bounded if there exists $M > 0$ such that $||\phi(x, y)|| \leq M ||x|| ||y||$ for all $x \in X$ and $y \in Y$. The norm of $\phi$ is then defined by

$$||\phi|| := \sup\{ ||\phi(x, y)|| : (x, y) \in B_X \times B_Y \},$$

where $B_X := \{ x \in X : ||x|| \leq 1 \}$. The set of all bounded bilinear mappings from $X \times Y$ to $Z$ is denoted by $B\mathcal{L}(X \times Y, Z)$. Let $X'$ and $Y'$ be dual spaces of $X$ and $Y$ respectively. For given $x \in X$ and $y \in Y$, $x \otimes y$ is an element of $B\mathcal{L}(X' \times Y', F)$ defined by $x \otimes y(f, g) := f(x)g(y)$ for all $f \in X'$ and $g \in Y'$. The algebraic tensor product of $X$ and $Y$, $X \otimes Y$, is defined to be the linear span of $\{ x \otimes y : x \in X, y \in Y \}$ in $B\mathcal{L}(X' \times Y', F)$ (see [3]).

A classical question in the theory of functional equations is the following (see [4], [6], [7], [9], [10], [8], [12], [14], [15], [20], [19], [17], [18], [21], [13], [22]): When is it true that a function which approximately satisfies a functional equation $\zeta$ must be close to an exact solution of $\zeta$? If the problem accepts a solution, we say that the equation $\zeta$ is stable. There are cases in which each approximate solution is actually a true solution. In such cases, we call the equation $\zeta$ superstable.

In this paper, under certain hypotheses and using Hyers–Ulam approach, we find a unique bounded bilinear mapping $v$ near to a given mapping $u$ such that

$$\|v\| \leq \|x \otimes y\|$$

for $x \in X$, $y \in Y$. Throughout this paper, it is assumed that $X$ and $Y$ are normed spaces over a complete field $F$ with dual spaces $X^0$ and $Y^0$ respectively.

2. Results

Theorem 1. Let $u : X' \times Y' \to \mathbb{F}$ be a mapping for which there exist positive real valued functions $\varphi_1, \varphi_2$, and $\varphi$ on $X' \times X' \times Y'$, $X' \times Y' \times Y'$, and $X' \times Y'$, respectively such that

$$\check{\varphi}(f, g) := \sum_{i=0}^{\infty} \frac{1}{2^i+1} \varphi_1(2^i f, 2^i f, g) < \infty, \quad (2.1)$$

$$\lim_{n \to \infty} \frac{1}{2^n} \varphi_1(2^n f_1, 2^n f_2, g) = \lim_{n \to \infty} \frac{1}{2^n} \varphi_2(2^n f, g_1, g_2) = \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n f, g) = 0, \quad (2.2)$$

$$|u(cf_1 + f_2, g) - cu(f_1, g) - u(f_2, g)| \leq \varphi_1(f_1, f_2, g), \quad (2.3)$$

$$|u(f, cg_1 + g_2) - cu(f, g_1) - u(f, g_2)| \leq \varphi_2(f, g_1, g_2) \quad (2.4)$$

for all $f, f_1, f_2 \in X'$, $g, g_1, g_2 \in Y'$, and $c \in \mathbb{F}$. Then, there exists a unique bilinear mapping $v$ from $X' \times Y'$ to $\mathbb{F}$ such that

$$|u(f, g) - v(f, g)| \leq \check{\varphi}(f, g) \quad (f \in X', g \in Y'). \quad (2.5)$$

