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HERMITE INTERPOLATING POLYNOMIALS

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ABSTRACT. A method is described to construct Hermite polynomials fitting on to 2, 4, 8 points of a 1D, 2D, 3D functions, respectively, up to their rth derivatives.

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1. INTRODUCTION

The Hermite interpolation is a well-known theoretical tool for approximating functions. Although the theory and practice of Hermite interpolation is well developed for univariate functions (see, e. g., [1, 4, 6]), fewer results is known for the multivariate case (see, e. g., [3, 5]). We emphasize the importance of the work of Lorentz [3], which surveys the well-known theory on the multivariate case. Here we describe a computational method to construct Hermite polynomials fitting on to 2, 4, 8 points of a 1D, 2D, 3D functions, respectively, up to their rth derivatives.

2. ONE-DIMENSIONAL POLYNOMIAL

The aim of this section is to construct a polynomial $H_r(x)$ on to the points x_0 and $x_1$of the function $y(x)$ up to its rth derivatives.

$$\frac{d^m H_r(x_l)}{dx^m} = \frac{d^m y(x_l)}{dx^m}, \qquad l = 0, 1, \quad 0 \le m \le r.$$

Let us seek for it in the form

$$H_r(x) = \boldsymbol{A}_{r,1}\boldsymbol{X}_{r,1}(x) + \boldsymbol{A}_{r,2}C_r(x)\boldsymbol{X}_{r,2}(x). \tag{1}$$

The number of the prescribed conditions is $2r + 2$. For the sake of a convenient form of writing the vectors, we define the $\boldsymbol{e}_{r,j+1}$ unit vector $(0 \le j \le r)$ the $(j + 1)$th entry of which is equal to 1 and the rest are equal zero. Then the vectors in (1) can be written in the form

$$\boldsymbol{A}_{r,1} = \sum_{j=0}^{r} a_{j+1}\boldsymbol{e}_{r,j+1}, \qquad \boldsymbol{A}_{r,2} = \sum_{j=0}^{r} a_{r+2+j}\boldsymbol{e}_{r,j+1},$$

$$\boldsymbol{X}_{r,1}(x) = \sum_{j=0}^{r} (x-x_0)^j\,\boldsymbol{e}_{r,j+1}, \qquad \boldsymbol{X}_{r,2}(x) = \sum_{j=0}^{r}(x-x_1)^j\boldsymbol{e}_{r,j+1},$$

where $C_r(x) = (x-x_0)^{r+1}$. To determine the constants a, the derivatives of the function $H_r(x)$ must be calculated at the values x_0 and x_1,

$$\boldsymbol{A}_{r,1}\boldsymbol{X}_{r,1}(x) = \sum_{j=0}^{r} a_{j+1}(x-x_0)^j,$$

$$\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x)}{dx^m} = \sum_{j=0}^{r-m}\frac{(m+j)!}{j!}(x-x_0)^j\,a_{m+1+j},$$

Let us set $x_1 - x_0 = h$. Then

$$\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x_0)}{dx^m} = m!a_{m+1},$$

$$\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x_1)}{dx^m} = \sum_{j=0}^{r-m}\frac{(m+j)!}{j!}h^j a_{m+1+j}, \qquad 0 \le m \le r.$$

To compute the derivatives of the second term in $H_r(x)$, the following notation will be used: $f_r(x) = (x-x_0)^{r+1}$, $g_j(x) = (x-x_1)^j$,

$$\frac{d^t f_r(x)}{dx^t} = f_r^{(t)}(x) = \frac{(r+1)!}{(r+1-t)!}(x-x_0)^{r+1-t},$$

$f_r^{(r+1)}(x) = (r+1)!$, $f_r^{(t)}(x_0) = 0$ if $t < r+1$. Let $t = m-l$. Then $f_r^{(m-l)}(x_0) = 0$ if $r+1 > m-l$, and

$$f_r^{(m-l)}(x_1) = \frac{(r+1)!}{(r+1-m+l)!}h^{r+1-m+l}, \qquad m-l \le r+1;$$

moreover,

$$g_j^{(l)}(x) = \frac{j!}{(j-l)!}(x-x_1)^{j-l}, \quad l \le j, \quad g_j^{(j)}(x) = j!,$$

$$g_j^{(l)}(x_1) = \begin{cases} 0 & \text{if } l \ne j, \\ j! & \text{if } l = j. \end{cases}$$

The derivative of their product is

$$\frac{d^m}{dx^m}\left[f_r(x)g_j(x)\right] = \sum_{l=0}^{m}\binom{m}{l} f_r^{(m-l)}(x)g_j^{(l)}(x),$$

and $(d^m/dx^m)[f_r(x_0)g_j(x_0)] = 0$ for $0 \le m \le r$.

One has $g_j^{(l)}(x_1) = 0$ if $l \ne j$ and $g_j^{(j)} = j!$; therefore $m - l = m - j$, and then

$$f_r^{(m-j)}(x_1) = \frac{(r+1)!}{(r+1-m+j)!} h^{r+1-m+j}, \qquad 0 \le m \le r.$$

The derivative at the point x_1 will be

$$\frac{d^m}{dx^m}\left[f_r(x_1)g_j(x_1)\right] = \sum_{j=0}^{m} \frac{m!}{(m-j)!} \frac{(r+1)!}{(r+1-m+j)!} h^{r+1-m+j}, \qquad 0 \le m \le r,$$

and at x_0

$$\frac{d^m}{dx^m}\left[f_r(x_0)g_j(x_0)\right] = \sum_{l=l_{\min}}^{l_{\max}} \binom{m}{l} f_r^{(m-l)}(x_0) g_j^{(l)}(x_0).$$

If $m - l \ne r + 1$, then $f_r^{(m-l)}(x_0) = 0$ and $f_r^{(r+1)}(x_0) = (r+1)!$, so $m - l = r + 1$, $l = m - (r+1)$, and

$$g_j^{(m-(r+1))}(x_0) = \frac{j!}{(r+1-m+j)!} (-h)^{r+1-m+j},$$

whence

$$\frac{d^m}{dx^m}\left[f_r(x_0)g_j(x_0)\right] = \\ = \binom{m}{m-(r+1)} (r+1)! \sum_{j=m-(r+1)}^{r} \frac{j!}{(r+1-m+j)!} (-h)^{r+1-m+j}$$

for $r + 1 \le m \le 2r + 1$. The derivatives at the point x_1 are $g_j^{(l)}(x_1) = 0$ if $l \ne j$, therefore, with $m - l = m - j$, there should be

$$f_r^{(m-j)}(x_1) = \frac{(r+1)!}{(r+1-m+j)!} h^{r+1-m+j},$$

whence

$$\frac{d^m}{dx^m}\left[f_r(x_1)g_j(x_1)\right] = = m!(r+1)! \sum_{j=m-(r+1)}^{r} \frac{h^{r+1-m+j}}{(m-j)!(r+1-m+j)!}$$

for $r + 1 \le m \le 2r + 1$. Returning to the original notation, we get

$$C_r(x)\boldsymbol{A}_{r,2}\boldsymbol{X}_{r,2}(x) = \sum_{j=0}^{r} a_{r+2+j} f_r(x) g_j(x).$$

