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MODULES OVER GROUP RINGS OF LOCALLY SOLUBLE GROUPS WITH A CERTAIN CONDITION OF MINIMALITY

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Abstract. Let A be an $\mathbf{R}G$-module, where $\mathbf{R}$ is an associative ring, $A/C_A(G)$ is an infinite $\mathbf{R}$-module, $C_G(A) = 1$, G is a locally soluble group. Let $L_{nf}(G)$ be the system of all subgroups $H \leq G$ such that quotient modules $A/C_A(H)$ are infinite $\mathbf{R}$-modules. The author studies an $\mathbf{R}G$-module A such that $L_{nf}(G)$ satisfies the minimal condition as an ordered set. It is proved that a locally soluble group G with these conditions is soluble. The structure of G is described.

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1. INTRODUCTION

Let A be a vector space over a field F. The subgroups of the group $GL(F,A)$ of all automorphisms of A are called linear groups. If A has a finite dimension over F then $GL(F,A)$ can be considered as the group of non-singular $(n \times n)$-matrices, where $n = dim_F A$. Finite dimensional linear groups have played an important role in various fields of mathematics, physics and natural sciences, and have been studied many times. When A is infinite dimensional over F, the situation is totally different. Infinite dimensional linear groups have been investigated little. The study of this class of groups requires additional restrictions. In [5] it was introduced the definition of the central dimension of an infinite dimensional linear group. Let H be a subgroup of $GL(F,A)$. H acts on the quotient space $A/C_A(H)$ in a natural way. The authors define $centdim_F H$ to be $dim_F(A/C_A(H))$. The subgroup H is said to have a finite central dimension if $centdim_F H$ is finite and H has infinite central dimension otherwise. Let $G \leq GL(F,A)$. In [5] it was considered the system $L_{id}(G)$ of all subgroups of G of infinite central dimension. In order to investigate infinite dimensional linear groups that are close to finite dimensional, it is natural to consider the case where the system $L_{id}(G)$ is "very small". The authors have studied locally soluble infinite dimensional linear groups such that $L_{id}(G)$ satisfies the minimal condition as an ordered set [5].

If $G \leq GL(F, A)$ then A can be considered as an FG-module. The natural generalization of this case is the consideration of an **R**G-module A, where **R** is a ring whose structure is near to a field. At this point the generalization of the notion of the central dimension of a subgroup of a linear group is the notion of the cocentralizer of a subgroup. This notion was introduced in [8]. Let A be an **R**G-module, **R** be an associative ring, G be a group. If $H \leq G$ then the quotient module $A/C_A(H)$ considered as an **R**-module is called the cocentralizer of H in the module A.

Modules over group rings of finite groups have been considered by many authors. Recently this class of modules was investigated in [6]. Study of modules over group rings of infinite groups requires some additional restrictions as in the case of infinite dimensional linear groups. In [2] it was investigated an **R**G-module A such that **R** is a dedekind domain and the cocentralizer of G in the module A is not an artinian **R**-module. It was considered the system $L_{nad}(G)$ of all subgroups of G such that their cocentralizers in the module A are not artinian **R**-modules which is ordered by the usual inclusion. It is investigated an **R**G-module A such that the system $L_{nad}(G)$ satisfies the minimal condition as an ordered set, G is a locally soluble group, $C_G(A) = 1$. The analogous problem for the ring of integers **R** was investigated in [3].

In [1] we have studied an **R**G-module A such that **R** is the ring of integers, the cocentralizer of G in the module A is not a noetherian **R**-module and $C_G(A) = 1$. Let $L_{nnd}(G)$ be the system of all subgroups of G such that their cocentralizers in the module A are not noetherian **R**-modules. It was investigated an **R**G-module A such that $L_{nnd}(G)$ satisfies the minimal condition as an ordered set and G is locally soluble.

In [4] we have considered the similar problem where **R** is the ring of integers and the noetherian condition is replaced by the minimax condition.

In this paper we investigate **R**G-module, where **R** is an associative ring, $A/C_A(G)$ is an infinite **R**-module, $C_G(A) = 1$, G is a locally soluble group. Let $L_{nf}(G)$ be the system of all subgroups $H \leq G$ such that $A/C_A(H)$ are infinite **R**-modules. We study an **R**G-module A such that $L_{nf}(G)$ satisfies the minimal condition as an ordered set. It is proved that a locally soluble group G with these conditions is soluble and the structure of G is described.