Moreover, if the mapping $u$ satisfies

$$\|u(f, g)\| - |f(x) g(y)| \leq \varphi(f, g) \quad (2.6)$$

for some fixed $x \in X$ and $y \in Y$, then $\|v\| = \|x \otimes y\|$ and so in particular $v$ is bounded.
Proof. Putting $c = 1$ and replacing $f_1$ and $f_2$ in (2.3) by $f$ and dividing both sides by 2, we get
\[
\left| \frac{1}{2} u(2f, g) - u(f, g) \right| \leq \frac{1}{2} \varphi_1(f, f, g) \tag{2.7}
\]
for all $f \in X'$ and $g \in Y'$. Replacing $f$ by $2f$ in (2.7) and dividing both sides by 2, we find that
\[
\left| \frac{1}{2^n} u(2^n f, g) - u(f, g) \right| \leq \frac{1}{2^n} \varphi_1(2^n f, 2^n f, g) \tag{2.8}
\]
for all $f \in X'$ and $g \in Y'$. Combining (2.7) with (2.8), we obtain
\[
\left| \frac{1}{2^n} u(2^n f, g) - u(f, g) \right| \leq \frac{1}{2^n} \varphi_1(f, f, g) + \frac{1}{2^n} \varphi_1(2^n f, 2^n f, g)
\]
for all $f \in X'$ and $g \in Y'$. By induction on $n$, we conclude that
\[
\left| \frac{1}{2^n} u(2^n f, g) - u(f, g) \right| \leq \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} \varphi_1(2^i f, 2^i f, g) \tag{2.9}
\]
for all $f \in X'$ and $g \in Y'$. We now turn to use the Cauchy convergence criterion. Replace $f$ by $2^k f$ in (2.9) and divide both sides by $2^k$, where $k$ is an arbitrary positive integer, to get
\[
\left| \frac{1}{2^{n+k}} u(2^{n+k} f, g) - \frac{1}{2^k} u(2^k f, g) \right| \leq \sum_{i=k}^{n-1} \frac{1}{2^{i+1}} \varphi_1(2^i f, 2^i f, g)
\]
for all $f \in X'$, $g \in Y'$, and all positive integers $n \geq k$. It follows from the last inequality and (2.1) that the sequence $\{ \frac{1}{2^n} u(2^n f, g) \}$ is a Cauchy sequence for all $f \in X'$ and $g \in Y'$. Since $F$ is a complete field, this sequence converges. Define $v(f, g) := \lim_{n \to \infty} \frac{1}{2^n} u(2^n f, g)$. Taking the limit as $n \to \infty$ in (2.9), we find that the inequality (2.5) holds for all $f \in X'$ and $g \in Y'$. Replace $f_1$ and $f_2$ in (2.3) by $2^n f_1$ and $2^n f_2$ respectively and divide both sides by $2^n$ and take the limit as $n \to \infty$ and apply then (2.2) to get the mapping $f \mapsto v(f, g)$ is linear. By a similar way one can replace $f$ in (2.4) by $2^n f$ and divide both sides by $2^n$ to deduce that the mapping $g \mapsto v(f, g)$ is linear. Consequently, the mapping $v$ is bilinear. Our next claim is to prove that $v$ is unique. Let $v'$ be another mapping satisfying (2.5). Hence,
\[
|v(f, g) - v'(f, g)| = \frac{1}{2k} |v(2^k f, g) - v'(2^k f, g)|
\]
\[
\leq \frac{2}{2k} \tilde{\varphi}(2^k f, g)
\]
\[
= 2 \sum_{i=k}^{\infty} \frac{1}{2^{i+1}} \varphi_1(2^i f, 2^i f, g)
\]
for all \( f \in X' \) and \( g \in Y' \). Passing to the limit as \( k \to \infty \), we conclude that \( v \) is unique. Replace \( f \) by \( 2^n f \) in (2.6) and divide both sides by \( 2^n \), to arrive at

\[
\left| \frac{1}{2^n} |u(2^n f, g)| - |f(x)g(y)| \right| \leq \frac{1}{2^n} \phi(2^n f, g) \tag{2.10}
\]

for all \( f \in X' \) and \( g \in Y' \). Taking the limit as \( n \to \infty \) in (2.10) and applying the definition of the norm, we conclude that \(||v|| = ||x \otimes y||\) and so \( v \) is bounded. \( \square \)

**Remark 1.** Under the same hypotheses of Theorem 1, with (2.1) and (2.2) replaced by

\[
\tilde{\phi}(f, g) := \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \phi_2(f, 2^i g, 2^i g) < \infty, \tag{2.11}
\]

\[
\lim_{n \to \infty} \frac{1}{2^n} \phi_1(f_1, f_2, 2^n g) = \lim_{n \to \infty} \frac{1}{2^n} \phi_2(f, 2^n f_1, 2^n g_2) = \lim_{n \to \infty} \frac{1}{2^n} \phi(f, 2^n g) = 0. \tag{2.12}
\]

there exists a unique mapping \( v \in \mathcal{BL}(X' \times Y', \mathbb{F}) \) satisfying (2.5). Note that by using (2.4) and the same method as in the proof of Theorem 1, we can define \( v(f, g) := \lim_{n \to \infty} \frac{1}{2^n} u(f, 2^n g) \).

In the following corollaries, as a consequence of Theorem 1, we show the Rassias stability of algebraic tensor products.