Summing up the results above, we get

$$\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x_0)}{dx^m} = m!a_{m+1}, \qquad 0 \le m \le r,$$

$$\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x_1)}{dx^m} = \sum_{j=0}^{r-m}\frac{(m+j)!}{j!}\, h^j\, a_{m+1+j}, \qquad 0 \le m \le r,$$

$$\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x_0)}{dx^m} = 0, \qquad m > r,$$

$$\boldsymbol{A}_{r,2}\frac{d^m}{dx^m}\left[C_r(x_0)\boldsymbol{X}_{r,2}(x_0)\right] = 0, \qquad 0 \le m \le r,$$

$$\boldsymbol{A}_{r,2}\frac{d^m}{dx^m}\left[C_r(x_1)\boldsymbol{X}_{r,2}(x_1)\right] = m!(r+1)!\sum_{j=0}^{m}\frac{h^{r+1-m+j}}{(m-j)!(r+1-m+j)!}\, a_{r+2+j}$$

for $0 \le m \le r$,

$$\boldsymbol{A}_{r,2}\frac{d^m}{dx^m}\left[C_r(x_0)\boldsymbol{X}_{r,2}(x_0)\right] = \binom{m}{m-(r+1)}(r+1)!\times$$
$$\times\sum_{j=m-(r+1)}^{r}\frac{j!\,(-h)^{r+1-m+j}}{(r+1-m+j)!}a_{r+2+j}$$

for $r+1 \le m \le 2r+1$, and

$$\boldsymbol{A}_{r,2}\frac{d^m}{dx^m}\left[C_r(x_1)\boldsymbol{X}_{r,2}(x_1)\right] = m!(r+1)!\sum_{j=m-(r+1)}^{r}\frac{h^{r+1-m+j}}{(m-j)!(r+1-m+j)!}a_{r+2+j}$$

for $r+1 \le m \le 2r+1$. Finally, the derivatives of the Hermite polynomial are

$$\frac{d^m H_r(x)}{dx^m} = \boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}}{dx^m} + \boldsymbol{A}_{r,2}\frac{d^m}{dx^m}\left[C_r(x)\boldsymbol{X}_{r,2}(x)\right].$$

Example 1. For $r = 2$,

$$H_2(x_0) = a_1,$$
$$\frac{dH_2(x_0)}{dx} = a_2,$$
$$\frac{d^2H_2(x_0)}{dx^2} = a_3 2,$$
$$H_2(x_1) = a_1 + a_2h + a_3h^2 + a_4h^3,$$
$$\frac{dH_2(x_1)}{dx} = a_2 + a_3 2h + a_4 3h^2 + a_5h^3,$$
$$\frac{d^2H_2(x_1)}{dx^2} = a_3 2 + a_4 6h + a_5 6h^2 + a_6 2h^3.$$

3. Two-dimensional polynomial

The aim of this section is to construct a polynomial $H_r(x, y)$ on to the points (x_0, y_0), (x_0, y_1), (x_1, y_0), (x_1, y_1), which fit the given values up to its rth derivatives. The conditions can be written as follows:

$$\frac{\partial^{m+n} H_r(x_s, y_t)}{\partial x^m \partial y^n} = \frac{\partial^{m+n} u(x_s, y_t)}{\partial x^m \partial y^n}$$

for $0 \le m, n \le r$, $0 \le m + n \le r$, $s, t = 0, 1$. The number of the prescribed conditions at one point is $\frac{1}{2}r(r + 3) + 1$, and in the four points is $2r^2 + 6r + 4$.

Let us seek for it in the form

$$H_r(x, y) = \boldsymbol{A}(x)\boldsymbol{Y}_{r,1}(y) + \boldsymbol{B}(x)D_r(y)\boldsymbol{Y}_{r,2}(y).$$

For the sake of a convenient form of writing the vectors, let us define the unit vector of $j + 1$ entries, $\boldsymbol{e}_{j,q+1}$ $(0 \le q \le j)$, the $(q + 1)$th entry of which is equal to 1 and the rest are equal to zero. Then the vectors in $H_r(x, y)$ can be written as follows:

$$\boldsymbol{A}(x) = \sum_{j=0}^{r} A_j(x)\boldsymbol{e}_{r,r+1-j}, \qquad \boldsymbol{B}(x) = \sum_{j=0}^{r} B_j(x)\boldsymbol{e}_{r,r+1-j},$$

$$\boldsymbol{Y}_{j,1}(y) = \sum_{i=0}^{j} (y - y_0)^i \boldsymbol{e}_{j,i+1}, \qquad \boldsymbol{Y}_{j,2}(y) = \sum_{i=0}^{j} (y - y_1)^i \boldsymbol{e}_{j,i+1},$$

where $D_j(y) = (y - y_0)^{j+1}$, $0 \le j \le r$. The functions $\boldsymbol{A}(x)$ and $\boldsymbol{B}(x)$ are composed as

$$A_j(x) = \boldsymbol{A}_{j,1}\boldsymbol{X}_{j,1}(x) + \boldsymbol{A}_{j,2}C_j(x)\boldsymbol{X}_{j,2}(x),$$
$$B_j(x) = \boldsymbol{B}_{j,1}\boldsymbol{X}_{j,1}(x) + \boldsymbol{B}_{j,2}C_j(x)\boldsymbol{X}_{j,2}(x),$$

where

$$\boldsymbol{X}_{j,1}(x) = \sum_{i=0}^{j} (x - x_0)^i \boldsymbol{e}_{j,i+1}, \qquad \boldsymbol{X}_{j,2}(x) = \sum_{i=0}^{j} (x - x_1)^i \boldsymbol{e}_{j,i+1},$$

and $C_j(x) = (x - x_0)^{j+1}$, $0 \le j \le r$. The distances between the coordinates are $x_1 - x_0 = h$, $y_1 - y_0 = g$.