The main results of this paper are Theorems 2 and 3.

2. Preliminary results

We reduce some elementary facts about **R**G-modules.

Later on it is considered an **R**G-module A such that $C_G(A) = 1$.

Let A be an **R**G-module where G is a group, **R** is an associative ring. Recall that if $K \leq H \leq G$ and the cocentralizer of H in the module A is a finite **R**-module then the cocentralizer of K in the module A is a finite **R**-module also. If U, V are subgroups of G such that their cocentralizers in the module A are finite **R**-modules, then $A/(C_A(U) \cap C_A(V))$ is a finite **R**-module also.

Suppose that a group G satisfies the condition $min-nf$. If $H_1 > H_2 > H_3 > \cdots$ is an infinite strictly descending chain of subgroups of G then there is the natural number n such that the cocentralizer of H_n in the module A is a finite **R**-module. Moreover, if N is a normal subgroup of G and the cocentralizer of N in the module A is an infinite **R**-module then G/N satisfies the minimal condition on subgroups.

Lemma 1. *Let A be an **R**G-module, G be a group, **R** be an associative ring. Suppose that G satisfies the condition $min-nf$, X, H are subgroups of G and Λ is an index set such that*

(1) $X = Dr_{\lambda \in \Lambda} X_\lambda$, *where $1 \neq X_\lambda$ is an H-invariant subgroup of X, for each $\lambda \in \Lambda$.*

(2) $H \cap X \leq Dr_{\lambda \in \Gamma} X_\lambda$ *for some subset Γ of Λ.*

*If the set $\Omega = \Lambda \setminus \Gamma$ is infinite, then the cocentralizer of H in the module A is a finite **R**-module.*

Proof. Suppose that the set Ω is infinite and let $\Omega_1 \supset \Omega_2 \supset \cdots$ be a strictly descending chain of subsets of the set Ω. Since $H \cap Dr_{\lambda \in \Omega} X_\lambda = 1$, the chain of subgroups $\langle H, X_\lambda | \lambda \in \Omega_1 \rangle > \langle H, X_\lambda | \lambda \in \Omega_2 \rangle > \cdots$ is strictly descending. It follows that for some natural number d the cocentralizer of the subgroup $\langle H, X_\lambda | \lambda \in \Omega_d \rangle$ in the module A is a finite **R**-module. Therefore the cocentralizer of H in the module A is a finite **R**-module also. □

Lemma 2. *Let A be an **R**G-module, G be a group, **R** be an associative ring, G satisfy the condition $min-nf$, H, K be subgroups of G such that K is a normal subgroup of H. Suppose that there exists an index set Λ and subgroups H_λ of G such that $K \leq H_\lambda$ for all $\lambda \in \Lambda$, $H/K = Dr_{\lambda \in \Lambda} H_\lambda/K$, and the set Λ is infinite. Then the cocentralizer of H in the module A is a finite **R**-module.*

Proof. Let Γ and Ω are infinite disjoint subsets of the set Λ such that $\Lambda = \Gamma \cup \Omega$. Let $U/K = Dr_{\lambda \in \Gamma} H_\lambda/K$, let $V/K = Dr_{\lambda \in \Omega} H_\lambda/K$, and let $\Gamma_1 \supset \Gamma_2 \supset \cdots$ be a strictly descending chain of subsets of the set Γ. Then we construct an infinite strictly descending chain of subgroups

$$\langle V, H_\lambda | \lambda \in \Gamma_1 \rangle \;>\; \langle V, H_\lambda | \lambda \in \Gamma_2 \rangle \;>\; \cdots .$$

It follows from the condition $min-nf$ that the cocentralizer of V in the module A is a finite **R**-module. Likewise, we obtain that the cocentralizer of U in the module A is a finite **R**-module. Since $H = UV$, it follows that the cocentralizer of H in the module A is a finite **R**-module also. □

Lemma 3. *Let A be an **R**G-module, G be a group, **R** be an associative ring, G satisfy the condition $min-nf$. If an element $g \in G$ has infinite order then the cocentralizer of $\langle g \rangle$ in the module A is a finite **R**-module.*