**Corollary 1.** Let \( x \in X, y \in Y, \) and \( u : X' \times Y' \to \mathbb{F} \) be a mapping such that

\[
|u(f, g) - |f(x)g(y)|| \leq \alpha + \beta(||f||^p + ||g||^p) + \gamma||f||^p ||g||^p, \tag{2.13}
\]

\[
|u(cf_1 + f_2, g) - cu(f_1, g) - u(f_2, g)| \leq \alpha + \beta(||f_1||^q + ||f_2||^q + ||g||^q)
+ \gamma||f_1||^q ||f_2||^q ||g||^q,
\]

\[
|u(cf_1g_1 + g_2) - cu(f_1, g_1) - u(f_2, g_2)| \leq \alpha + \beta(||f||^r + ||g_1||^r + ||g_2||^r)
+ \gamma||f||^r ||g_1||^2 ||g_2||^2
\]

for all \( f, f_1, f_2 \in X', g, g_1, g_2 \in Y', \) and \( c \in \mathbb{F} \), where \( p, q, r < 1, \alpha > 0, \) and \( \beta, \gamma \geq 0 \). Then, there exists a unique mapping \( v \in \mathcal{BL}(X' \times Y', \mathbb{F}) \) such that \(||v|| = ||x \otimes y||\) and

\[
|u(f, g) - v(f, g)| \leq \alpha + \beta(2k||f||^q + ||g||^q) + \gamma k ||f||^q ||g||^q \tag{2.14}
\]

for all \( f \in X' \) and \( g \in Y' \), where \( k = \frac{1}{\alpha - 2\beta} \).

**Remark 2.** Under the hypotheses of Corollary 1 and using Remark 1, there exists a unique mapping \( v \in \mathcal{BL}(X' \times Y', \mathbb{F}) \) such that \(||v|| = ||x \otimes y||\) and

\[
|u(f, g) - v(f, g)| \leq \alpha + \beta(||f||^r + 2k||g||^r) + \gamma k ||f||^r ||g||^r
\]

for all \( f \in X' \) and \( g \in Y' \), where \( k = \frac{1}{\alpha - 2\beta} \).
Theorem 2. Let \( \{x_i\}_{i=1}^m \) and \( \{y_i\}_{i=1}^m \) be linearly independent sets in \( X \) and \( Y \) respectively and \( u \) be a mapping from \( X' \times Y' \) to \( \mathbb{F} \) for which there exist mappings \( \varphi_1 : X' \times X' \times Y' \rightarrow \mathbb{R}^+ \), \( \varphi_2 : X' \times Y' \times Y' \rightarrow \mathbb{R}^+ \), and \( \varphi : X' \times Y' \rightarrow \mathbb{R}^+ \) satisfying (2.1), (2.2), (2.3), (2.4) and

\[
|u(f, g) - \sum_{i=1}^m f(x_i)g(y_i)| \leq \varphi(f, g) \tag{2.15}
\]

for all \( f \in X' \), \( g \in Y' \). Then, there exists a unique mapping \( v \in \mathcal{B}\mathcal{L}(X' \times Y', \mathbb{F}) \) such that

\[
|u(f, g) - v(f, g)| \leq \bar{\varphi}(f, g) \quad (f \in X', g \in Y'), \quad ||v|| \leq \sum_{i=1}^m ||x_i \otimes y_i||. \tag{2.16}
\]

In the following our interest is to provide a dual for Theorem 1.

Theorem 3. Let \( x \in X \), \( y \in Y \), and let \( u : X' \times Y' \rightarrow \mathbb{F} \) be a mapping for which there exist mappings \( \varphi_1 : X' \times X' \times Y' \rightarrow \mathbb{R}^+ \), \( \varphi_2 : X' \times Y' \times Y' \rightarrow \mathbb{R}^+ \), and \( \varphi : X' \times Y' \rightarrow \mathbb{R}^+ \) satisfying (2.3), (2.4), (2.6), and

\[
\bar{\varphi}(f, g) := \sum_{i=0}^{\infty} 2^i \varphi_1(\frac{f}{2^{i+1}}, \frac{g}{2^{i+1}}) < \infty. \tag{2.17}
\]

\[
\lim_{n \to \infty} 2^n \varphi_1(\frac{f_1}{2^n}, \frac{f_2}{2^n}, g) = \lim_{n \to \infty} 2^n \varphi_2(\frac{f}{2^n}, g_1, g_2) = \lim_{n \to \infty} 2^n \varphi(\frac{f}{2^n}, g) = 0 \tag{2.18}
\]

for all \( f, f_1, f_2 \in X' \), \( g, g_1, g_2 \in Y' \). Then, there exists a unique mapping \( v \in \mathcal{B}\mathcal{L}(X' \times Y', \mathbb{F}) \) satisfying (2.5).

Proof. By induction on \( n \), we conclude that

\[
|u(f, g) - 2^n u(\frac{f}{2^n}, g)| \leq \sum_{i=0}^{n-1} 2^i \varphi_1(\frac{f}{2^{i+1}}, \frac{f}{2^{i+1}}, g) \tag{2.19}
\]

for all \( f \in X' \) and \( g \in Y' \). Replace \( f \) by \( \frac{f}{2^n} \) in (2.19) and multiply both sides by \( 2^k \), where \( k \) is an arbitrary positive integer, to get

\[
|2^k u(\frac{f}{2^k}, g) - 2^n u(\frac{f}{2^n+k}, g)| \leq \sum_{i=k}^{n-k-1} 2^i \varphi_1(\frac{f}{2^{i+1}}, \frac{f}{2^{i+1}}, g) \tag{2.20}
\]