For writing the vectors $\boldsymbol{A}_{j,1}$, $\boldsymbol{A}_{j,2}$, $\boldsymbol{B}_{j,1}$, $\boldsymbol{B}_{j,2}$ conveniently, we introduce the notation

$$l_j = (r - j)\frac{r + 3 + j}{2}, \qquad n_r = 1/2(r + 1)(r + 2), \qquad L_j = l_j + n_r.$$

With this notation, we get

$$\boldsymbol{A}_{j,1} = \sum_{i=0}^{j} a_{l_j+i+1}\boldsymbol{e}_{j,i+1}, \qquad \boldsymbol{A}_{j,2} = \sum_{i=0}^{j} a_{L_j+i+1}\boldsymbol{e}_{j,i+1},$$

$$\boldsymbol{B}_{j,1} = \sum_{i=0}^{j} b_{l_j+i+1}\boldsymbol{e}_{j,i+1}, \qquad \boldsymbol{B}_{j,2} = \sum_{i=0}^{j} b_{L_j+i+1}\boldsymbol{e}_{j,i+1}.$$

The number of the constants a, b in $\boldsymbol{A}(x)$, $\boldsymbol{B}(x)$ is $4n_r = 2r^2+6r+4$, which agrees with the number of conditions; consequently, $H_r(x,y)$ is the minimal degree polynomial satisfying the requirements.

The derivatives of the above vectors should be computed,

$$\frac{d^m \boldsymbol{X}_{j,1}(x)}{dx^m} = \sum_{q=0}^{j-m} \frac{(m+q)!}{q!}(x-x_0)^q \boldsymbol{e}_{j,m+1+q}, \qquad 0 \le m \le j,$$

$$\frac{d^t \boldsymbol{X}_{j,2}(x)}{dx^t} = \sum_{q=0}^{j-t} \frac{(t+q)!}{q!}(x-x_1)^q \boldsymbol{e}_{j,t+1+q}, \qquad 0 \le t \le j,$$

$$\frac{d^l C_j(x)}{dx^l} = C_j^{(l)}(x) = \frac{(j+1)!}{(j+1-l)!}(x-x_0)^{j+1-l}, \qquad 0 \le l \le j+1,$$

$$\frac{d^m}{dx^m}\left[C_j(x)\boldsymbol{X}_{j,2}(x)\right] = \sum_{l=l_{\min}}^{l_{\max}} \binom{m}{l} C_j^{(l)}(x)\boldsymbol{X}_{j,2}^{(m-l)}(x).$$

The limits of the sums are

$$l_{\min} = 0,\ l_{\max} = m \qquad \text{if } 0 \le m \le j,$$
$$l_{\min} = m-j,\ l_{\max} = j+1 \qquad \text{if } j+1 \le m \le 2j+1.$$

The derivatives of $A_j(x)$ and $B_j(x)$ are

$$\frac{d^m A_j(x)}{dx^m} = \boldsymbol{A}_{j,1}\frac{d^m \boldsymbol{X}_{j,1}(x)}{dx^m} + \boldsymbol{A}_{j,2}\frac{d^m}{dx^m}\left[C_j(x)\boldsymbol{X}_{j,2}(x)\right],$$
$$\frac{d^m B_j(x)}{dx^m} = \boldsymbol{B}_{j,1}\frac{d^m \boldsymbol{X}_{j,1}(x)}{dx^m} + \boldsymbol{B}_{j,2}\frac{d^m}{dx^m}\left[C_j(x)\boldsymbol{X}_{j,2}(x)\right].$$

To compute the coefficients a and b at the points x_0 and x_1, the following formulas are used:

$$\frac{d^m A_j(x_0)}{dx^m} = m!\, a_{l_j+m+1}, \qquad 0 \le m \le j,$$

$$\frac{d^m A_j(x_1)}{dx^m} = \sum_{q=0}^{j-m} \frac{(m+q)!}{q!} h^q a_{l_j+m+1+q} + m! \sum_{l=0}^{m} \frac{(j+1)!}{l!\,(j+1-l)!} h^{j+1-l} a_{L_j+m+1-l}, \qquad 0 \le m \le j,$$

$$\frac{d^m A_j(x_0)}{dx^m} = \frac{m!}{[m-(j+1)]!} \times \sum_{q=0}^{2j+1-m} \frac{[m-(j+1)+q]!}{q!} (-h)^q\, a_{L_j+m-j+q},$$

$$j+1 \le m \le 2j+1,$$

$$\frac{d^m A_j(x_1)}{dx^m} = m! \sum_{l=m-j}^{j+1} \frac{(j+1)!}{(j+1-l)!} h^{j+1-l} a_{L_j+m+1-l}$$

$$j+1 \le m \le 2j+1.$$

Analogous formulas are valid for $B_j(x)$. The derivatives of the Hermite polynomial are

$$\frac{\partial^{m+n} H_r(x,y)}{\partial x^m \partial y^n} = \frac{d^m \boldsymbol{A}(x)}{dx^m} \frac{d^n \boldsymbol{Y}_{r,1}(y)}{dy^n} + \frac{d^m \boldsymbol{B}(x)}{dx^m} \frac{d^n}{dy^n} \left[D_r(y) \boldsymbol{Y}_{r,2}(y) \right].$$

The following formulas are also needed:

$$\frac{d^n \boldsymbol{Y}_{j,1}(y)}{dy^n} = \sum_{s=0}^{j-n} \frac{(n+s)!}{s!} (y-y_0)^s \boldsymbol{e}_{j,n+1+s}, \qquad 0 \le n \le j,$$

$$\frac{d^t \boldsymbol{Y}_{j,2}(y)}{dy^t} = \sum_{s=0}^{j-t} \frac{(t+s)!}{s!} (y-y_1)^s \boldsymbol{e}_{j,t+1+s}, \qquad 0 \le t \le j,$$

$$\frac{d^l D_j(y)}{dy^l} = D_j^{(l)}(y) = \frac{(j+1)!}{(j+1-l)!} (y-y_0)^{j+1-l}, \qquad 0 \le l \le j+1.$$

Thus, we can write

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\boldsymbol{A}(x) \boldsymbol{Y}_{r,1}(y) \right] = \sum_{j=0}^{r-n} A_j(x) \frac{(r-j)!}{(r-n-j)!} (y-y_0)^{r-n-j}, \tag{2}$$

and the values of these are

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\boldsymbol{A}(x_0) \boldsymbol{Y}_{r,1}(y_0) \right] = n! \frac{d^m A_{r-n}(x_0)}{dx^m}, \tag{3}$$