Proof. Let p, q are distinct primes greater than 3 and let $u = g^p$, $v = g^q$. Then there is an infinite descending chain of subgroups $\langle u\rangle > \langle u^2\rangle > \langle u^4\rangle > \cdots$. It follows from the condition $min-nf$ that there exists the natural number k such that the cocentralizer of the subgroup $\langle u^{2^k}\rangle$ in the module A is a finite **R**-module. Similarly, there exists a natural number l such that the cocentralizer of the subgroup $\langle v^{3^l}\rangle$ in the module A is a finite **R**-module. Therefore the cocentralizer of the subgroup $\langle g\rangle = \langle u^{2^k}\rangle\langle v^{3^l}\rangle$ in the module A is a finite **R**-module.

□

The following result gives an important information about the derived quotient group under the condition $min-nf$.

Lemma 4. *Let A be an $\mathbf{R}G$-module, G be a group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module, and G satisfies the condition $min-nf$. Then the quotient group G/G' is Chernikov.*

Proof. Suppose that the quotient group G/G' is not Chernikov group. Let $\mathfrak{S}$ be the family of all subgroups $H \leq G$ such that the quotient group H/H' is not Chernikov and the cocentralizer of H in the module A is an infinite **R**-module. Since $G \in\mathfrak{S}$ then $\mathfrak{S}\neq \varnothing$. Since the set $\mathfrak{S}$ satisfies the minimal condition, then it has a minimal element. Let D be this minimal element. If U, V are proper subgroups of the group D such that $D = UV$ and $U \cap V = D'$, then at least one of these subgroups, U say, such that its cocentralizer in the module A is an infinite **R**-module. The choice of D implies that U/U' is Chernikov. It follows with regard to the isomorphism $U/D' \simeq (U/U')/(D'/U')$ that U/D' is also Chernikov. Since the cocentralizer of U in the module A is an infinite **R**-module it follows that the abelian quotient group D/U is also Chernikov. Hence the quotient group D/D' is Chernikov. Contrary to the choice of D. Therefore D/D' is indecomposable. Hence D/D' is isomorphic to a subgroup of quasi-cyclic group C_{q^∞}, for some prime q. Contradiction.

□

Let A be an $\mathbf{R}G$-module, G be a group, **R** be an associative ring. Let $FFD(G)$ be the set of all elements $x \in G$ such that the cocentralizer of $\langle x\rangle$ in the module A is a finite **R**-module. Since $C_A(x^g) = C_A(x)g$ for all $x, g \in G$, it follows that $FFD(G)$ is a normal subgroup of G.

Lemma 5. *Let A be an $\mathbf{R}G$-module, G be a group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module, and G satisfies the condition $min-nf$. Then G is either periodic or $G = FFD(G)$.*

Proof. We suppose to the contrary that G is neither periodic nor $G \neq FFD(G)$. Let $\mathfrak{S}$ be the family of all subgroups $H \leq G$ such that H is not periodic and $H \neq FFD(H)$. $\mathfrak{S}$ is non-empty. If $H \neq FFD(H)$ then there is an element $h \in H$ such that the quotient module $A/C_A(h)$ is an infinite **R**-module. Hence $\mathfrak{S} \subseteq L_{nf}(G)$, and therefore $\mathfrak{S}$ satisfies the minimal condition. Let D be the minimal element of $\mathfrak{S}$, let $L = FFD(D)$. Note that $L \neq 1$, since D is not a periodic group. If $L \leq S \leq D$ and $S \neq D$, then $S = FFD(S)$ so $S \leq L$. Hence D/L has order q for some prime q. Let $x \in D \setminus L$. If an element a has infinite order, then the choice of D implies that $\langle x, a\rangle = D$. It follows that L is finitely generated and since $L = FFD(L)$, the quotient module $A/C_A(L)$ is a finite $\mathbf{R}$-module. Since the subgroup L is normal in D, then $C = C_A(L)$ is an $\mathbf{R}D$-submodule of A. It follows that A has the finite series of $\mathbf{R}D$-submodules

$$\langle 0\rangle \leq C \leq A,$$

such that A/C is a finite $\mathbf{R}$-module. Since A/C is a finite $\mathbf{R}$-module then $D/C_D(A/C)$ is finite. As $C = C_A(L)$ then $L \leq C_D(C)$. It follows that $D/C_D(C)$ is finite too.