for all \( f \in X' \), \( g \in Y' \), and all positive integers \( n \geq k \). In order to use the Cauchy convergence criterion, the last inequality and (2.17) imply the sequence \( \{2^n u(\frac{f}{2^n}, g)\} \) is a Cauchy sequence for all \( f \in X' \) and \( g \in Y' \). Due to completeness of \( \mathbb{F} \), this sequence converges. Define \( \nu(f, g) := \lim_{n \to \infty} 2^n u(\frac{f}{2^n}, g) \). Taking the limit as \( n \to \infty \) in (2.19), we deduce that the inequality (2.5) holds for all \( f \in X' \) and \( g \in Y' \). The rest of the proof is similar to that of Theorem 1. \( \square \)
Remark 3. Under the same hypotheses of Theorem 3, with (2.17) and (2.18) replaced by
\[\tilde{\varphi}(f, g) := \sum_{i=0}^{\infty} 2^i \varphi_2(f, \frac{g}{2i+1}, \frac{g}{2i+1}) < \infty, \quad (2.20)\]
\[
\lim_{n \to \infty} 2^n \varphi_1(f_1, f_2, \frac{g}{2^n}) = \lim_{n \to \infty} 2^n \varphi_2(f, \frac{g_2}{2^n}, \frac{g_2}{2^n}) = \lim_{n \to \infty} 2^n \psi(f, \frac{g}{2^n}) = 0, \quad (2.21)
\]
there exists a unique mapping \(v \in \mathcal{B}L(X' \times Y', \mathbb{F})\) satisfying (2.5). We remark that by using (2.4) and the same method as in the proof of Theorem 3, one can define \(v(f, g) := \lim_{n \to \infty} 2^n u(f, \frac{g}{2^n})\).

**Corollary 2.** Let \(x \in X, y \in Y,\) and \(u: X' \times Y' \to \mathbb{F}\) be a mapping such that
\[||u(f, g)| - |f(x)g(y)|| \leq \alpha ||f||^p ||g||^p, \quad (2.22)\]
\[|u(cf_1 + f_2, g) - cu(f_1, g) - u(f_2, g)| \leq \beta ||f_1||^2 ||f_2||^2 ||g||^q, \quad (2.23)\]
\[|u(f, cg_1 + g_2) - cu(f, g_1) - u(f, g_2)| \leq \gamma ||f||^r ||g_1||^2 ||g_2||^q, \quad (2.24)\]
for all \(f, f_1, f_2 \in X', g, g_1, g_2 \in Y',\) and \(c \in \mathbb{F},\) where \(p, q, r > 1,\) and \(\alpha, \beta, \gamma > 0.\) Then, there exists a unique mapping \(v \in \mathcal{B}L(X' \times Y', \mathbb{F})\) such that \(||v|| = ||x \otimes y||\) and
\[|u(f, g) - v(f, g)| \leq \frac{\beta}{2q - 2} ||f||^q ||g||^q (f \in X', g \in Y'). \quad (2.25)\]

**Proof.** It is enough to define \(\varphi(f, g) := \alpha ||f||^p ||g||^p,\)
\[\varphi_1(f_1, f_2, g) := \beta ||f_1||^2 ||f_2||^2 ||g||^q, \quad \text{and} \quad \varphi_2(f, g_1, g_2) := \gamma ||f||^r ||g_1||^2 ||g_2||^q \quad (2.26)\]
for all \(f, f_1, f_2 \in X'\) and \(g, g_1, g_2 \in Y'\) and then apply Theorem 3. \(\square\)

Remark 4. Under the hypotheses of Corollary 2 and using Remark 3, there exists a unique mapping \(v \in \mathcal{B}L(X' \times Y', \mathbb{F})\) such that \(||v|| = ||x \otimes y||\) and
\[|u(f, g) - v(f, g)| \leq \frac{\gamma}{2r - 2} ||f||^r ||g||^r (f \in X', g \in Y'). \quad (2.27)\]

**Theorem 4.** Let \(\{x_i\}_{i=1}^m\) and \(\{y_j\}_{j=1}^n\) be linearly independent sets in \(X\) and \(Y\) respectively and \(u\) be a mapping from \(X' \times Y'\) to \(\mathbb{F}\) for which there exist mappings \(\varphi_1: X' \times X' \times Y' \to \mathbb{R}^+,\)
\(\varphi_2 : X' \times Y' \times Y' \to \mathbb{R}^+,\) and \(\varphi : X' \times Y' \to \mathbb{R}^+\) satisfying (2.17), (2.18), (2.19), (2.20). Then, there exists a unique mapping \(v \in \mathcal{B}L(X' \times Y', \mathbb{F})\) satisfying (2.16).

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