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\boldsymbol{A}(x_0) \boldsymbol{Y}_{r,1}(y_1) \right] = \sum_{j=0}^{r-n} \frac{d^m A_j(x_0)}{dx^m} \frac{(r-j)!}{(r-n-j)!} g^{r-n-j}, \tag{4}$$

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\boldsymbol{A}(x_1) \boldsymbol{Y}_{r,1}(y_0) \right] = n! \frac{d^m A_{r-n}(x_1)}{dx^m}, \tag{5}$$

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\boldsymbol{A}(x_1) \boldsymbol{Y}_{r,1}(y_1) \right] = \sum_{j=0}^{r-n} \frac{d^m A_j(x_1)}{dx^m} \frac{(r-j)!}{(r-n-j)!} g^{r-n-j} \tag{6}$$

for $0 \le n \le r$, $0 \le m \le 2r+1$.

To compute the derivatives of the second term in $H_r(x,z)$, the next *interim* notation will be used: $f_r(y) = D_r(y)$, $g_j(y) = (y-y_1)^{r-j}$. The derivatives of these are

$$f_r^{(l)}(y) = \frac{(r+1)!}{(r+1-l)!} (y-y_0)^{r+1-l}, \qquad 0 \le l \le r+1,$$

$f_r^{(r+1)}(y) = (r+1)!$, $f_r^{(l)}(y_0) = 0$ if $0 \le l < r+1$, $f_r^{(l)}(y) = 0$ if $l > r+1$, $g_j^{(t)}(y) = (r-j)!((r-j-t)!)^{-1}(y-y_1)^{r-j-t}$,

$$g_j^{(t)}(y) = \begin{cases} (r-j)! & \text{if } t = r-j, \\ 0 & \text{if } t \ne r-j, \end{cases}$$

$$\frac{d^n f_r(y)}{dy^n} = \sum_{l=l_{\min}}^{l_{\max}} \binom{n}{l} f_r^{(l)}(y) g_j^{(n-l)}(y),$$

$l_{\min} = \max\{n-(r-j),\ 0\}$, $l_{\max} = n$ if $r-n \le j \le r$ and $0 \le n \le r$; $l_{\min} = n+j-r$, $l_{\max} = r+1$ if $0 \le j \le 2r+1-n$, $r+1 \le n \le 2r+1$. It is easy to see that

$$\frac{d^n}{dy^n}\left[f_r(y_0)g_j(y_0)\right] = 0, \qquad 0 \le n \le r,$$

$$\frac{d^n}{dy^n}\left[f_r(y_1)g_j(y_1)\right] = \binom{n}{n+j-r}\frac{(r+1)!(r-j)!}{(2r+1-n-j)!}g^{2r+1-n-j}, \qquad 0 \le n \le r,\ r-n \le j \le r,$$

$$\frac{d^n}{dy^n}\left[f_r(y_0)g_j(y_0)\right] = \binom{n}{r+1}\frac{(r+1)!(r-j)!}{(2r+1-n-j)!}(-g)^{2r+1-n-j}$$

for $r+1 \le n \le 2r+1$, $0 \le j \le 2r+1-n$,

$$\frac{d^n}{dy^n}\left[f_r(y_1)g_j(y_1)\right] = \binom{n}{n+j-r}\frac{(r+1)!(r-j)!}{(2r+1-n-j)!}g^{2r+1-n-j}$$

for $r+1 \le n \le 2r+1$, $0 \le j \le 2r+1-n$. The derivatives of the second term in $H_r(x,y)$ are

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\boldsymbol{B}(x)\,D_r(y)\boldsymbol{Y}_{r,2}(y)\right] = \sum_{j=0}^{r} \frac{d^m B_j(x)}{dx^m}\,\frac{d^n}{dy^n}\left[f_r(y)g_j(y)\right],$$

and their values of at x, y_0 and y_1 are

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\boldsymbol{B}(x)\,D_r(y_0)\boldsymbol{Y}_{r,2}(y_0)\right] = 0 \tag{7}$$

for $0 \le m \le 2r+1$, $0 \le n \le r$;

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\boldsymbol{B}(x)\,D_r(y_1)\boldsymbol{Y}_{r,2}(y_1)\right] = (r+1)!\sum_{j=r-n}^{r}\frac{d^m B_j(x)}{dx^m}\times$$
$$\times\binom{n}{n+j-r}\frac{(r-j)!}{(2r+1-n-j)!}g^{2r+1-n-j} \tag{8}$$

for $0 \le m \le 2r+1$, $0 \le n \le r$;

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\boldsymbol{B}(x)\, D_r(y_0) \boldsymbol{Y}_{r,2}(y_0)\right] = (r+1)! \binom{n}{r+1} \sum_{j=0}^{2r+1-n} \frac{d^m B_j(x)}{dx^m} \times \\ \times \frac{(r-j)!}{(2r+1-n-j)!}(-g)^{2r+1-n-j} \tag{9}$$

for $0 \le m \le 2r+1$, $r+1 \le n \le 2r+1$;

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\boldsymbol{B}(x)\, D_r(y_1) \boldsymbol{Y}_{r,2}(y_1)\right] = (r+1)! \sum_{j=0}^{2r+1-n} \frac{d^m B_j(x)}{dx^m} \binom{n}{n+j-r} \times \\ \times \frac{(r+1)!}{(2r+1-n-j)} g^{2r+1-n-j} \tag{10}$$

for $0 \le m \le 2r+1$, $r+1 \le n \le 2r+1$. Applying the formulas (2), (3), (4), (5), (6) and (7), (8), (9), (10), the derivatives of $H_r(x, y)$ can be calculated and upon these the coefficients a, b can be determined.