Let $W = C_D(C) \cap C_D(A/C)$. By Remak theorem

$$D/W \leq D/C_D(C) \times D/C_D(A/C).$$

It follows that the quotient group D/W is finite. W acts trivially on each factor of the series $\langle 0\rangle \leq C \leq A$. Therefore W is abelian.

Let U be a normal subgroup of finite index of D. The subgroup U is not periodic and so $\langle U, x\rangle$ is neither periodic nor $\langle U, x\rangle \neq FFD(\langle U, x\rangle)$. The choice of D implies that $D = \langle U, x\rangle$ and hence the quotient group D/U is abelian. If E is the finite residual of D, it follows that the quotient group D/E is abelian. Since $E \leq W$ then D/W is also abelian. It follows that $D/(W \cap L)$ is abelian. Since $W \cap L \leq W$, then the subgroup $W \cap L$ is abelian, and so D is a finitely generated metabelian subgroup. By theorem of P.Hall (Theorem 9.51 [9]) D is residually finite. As above, D is therefore abelian. Since $D = U\langle x\rangle$ for every subgroup U of finite index, it follows that the group D is infinite cyclic. By Lemma 3 $D = FFD(D)$. We have the contradiction with the choice of D.

□

3. Locally soluble groups with the condition $min-nf$

Lemma 6. *Let A be an $\mathbf{R}G$-module, G be a periodic locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module and G satisfies the condition $min-nf$. Then G either satisfies the minimal condition on subgroups or $G = FFD(G)$.*

Proof. We suppose to the contrary that G is neither satisfies the minimal condition on subgroups nor $G \neq FFD(G)$. Let $\mathfrak{S}$ be the family of all subgroups $H \leq G$ such that H does not satisfy the minimal condition on subgroups and $H \neq FFD(H)$. Then $\mathfrak{S} \neq \varnothing$. If $H \neq FFD(H)$ then the cocentralizer of H in the module A is an infinite $\mathbf{R}$-module and hence $\mathfrak{S} \subseteq L_{nf}(G)$. Therefore $\mathfrak{S}$ satisfies the minimal condition. Let D be the minimal element and let $L = FFD(D)$. There exists an infinite strictly descending chain of subgroups of D:

$$H_1 > H_2 > H_3 > \cdots .$$

Since D satisfies the condition $min-nf$ then there exists the natural number k such that the cocentralizer of H_k in the module A is a finite $\mathbf{R}$-module. Therefore $H_k \leq L$, and hence L does not satisfy the minimal condition. If $x \in D \setminus L$ then it follows from the choice of the subgroup D that $\langle x, L\rangle = D$. Hence the quotient group D/L has the order q for prime q. If it is necessary we replace x by the suitable power and obtain that x has the order q^r for some natural number r. Since the group D is not Chernikov then by D.I.Zaicev's theorem [10], D contains $\langle x\rangle$-invariant abelian subgroup $B = Dr_{n\in\mathbf{N}}\langle b_n\rangle$ and we may assume that the elements b_n have prime orders for all $n \in \mathbf{N}$. Let $1 \neq c_1 \in B$ and $C_1 = \langle c_1\rangle^{\langle x\rangle}$. Then C_1 is finite and there is the subgroup E_1 such that $B = C_1 \times E_1$. Let $U_1 = core_{\langle x\rangle} E_1$. Therefore U_1 has finite index in B. If $1 \neq c_2 \in U_1$ and $C_2 = \langle c_2\rangle^{\langle x\rangle}$ then C_2 is a finite $\langle x\rangle$-invariant subgroup and $\langle C_1, C_2\rangle = C_1 \times C_2$. Continuing in this manner, we can construct a family of subgroups $\{C_n | n \in \mathbf{N}\}$ for which $\langle C_n | n \in \mathbf{N}\rangle = Dr_{n\in\mathbf{N}} C_n$. By Lemma 1 $x \in L$. Contradiction.

□

From Lemmas 5 and 6 it follows the theorem.

Theorem 1. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module, and G satisfies the condition $min-nf$. Then G either satisfies the minimal condition on subgroups or $G = FFD(G)$.*

Lemma 7. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group. Suppose that the cocentralizer of G in the module A is a finite $\mathbf{R}$-module. Then G is almost abelian.*

Proof. Let $C = C_A(G)$. Then A has the series of $\mathbf{R}G$-submodules $\langle 0\rangle \leq C \leq A$, where A/C is a finite $\mathbf{R}$-module. Since $G \leq C_G(C)$ then $G/C_G(C)$ is trivial. As A/C is a finite $\mathbf{R}$-module then $G/C_G(A/C)$ is finite.

Let $H = C_G(C) \cap C_G(A/C)$. Each element of H acts trivially on every factor of the series $\langle 0\rangle \leq C \leq A/C$. By Kaluzhnin Theorem (p. 144 [7]) H is abelian. By Remak's Theorem

$$G/H \leq G/C_G(C) \times G/C_G(A/C).$$

It follows that G/H is finite. Then G is an almost abelian group.

□

Lemma 8. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $\mathbf{R}$ be an associative ring, and if the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module then G satisfies the condition $min-nf$. Then either G is soluble or G has an ascending series of normal subgroups $1 = W_0 \leq W_1 \leq \cdots \leq W_n \leq \cdots \leq W_\omega = \cup_{n\in\mathbf{N}} W_n \leq G$,* *such that the cocentralizer of each subgroup W_n in the module A is a finite* **R***-module, the factors W_{n+1}/W_n are abelian for $n = 1, 2, \cdots$, and G/W_ω is a Chernikov group.*

Proof. If the quotient module $A/C_A(G)$ is a finite **R**-module then G is soluble by Lemma 7. Therefore it seemed reasonable to study locally soluble groups G such that $A/C_A(G)$ is an infinite **R**-module.

Later we consider the case when the cocentralizer of G in the module A is an infinite **R**-module. At first we prove that G is hyperabelian. To accomplish this we show that every non-trivial image of G contains a non-trivial normal abelian subgroup.

Let H be a proper normal subgroup of G. Suppose that the cocentralizer of H in the module A is an infinite **R**-module. Then G/H satisfies the minimal condition on subgroups. Therefore G/H is Chernikov group, and contains a non-trivial normal abelian subgroup. Now we suppose that the the cocentralizer of H in the module A is a finite **R**-module. Let $\mathfrak{L} = \{M_\sigma/H | \sigma \in \Sigma\}$ be the family of all non-trivial normal subgroups of the quotient group G/H. At first we consider the case when for all $\sigma \in \Sigma$ the cocentralizer of M_σ in the module A is an infinite **R**-module. We shall prove that the quotient group G/H satisfies the minimal condition on normal subgroups. Let $\{M_\delta/H\}$ be a non-empty subset of $\mathfrak{L}$. The cocentralizer of a subgroup M_δ in the module A is an infinite **R**-module for all δ. By the condition $min-nf$ the set $\{M_\delta\}$ has the minimal element M. Then M/H is the minimal element of subset $\{M_\delta/H\}$. Therefore G/H satisfies the minimal condition on normal subgroups. It follows that the quotient group G/H is hyperabelian and contains a non-trivial normal abelian subgroup. In the case when for some $\gamma \in \Sigma$ the cocentralizer of M_γ in the module A is a finite **R**-module, the subgroup M_γ is soluble. Then M_γ/H is a non-trivial normal soluble subgroup of G/H. Therefore the quotient group G/H contains a non-trivial normal abelian subgroup and so G is hyperabelian.

Let $1 = H_0 \leq H_1 \leq \cdots \leq H_\alpha \leq \cdots \leq G$ be a normal ascending series with abelian factors and let α be the least ordinal such that the cocentralizer of H_α in the module A is an infinite **R**-module. Then, as above, the subgroup H_β is soluble for all $\beta < \alpha$. Moreover, the quotient group G/H_α satisfies the minimal condition on subgroups, and so is a soluble Chernikov group.