Example 2 ($r = 2$).

$$\begin{aligned}
H_2(x_0, y_0) &= \underline{a}_1, \\
H_2^{(x)}(x_0, y_0) &= \underline{a}_2, \\
H_2^{(x,x)}(x_0, y_0) &= \underline{a}_3 2, \\
H_2^{(y)}(x_0, y_0) &= \underline{a}_4, \\
H_2^{(x,y)}(x_0, y_0) &= \underline{a}_5, \\
H_2^{(y,y)}(x_0, y_0) &= \underline{a}_6 2, \\
H_2(x_1, y_0) &= a_1 + a_2 h + a_3 h^2 + \underline{a}_7 h^3, \\
H_2^{(x)}(x_1, y_0) &= a_2 + a_3 2h + a_7 3h^2 + \underline{a}_8 h^3, \\
H_2^{(x,x)}(x_1, y_0) &= a_3 2 + a_7 6h + a_8 6h^2 + \underline{a}_9 2h^3, \\
H_2^{(y)}(x_1, y_0) &= a_4 + a_5 h + \underline{a}_{10} h^2, \\
H_2^{(x,y)}(x_1, y_0) &= a_5 + a_{10} 2h + \underline{a}_{11} h^2, \\
H_2^{(y,y)}(x_1, y_0) &= a_6 2 + \underline{a}_{12} 2h, \\
H_2(x_0, y_1) &= a_1 + a_4 g + a_6 g^2 + \underline{b}_1 g^3, \\
H_2^{(x)}(x_0, y_1) &= a_2 + a_5 g + a_{12} g^2 + \underline{b}_2 g^3, \\
H_2^{(x,x)}(x_0, y_1) &= a_3 2 + a_{10} 2g - a_{11} 2hg + \underline{b}_3 2g^3, \\
H_2^{(y)}(x_0, y_1) &= a_4 + a_6 2g + b_1 3g^2 + \underline{b}_4 g^3,
\end{aligned}$$

$$H_2^{(x,y)}(x_0,y_1) = a_5 + a_{12}2g + b_2 3g^2 + \underline{b}_5 2g^3,$$
$$H_2^{(y,y)}(x_0,y_1) = a_6 2 + b_1 6g + b_4 6g^2 + \underline{b}_6 2g^3,$$
$$H_2(x_1,y_1) = a_1 + a_2 h + a_3 h^2 + a_4 g + a_5 hg + a_6 g^2 + a_7 h^3 + a_{10} h^2 g + a_{12} hg^2 +, \\ + b_1 g^3 + b_2 hg^3 + b_3 h^2 g^3 + \underline{b}_7 h^3 g^3,$$
$$H_2^{(x)}(x_1,y_1) = a_2 + a_3 2h + a_5 g + a_7 3h^2 + a_8 h^3 + a_{10} 2hg + a_{11} h^2 g + a_{12} g^2 +, \\ + b_2 g^3 + b_3 2hg^3 + b_7 3h^2 g^3 + \underline{b}_8 h^3 g^3,$$
$$H_2^{(x,x)}(x_1,y_1) = a_3 2 + a_7 6h + a_8 6h^2 + a_9 2h^3 + a_{10} 2g + a_{11} 4hg + b_3 2g^3 + b_7 6hg^3 +, \\ + b_8 6h^2 g^3 + \underline{b}_9 2h^3 g^3,$$
$$H_2^{(y)}(x_1,y_1) = a_4 + a_5 h + a_6 2g + a_{10} h^2 + a_{12} 2hg + b_1 3g^2 + b_2 3hg^2 + b_3 3h^2 g^2 +, \\ + b_4 g^3 + b_5 hg^3 + b_7 3h^3 g^2 + \underline{b}_{10} h^2 g^3,$$
$$H_2^{(x,y)}(x_1,y_1) = a_5 + a_{10} 2h + a_{11} h^2 + a_{12} 2g + b_2 3g^2 + b_3 6hg^2 + b_5 g^3 + b_7 9h^2 g^2 +, \\ + b_8 3h^3 g^2 + b_{10} 2hg^3 + \underline{b}_{11} h^2 g^3,$$
$$H_2^{(y,y)}(x_1,y_1) = a_6 2 + a_{12} 2h + b_1 6g + b_2 6hg + b_3 6h^2 g + b_4 6g^2 + b_5 6hg^2 + b_6 2g^3 +, \\ + b_7 6h^3 g + b_{10} 6h^2 g^2 + \underline{b}_{12} 2hg^3.$$

4. Three-dimensional polynomial

The aim of this section is to construct a polynomial $H_r(x,y,z)$ which fits the given values on to the coordinate points $x_0, x_1, y_0, y_1, z_0, z_1$ up to its rth derivatives. The conditions can be written as follows:

$$\frac{\partial^{m+n+p} H_r\,(x_l, y_s, z_t)}{\partial x^m \partial y^n \partial z^p} = \frac{\partial^{m+n+p} u\,(x_l, y_s, z_t)}{\partial x^m \partial y^n \partial z^p}$$

for $l, s, t = 0, 1$; $0 \le m, n, p \le r$; $0 \le m+n+p \le r$. The number of the jth derivatives of a function having three independent variables is the number of conditions at one point, $s_j = \frac{1}{2}(j^2 + 3j) + 1$; up to its rth derivatives, the number is equal to $S_r = \sum_{j=0}^{r} s_j = \frac{1}{6}(r^3 + 6r^2 + 11r + 6)$, and for the eight points we have $8S_r$.

Let us seek for it in the form

$$H_r(x,y,z) = \boldsymbol{A}(x,z)\mathbf{Y}_{r,1}(y) + \boldsymbol{B}(x,z)D_r(y)\mathbf{Y}_{r,2}(y).$$

For the sake of a convenient form of writing the vectors, let us define the unit vector of $j+1$ entries $\boldsymbol{e}_{j,q+1}$ $(0 \le q \le j)$ the $q+1$-*th* entry of which is 1, and all the rest are zero. Then the vectors in $H_r(x, y, z)$ can be written by using the formulas

$$\boldsymbol{Y}_{j,1}(y) = \sum_{i=0}^{j} (y - y_0)^i \, \boldsymbol{e}_{j,i+1}, \qquad \boldsymbol{Y}_{j,2}(y) = \sum_{i=0}^{j} (y - y_1)^i \boldsymbol{e}_{j,i+1},$$

$$D_j(y) = (y - y_0)^{j+1}, \quad 0 \le j \le r.$$

$$\boldsymbol{A}(x, z) = \sum_{j=0}^{r} A_j(x, z)\boldsymbol{e}_{r,r+1-j}, \qquad \boldsymbol{B}(x, z) = \sum_{j=0}^{r} B_j(x, z)\boldsymbol{e}_{r,r+1-j},$$

$$A_j(x, z) = \boldsymbol{A}_{j,1}(z)\boldsymbol{X}_{j,1}(x) + \boldsymbol{A}_{j,2}(z)C_j\,(x)\,\boldsymbol{X}_{j,2}(x), \quad 0 \le j \le r,$$

$$B_j(x, z) = \boldsymbol{B}_{j,1}(z)\boldsymbol{X}_{j,1}(x) + \boldsymbol{B}_{j,2}(z)C_j\,(x)\,\boldsymbol{X}_{j,2}(x), \quad 0 \le j \le r,$$