At first we suppose that α is not a limit ordinal. Then the subgroup H_α is soluble and it follows that G is soluble also. Now we consider the case when α is a limit ordinal, and G is not soluble. For all natural numbers k there exists an ordinal β_k such that $\beta_k < \alpha$, H_{β_k} has derived length at least k. Moreover, we may assume that $\beta_i < \beta_{i+1}$ for all natural numbers i. Let $T_i = H_{\beta_i}$ for all natural numbers i. It follows that G has an ascending series of normal soluble subgroups $1 = T_0 \leq T_1 \leq \cdots \leq \cdots$. Then the subgroup $T_\omega = \cup_{n\in\mathbf{N}} T_n$ is not soluble and so $T_\omega = H_\alpha$. A series $1 = W_0 \leq W_1 \leq \cdots \leq W_n \leq \cdots \leq W_\omega = \cup_{n\in\mathbf{N}} W_n \leq G$ with the propeties referred in the theorem can be obtained from the series $1 = T_0 \leq T_1 \leq \cdots \leq T_\omega \leq G$.

□

Lemma 9. *Let A be an $\mathbf{R}G$-module, G be a group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module, G satisfies the condition $min-nf$ and $G = FFD(G)$. Then the quotient group $G/G^{\mathfrak{F}}$ is finite.*

Proof. We suppose for a contradiction that the quotient group $G/G^{\mathfrak{F}}$ is infinite. Then G has an infinite strictly descending series of normal subgroups $G > N_1 > N_2 > \cdots$, such that the quotient groups G/N_i are finite for each i. Therefore there exists k for which the quotient group G/N_k is finite and the cocentralizer of N_k in the module A is a finite $\mathbf{R}$-module. Since $G = FFD(G)$, there is the subgroup H such that its cocentralizer in the module A is a finite $\mathbf{R}$-module and $G = HN_k$. Hence the cocentralizer of G in the module A is a finite $\mathbf{R}$-module. Contradiction.

□

Lemma 10. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module and G satisfies the condition $min-nf$. If G has an ascending series of normal subgroups $1 = W_0 \le W_1 \le \cdots \le W_n \le \cdots \le \cup_{n\ge 1} W_n = G$, in which the cocentralizer of each subgroup W_n in the module A is a finite $\mathbf{R}$-module, and each factor W_{n+1}/W_n is abelian, then G is soluble.*

Proof. Since the quotient module $A/C_A(W_k)$ is a finite $\mathbf{R}$-module for each $k \in \mathbb{N}$ then there is the series of $\mathbf{R}G$-submodules $A = A_0 \ge A_1 \ge A_2 \ge \cdots \ge A_k \ge \cdots \ge A_\omega = C_A(G)$, such that $A_k = C_A(W_k)$ and each factor A_k/A_{k+1} is a finite $\mathbf{R}G$-module. Let $H = \cap_{j\ge 0} C_G(A_j/A_{j+1})$. Then $G/C_G(A_j/A_{j+1})$ is finite for each $j \in \mathbb{N}$. Since G/H embeds in the Cartesian product of the quotient groups $G/C_G(A_j/A_{j+1})$, it follows that G/H is residually finite. Moreover, G is a union of subgroups such that their cocentralizers in the module A are finite $\mathbf{R}$-modules. Hence $G = FFD(G)$. By Lemma 9 the quotient group G/H is finite.

Since $G = FFD(G)$ then the cocentralizer of H in the module A is an infinite $\mathbf{R}$-module. We shall prove that H is soluble. Let $L_j = C_H(A/A_j)$, $j = 1, 2, \cdots$. Let $H \ne L_j$ for some j. The quotient group H/L_j is finite for each $j = 1, 2, \cdots$. We suppose that there is the number j such that the cocentralizer of L_j in the module A is a finite $\mathbf{R}$-module. Let j be the least number with this property. It follows that the cocentralizer of L_{j-1} in the module A is an infinite $\mathbf{R}$-module. On the other hand since the quotient group L_{j-1}/L_j is finite and $G = FFD(G)$, then the cocentralizer of L_{j-1} in the module A is a finite $\mathbf{R}$-module. We have contradiction. Therefore the cocentralizer of each subgroup L_j in the module A is an infinite $\mathbf{R}$-module. Since H satisfies the condition $min-nf$ then there exists the number m such that $L_j = L_m$ for all $j \ge m$. From this fact and from the choice of subgroup L_j it follows that the subgroup L_m is soluble. Since the quotient group H/L_m is finite then H is also soluble. Then G is soluble.