$$\boldsymbol{X}_{j,1}(x) = \sum_{i=0}^{j} (x - x_0)^i \, \boldsymbol{e}_{j,i+1}, \qquad \boldsymbol{X}_{j,2}(x) = \sum_{i=0}^{j} (x - x_1)^i \boldsymbol{e}_{j,i+1},$$

$$C_j(x) = (x - x_0)^{j+1}, \quad 0 \le j \le r,$$

$$\boldsymbol{A}_{j,1}(z) = \sum_{q=0}^{j} A_{j,1,q}(z)\boldsymbol{e}_{j,j+1-q}, \qquad \boldsymbol{A}_{j,2}(z) = \sum_{q=0}^{j} A_{j,2,q}(z)\boldsymbol{e}_{j,j+1-q},$$

$$\boldsymbol{B}_{j,1}(z) = \sum_{q=0}^{j} B_{j,1,q}\,(z)\,\boldsymbol{e}_{j,j+1-q}, \qquad \boldsymbol{B}_{j,2}(z) = \sum_{q=0}^{j} B_{j,2,q}(z)\boldsymbol{e}_{j,j+1-q},$$

$$A_{j,1,q}(z) = \boldsymbol{\alpha}_{j,1,2k}\boldsymbol{Z}_{q,1}\,(z) + \boldsymbol{\alpha}_{j,1,2k+1}E_j(z)\boldsymbol{Z}_{q,2}(z), \quad 0 \le q \le j,$$

$$A_{j,2,q}(z) = \boldsymbol{\alpha}_{j,2,2k}\boldsymbol{Z}_{q,1}\,(z) + \boldsymbol{\alpha}_{j,2,2k+1}E_j(z)\boldsymbol{Z}_{q,2}(z), \quad 0 \le q \le j,$$

$$B_{j,1,q}(z) = \boldsymbol{\beta}_{j,1,2k}\boldsymbol{Z}_{q,1}\,(z) + \boldsymbol{\beta}_{j,1,2k+1}E_j(z)\boldsymbol{Z}_{q,2}(z), \quad 0 \le q \le j,$$

$$B_{j,2,q}(z) = \boldsymbol{\beta}_{j,2,2k}\boldsymbol{Z}_{q,1}\,(z) + \boldsymbol{\beta}_{j,2,2k+1}E_j(z)\boldsymbol{Z}_{q,2}(z), \quad 0 \le q \le j,$$

and $\boldsymbol{Z}_{q,1}(z) = \sum_{i=0}^{q} (z - z_0)^i \, \boldsymbol{e}_{q,i+1}$, $\boldsymbol{Z}_{q,2}(z) = \sum_{i=0}^{q} (z - z_1)^i \, \boldsymbol{e}_{q,i+1}$, $E_q(z) = (z - z_0)^{q+1}$, $0 \le q \le j$. The vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are composed by constant components a and b. To give the connection, the following notation will be applied:

$$l_{r,q} = (r - q)\frac{r + 3 + q}{2}, \qquad l_{j,q} = (j - q)\frac{r + 3 + q}{2} + \sum_{i=j+1}^{r} n_i,$$

$$n_i = \frac{1}{2}\,(i + 1)\,(i + 2)\,, \qquad N_r = \sum_{i=0}^{r} n_i = \frac{1}{6}\left(r^3 + 6r^2 + 11r + 6\right)$$

$$L_{r,q} = 2N_r + l_{r,q}, \qquad L_{j,q} = 2N_r + l_{j,q}, \quad k = j - q;$$

$$\boldsymbol{\alpha}_{j,1,2k} = \sum_{i=0}^{q} a_{l_{j,q}+i+1}\boldsymbol{e}_{q,i+1}, \qquad \boldsymbol{\alpha}_{j,1,2k+1} = \sum_{i=0}^{q} a_{N_r+l_{j,q}+i+1}\boldsymbol{e}_{q,i+1},$$

$$\boldsymbol{\alpha}_{j,2,2k} = \sum_{i=0}^{q} a_{L_{j,q}+i+1}\boldsymbol{e}_{q,i+1}, \qquad \boldsymbol{\alpha}_{j,2,2k+1} = \sum_{i=0}^{q} a_{N_r+L_{j,q}+i+1}\boldsymbol{e}_{q,i+1}.$$

The relations between $\boldsymbol{\beta}$ and b are analogous to the previously given formulas where a is replaced by b.

The number of the constants a, b in $\boldsymbol{A}_{j,1}$, $\boldsymbol{A}_{j,2}$, $\boldsymbol{B}_{j,1}$, $\boldsymbol{B}_{j,2}$ is $8N_r$, which agrees with the number of conditions $8N_r = 8S_r$. Consequently, $H_r(x, y, z)$ is the minimal degree polynomial satisfying the requirements.

The distances between the coordinates are$x_1 - x_0 = h$, $y_1 - y_0 = g$, $z_1 - z_0 = l$.The derivatives of the $H_r(x, y, z)$ function are needed to compute the coefficients a and b (rather tedious calculations are omitted here to save space):

$$\frac{d^p A_{j,1,q}(z_0)}{dz^p} = p! a_{l_{j,q}+p+1}, \quad 0 \le p \le q, \tag{11}$$

$$\frac{d^p A_{j,1,q}(z_1)}{dz^p} = \sum_{i=0}^{q-p} \frac{(p+i)!}{i!} l^i a_{l_{j,q}+p+i+1} + \\ + p!\,(q+1)! \sum_{i=0}^{p} \frac{l^{q+1-p+i} a_{N_r+l_{j,q}+i+1}}{(p-i)!(q+1-p+i)!}, \quad 0 \le p \le q,$$

$$\frac{d^p A_{j,1,q}(z_0)}{dz^p} = \binom{p}{p-(q+1)}(q+1)! \sum_{i=p-(q+1)}^{q} \frac{i!\,(-l)^{q+1-p+i}}{(q+1-p+i)!} a_{N_r+l_{j,q}+i+1}, \tag{12}$$

$$\frac{d^p A_{j,1,q}(z_1)}{dz^p} = p!(q+1)! \sum_{i=p-(q+1)}^{q} \frac{l^{q+1-p+i}}{(p-i)!(q+1-p+i)!} a_{N_r+l_{j,q}+i+1}, \\ q+1 \le p \le 2q+1.$$

The derivatives of (11) are similar to the previous ones,

$$\frac{d^p A_{j,2,q}(z_0)}{dz^p} = p! a_{L_{j,q}+p+1}, \tag{13}$$

$$\frac{d^p A_{j,2,q}(z_1)}{dz^p} = \sum_{i=0}^{q-p} \frac{(p+i)!}{i!} l^i a_{L_{j,q}+p+i+1} + \\ + p!\,(q+1)! \sum_{i=0}^{p} \frac{l^{q+1-p+i} a_{N_r+L_{j,q}+i+1}}{(p-i)!\,(q+1-p+i)!}, \quad 0 \le p \le q,$$

$$\frac{d^p A_{j,2,q}(z_0)}{dz^p} = \binom{p}{p-(q+1)}(q+1)! \sum_{i=p-(q+1)}^{q} \frac{i!\,(-l)^{q+1-p+i}\, a_{N_r+L_{j,q}+i+1}}{(q+1-p+i)!}, \tag{14}$$

$$\frac{d^p A_{j,2,q}(z_1)}{dz^p} = p!\,(q+1)! \sum_{i=p-(q+1)}^{q} \frac{l^{q+1-p+i} a_{N_r+L_{j,q}+i+1}}{(p-i)!\,(q+1-p+i)!}, \quad q+1 \le p \le 2q+1.$$

To obtain similar formulas for $B_{j,1,q}$ and $B_{j,2,q}$, a should be replaced by b.

For the derivatives of $A_j(x,z)$, expressions (11), (12), (13), (14) are to be used,

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,1}(z)\boldsymbol{X}_{j,1}(x_0)\right] = 0 \quad \text{for } m \neq j-q, \tag{15}$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,1}(z)\boldsymbol{X}_{j,1}(x_0)\right] = \frac{d^p A_{j,1,j-m}(z)}{dz^p} m!,$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,1}(z)\boldsymbol{X}_{j,1}(x_1)\right] = \sum_{q=0}^{j-m} \frac{d^p A_{j,1,q}(z)}{dz^p} \frac{(j-q)!\, h^{j-q-m}}{(j-q-m)!}, \quad 0 \le m \le j;$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,2}(z)C_j(x_0)\boldsymbol{X}_{j,2}(x_0)\right] = 0, \quad 0 \le m \le j, \tag{16}$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,2}(z)C_j(x_1)\boldsymbol{X}_{j,2}(x_1)\right] = \sum_{q=j-m}^{j} \frac{d^p A_{j,2,q}(z)}{dz^p} \times$$

$$\times \frac{m!\,(j+1)!\, h^{2j+1-m-q}}{(m-(j-q))!(2j+1-m-q)!}, \quad 0 \le m \le j,\, j-q < m.$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,2}(z)C_j(x_0)\boldsymbol{X}_{j,2}(x_0)\right] = \frac{(m)!}{(m-(j+1))!} \times \tag{17}$$

$$\times \sum_{q=0}^{2j+1-m} \frac{d^p A_{j,2,q}(z)}{dz^p} \frac{(j-q)!\,(-h)^{2j+1-m-q}}{(2j+1-m-q)!},$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,2}(z)C_j(x_1)\boldsymbol{X}_{j,2}(x_1)\right] = m!(j+1)! \times$$

$$\times \sum_{q=0}^{2j+1-m} \frac{d^p A_{j,2,q}(z)}{dz^p} \frac{h^{2j+1-m-q}}{(m-(j-q))!(2j+1-m-q)},$$

where $j+1 \le m \le 2j+1$. The derivatives of $B_j(x,z)$ will be obtained by replacing A by B in formulas (15), (16), and (17).

Finally, the derivatives of the first and second terms in the Hermite polynomials are to be given, and these are as follows:

$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{A}(x,z)\boldsymbol{Y}_{r,1}(y_0)\right] = n! \frac{\partial^{m+p} A_{r-n}(x,z)}{\partial x^m \partial z^p},$$

$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{A}(x,z)\boldsymbol{Y}_{r,1}(y_1)\right] = \sum_{j=0}^{r-n} \frac{\partial^{m+p} A_j(x,z)}{\partial x^m \partial z^p} \frac{(r-j)!g^{r-j-n}}{(r-j-n)!}, \quad 0 \le n \le r;$$

$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{B}(x,z)D_r(y_0)\boldsymbol{Y}_{r,2}(y_0)\right] = 0, \quad 0 \le n \le r,$$

$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{B}(x,z)D_r(y_1)\boldsymbol{Y}_{r,2}(y_1)\right] = (r+1)!\sum_{j=r-n}^{r}\frac{\partial^{m+p}B_j(x,z)}{\partial x^m \partial z^p}\times$$

$$\times\binom{n}{n+j-r}\frac{(r-j)!\,g^{2r+1-n-j}}{(2r+1-n-j)!},\quad 0\le n\le r;$$

$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{B}(x,z)D_r(y_0)\boldsymbol{Y}_{r,2}(y_0)\right] = \binom{n}{r+1}(r+1)!\sum_{j=0}^{2r+1-n}\frac{\partial^{m+p}B_j(x,z)}{\partial x^m \partial z^p}\times$$

$$\times\frac{(r-j)!(-g)^{2r+1-n-j}}{(2r+1-n-j)!},$$

$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{B}(x,z)D_r(y_1)\boldsymbol{Y}_{r,2}(y_1)\right] = (r+1)!\sum_{j=0}^{2r+1-n}\frac{\partial^{m+p}B_j(x,z)}{\partial x^m \partial z^p}\binom{n}{n+j-r}\times$$

$$\times\frac{(r-j)!\,g^{2r+1-n-j}}{(2r+1-n-j)!},\quad r+1\le n\le 2r+1.$$

Example 3 ($r = 2$). In order to save space, we show below only the first 40 coefficients:

$$H_2(x_0,y_0,z_0) = \underline{a}_1,$$
$$H_2^{(z)}(x_0,y_0,z_0) = \underline{a}_2,$$
$$H_2^{(z,z)}(x_0,y_0,z_0) = \underline{a}_3 2,$$
$$H_2^{(x)}(x_0,y_0,z_0) = \underline{a}_4,$$
$$H_2^{(x,z)}(x_0,y_0,z_0) = \underline{a}_5,$$
$$H_2^{(x,x)}(x_0,y_0,z_0) = \underline{a}_6 2,$$
$$H_2^{(y)}(x_0,y_0,z_0) = \underline{a}_7,$$
$$H_2^{(y,z)}(x_0,y_0,z_0) = \underline{a}_8,$$
$$H_2^{(x,y)}(x_0,y_0,z_0) = \underline{a}_9,$$
$$H_2^{(y,y)}(x_0,y_0,z_0) = \underline{a}_{10} 2,$$
$$H_2(x_0,y_0,z_1) = a_1 + a_2 l + a_3 l^2 + \underline{a}_{11} l^3,$$
$$H_2^{(z)}(x_0,y_0,z_1) = a_2 + a_3 2l + a_{11} 3l^2 + \underline{a}_{12} l^3,$$
$$H_2^{(z,z)}(x_0,y_0,z_1) = a_3 2 + a_{11} 6l + a_{12} 6l^2 + \underline{a}_{13} 2l^3,$$
$$H_2^{(x)}(x_0,y_0,z_1) = a_4 + a_5 l + \underline{a}_{14} l^2,$$
$$H_2^{(x,z)}(x_0,y_0,z_1) = a_5 + a_{14} 2l + \underline{a}_{15} l^2,$$
$$H_2^{(x,x)}(x_0,y_0,z_1) = a_6 2 + \underline{a}_{16} 2l,$$

$$
\begin{aligned}
H_2^{(y)}(x_0, y_0, z_1) &= a_7 + a_8 l + \underline{a}_{17} l^2,\\
H_2^{(y,z)}(x_0, y_0, z_1) &= a_8 + a_{17} 2l + \underline{a}_{18} l^2,\\
H_2^{(x,y)}(x_0, y_0, z_1) &= a_9 + \underline{a}_{19} l,\\
H_2^{(y,y)}(x_0, y_0, z_1) &= a_{10} 2 + \underline{a}_{20} 2l,\\
H_2(x_1, y_0, z_0) &= a_6 h^2 + a_4 h + a_1 + \underline{a}_{21} h^3,\\
H_2^{(z)}(x_1, y_0, z_0) &= a_{16} h^2 + a_5 h + a_2 + \underline{a}_{22} h^3,\\
H_2^{(z,z)}(x_1, y_0, z_0) &= (a_{14} 2 - a_{15} 2l)\, h + a_3 2 + \underline{a}_{23} 2h^3,\\
H_2^{(x)}(x_1, y_0, z_0) &= a_6 2h + a_4 + a_{21} 3h^2 + \underline{a}_{24} h^3,\\
H_2^{(x,z)}(x_1, y_0, z_0) &= a_{16} 2h + a_5 + a_{22} 3h^2 + \underline{a}_{25} h^3,\\
H_2^{(x,x)}(x_1, y_0, z_0) &= a_6 2 + a_{21} 6h + a_{24} 6h^2 + \underline{a}_{26} h^3,\\
H_2^{(y)}(x_1, y_0, z_0) &= a_9 h + a_7 + \underline{a}_{27} h^2,\\
H_2^{(y,z)}(x_1, y_0, z_0) &= a_{19} h + a_8 + \underline{a}_{28} h^2,\\
H_2^{(x,y)}(x_1, y_0, z_0) &= a_9 + a_{27} 2h + \underline{a}_{29} h^2,\\
H_2^{(y,y)}(x_1, y_0, z_{01}) &= \left(a_{10} + \underline{a}_{30} h\right) 2,\\
H_2(x_1, y_0, z_1) &= a_1 + a_2 l + a_3 l^2 + a_{11} l^3 + \left(a_4 + a_5 l + a_{14} l^2\right) h + (a_6 + a_{16} l)\, h^2 +,\\
&\quad + \left(a_{21} + a_{22} l + a_{23} l^2 + \underline{a}_{31} l^3\right) h^3,\\
H_2^{(z)}(x_1, y_0, z_1) &= a_2 + a_3 2l + a_{11} 3l^2 + a_{12} l^3 + a_{16} h^2 + \left(a_5 + a_{14} 2l + a_{15} l^2\right) h +,\\
&\quad + \left(a_{22} + a_{23} 2l + a_{31} 3l^2 + \underline{a}_{32} l^3\right) h^3,\\
H_2^{(z,z)}(x_1, y_0, z_1) &= a_3 2 + a_{11} 6l + a_{12} 6l^2 + a_{13} 2l^3 + (a_{14} 2 + a_{15} 4l)\, h +,\\
&\quad + \left(a_{23} 2 + a_{31} 6l + a_{32} 6l^2 + \underline{a}_{33} 2l^3\right) h^3,\\
H_2^{(x)}(x_1, y_0, z_1) &= a_4 + a_5 l + a_{14} l^2 + (a_6 + a_{16} l)\, 2h +,\\
&\quad \left(a_{21} + a_{22} l + a_{23} l^2 + a_{31} l^3\right) 3h^2 + \left(a_{24} + a_{25} l + \underline{a}_{34} l^2\right) h^3,\\
H_2^{(x,z)}(x_1, y_0, z_1) &= a_5 + a_{14} 2l + a_{15} l^2 + a_{16} 2h +,\\
&\quad + \left(a_{22} + a_{23} 2l + a_{31} 3l^2 + a_{32} l^3\right) 3h^2 + \left(a_{25} + a_{34} 2l + \underline{a}_{35} l^2\right) h^3,\\
H_2^{(x,x)}(x_1, y_0, z_1) &= a_6 2 + a_{16} 2l + \left(a_{24} + a_{25} l + a_{34} l^2\right) 6h^2 +,\\
&\quad + \left(a_{21} + a_{22} l + a_{23} l^2 + a_{31} l^3\right) 6h + \left(a_{26} + \underline{a}_{36} l\right) h^3,\\
H_2^{(y)}(x_1, y_0, z_1) &= a_7 + a_8 l + a_{17} l^2 + (a_9 + a_{19} l)\, h +,
\end{aligned}
$$

$$+\left(a_{27}+a_{28}l+\underline{a}_{37}l^2\right)h^2,$$

$$H_2^{(y,z)}(x_1,y_0,z_1)=a_8+a_{17}2l+a_{18}l^2+a_{19}h+,$$

$$+\left(a_{28}+a_{37}2l+\underline{a}_{38}l^2\right)h^2,$$

$$H_2^{(x,y)}(x_1,y_0,z_1)=a_9+a_{19}l+\left(a_{27}+a_{28}l+a_{37}l^2\right)2h+\left(a_{29}+\underline{a}_{39}l\right)h^2,$$

$$H_2^{(y,y)}(x_1,y_0,z_1)=a_{10}2+a_{20}2l+\left(a_{30}+\underline{a}_{40}l\right)2h.$$

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