□

From the obtained results it follows Theorem 2.

Theorem 2. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that if the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module, G satisfies the condition $min-nf$. Then G is soluble.*

Theorem 3. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module and G satisfies the condition $min-nf$. Then G has the normal abelian subgroup H such that G/H is Chernikov.*

Proof. It should be noted that by Theorem 2 the group G is soluble. To accomplish this proof we consider the case when G is not Chernikov.

Let $G = D_0 \geq D_1 \geq D_2 \geq \cdots \geq D_n = 1$ be the derived series of G. There exists the number m such that the cocentralizer of D_m in the module A is an infinite $\mathbf{R}$-module but the cocentralizer of D_{m+1} in the module A is a finite $\mathbf{R}$-module. By Lemma 4 the quotient groups D_i/D_{i+1}, $i = 0, 1, \ldots, m$, are Chernikov. Let $U = D_{m+1}$. Then the quotient group G/U is Chernikov. Let $C = C_A(U)$. C is an $\mathbf{R}G$-submodule of A. Therefore there exists the series of -submodules

$$\langle 0 \rangle \leq C \leq A,$$

such that A/C is a finite $\mathbf{R}$-module. Then $G/C_G(A/C)$ is finite.

Let $H = C_G(C) \cap C_G(A/C)$. The subgroup H acts trivially on each factor of the series $\langle 0 \rangle \leq C \leq A$. Therefore H is abelian. Since the quotient group G/U is Chernikov and $U \leq C_G(C)$ then the quotient group $G/C_G(C)$ is also Chernikov. By Remak theorem $G/H \leq G/C_G(C) \times G/C_G(A/C)$. It follows that G/H is Chernikov. Therefore G contains the normal abelian subgroup H such that G/H is Chernikov. □

In the paper the author have used the methods of the proofs of [5].

References

[1] O. Dashkova, "Modules over integer group rings of locally solvable groups with the restrictions on some systems of subgroups. (Russian)," *Dopov. Nats. Akad. Nauk Ukr.*, no. 2, pp. 14–19, 2009.

[2] O. Dashkova, "On a class of modules over group rings of locally soluble groups. (Russian)," *Trudy Inst. Mat. i Mech. Ural Otd. Ros. Akad. Nauk*, vol. 15, no. 2, pp. 94–98, 2009. [Online]. Available: http://dx.doi.org/10.1134/S0081543809070062

[3] O. Dashkova, "On one class of modules over integer group rings of locally solvable groups. (Russian)," *Ukrainian Math. J.*, vol. 61, no. 1, pp. 44–51, 2009. [Online]. Available: http://dx.doi.org/10.1007/s11253-009-0197-x

[4] O. Dashkova, "Modules over integer group rings of locally soluble groups with minimax restriction. (Russian)," *Fundam. Prikl. Mat.*, vol. 17, no. 3, pp. 25–37, 2011/2012. [Online]. Available: http://dx.doi.org/10.1007/s10958-012-1055-1

[5] M. Dixon, M. Evans, and L. Kurdachenko, "Linear groups with the minimal condition on subgroups of infinite central dimension," *J. Algebra*, vol. 277, no. 1, pp. 172–186, 2004. [Online]. Available: http://dx.doi.org/10.1016/j.jalgebra.2004.02.029

[6] P. M. Gudivok, V. P. Rud'ko, and V. A. Bovdi, *Crystallographic groups (Ukrainian)*. Uzhgorod: Uzhgorods'kiĭ Natsional'niĭ Universitet, 2006.
[7] M. Kargapolov and Y. Merzlyakov, *Fundamentals of group theory. (Russian)*. Moscow: Izdat. "Nauka", 1972.
[8] L. Kurdachenko, "On groups with minimax classes of conjugate elements," *Infinite groups and adjoining algebraic structures*, pp. 160–177, 1993.
[9] D. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, ser. Ergebnisse der Mathematik und ihrer Grenzgebiete. Berlin, Heidelberg, New York: Springer-Verlag, 1972, vol. 1,2.
[10] D. Zaicev, "Solvable subgroups of locally solvable groups," *Dokl. Akad. Nauk SSSR*, vol. 214, no. 6, pp. 1250–1253, 1974.

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