A one-sided theorem for the product of Abel and Cesáro summability methods

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A ONE-SIDED THEOREM FOR THE PRODUCT OF ABEL AND CESÀRO SUMMABILITY METHODS

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Abstract. In this paper, a one-sided condition is given to recover $(C, \alpha)$ summability of a sequence from its $(A)(C, \alpha + 1)$ summability. Our result extends and generalizes the well known classical Tauberian theorems given for Abel and Cesàro summability methods.

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1. INTRODUCTION

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series of real numbers with partial sums $s_n = \sum_{k=0}^{n} a_k$. For all nonnegative integers $m$, we define

$$(n \Delta)_m s_n = n \Delta ((n \Delta)_{m-1} s_n),$$

where $(n \Delta)_0 s_n = s_n$ and $(n \Delta)_1 s_n = n \Delta s_n$.

The backward difference $\Delta s_n$ of $s_n$ is defined to be $\Delta s_n = s_n - s_{n-1}$, $n \geq 1$, with $\Delta s_0 = s_0$.

Let $A_0^\alpha = 0$, $A_1^\alpha = \frac{\alpha(\alpha + 1) \cdots (\alpha + n)}{n!}$, and $A_n^\alpha = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha + 1)}$ for $\alpha > -1$.

A sequence $(s_n)$ is said to be summable by the Cesàro mean of order $\alpha$, or $(C, \alpha)$ summable to $s$, where $\alpha > -1$, and we write $s_n \rightarrow s (C, \alpha)$ if

$$s_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha} = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k \rightarrow s$$
as $n \rightarrow \infty$.

We write $\tau_n = na_n$ and denote the $(C, \alpha)$ mean of $(\tau_n)$ by $\tau_n^{\alpha}$. Borwein [4] showed that if a sequence is $(C, \alpha)$ summable to $s$ for any $\alpha > -1$, it is $(C, \beta)$ summable to

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for any \( \beta > \alpha \). It is also well known that the \((C, \alpha)\) summability method is regular (see [3]). Note that \((C, 0)\) summability reduces to the ordinary convergence.

A sequence \((s_n)\) is said to be Abel summable to \(s\), and we write \(s_n \rightarrow s (A)\) if the series \(\sum_{n=0}^{\infty} a_n x^n\) is convergent for \(0 \leq x < 1\) and tends to \(s\) as \(x \rightarrow 1^-\). It is well known that if a sequence is \((C, \alpha)\) summable to \(s\) for any \(\alpha > -1\), then it is Abel summable to \(s\) (see [2]).

A sequence \((s_n)\) is said to be \((A)(C, \alpha)\) summable to \(s\), and we write \(s_n \rightarrow s (A)(C, \alpha)\) if the series \(\sum_{n=0}^{\infty} (s^n_n - s^n_{n-1}) x^n\), with \(s^n_{-1} = 0\), is convergent for \(0 \leq x < 1\) and tends to \(s\) as \(x \rightarrow 1^-\). Note that \((A)(C, \alpha)\) summability reduces to the Abel summability when \(\alpha = 0\).

The identity \(s_n - s^n_n = \tau^n_n\) is known as the Kronecker identity and it will be used in the proof of the main result. Throughout this paper we use the symbols \(s_n = o(1)\) and \(s_n = O(1)\) to mean that \(s_n \rightarrow 0\) as \(n \rightarrow \infty\) and \((s_n)\) is bounded for large enough \(n\).

2. Preliminary results

A theorem due to Abel [1] states that if \((s_n)\) converges to \(s\), then it is Abel summable to \(s\). The converse Abel’s theorem is not necessarily true. For example the series \(\sum_{n=0}^{\infty} (-1)^n\) is not convergent, but it is Abel summable to \(1/2\). However, the converse of Abel’s theorem may be valid under some condition which we call Tauberian condition. Any theorem stating that convergence follows from a summability method and a Tauberian condition is called a Tauberian theorem.

By imposing some restriction on \(a_n\), Tauber [17] obtained the first partial converses of Abel’s theorem.

**Theorem 1.** If \((s_n)\) is Abel summable to \(s\) and \(\tau_n = o(1)\), then \((s_n)\) converges to \(s\).

**Theorem 2.** If \((s_n)\) is Abel summable to \(s\) and \(\tau^n_n = o(1)\), then \((s_n)\) converges to \(s\).

Littlewood [12] replaced the condition \(\tau_n = o(1)\) by \(\tau_n = O(1)\) and later Hardy and Littlewood [9] obtained the following one-sided Tauberian theorem.

**Theorem 3.** If \((s_n)\) is Abel summable to \(s\) and \(\tau_n \geq -H\) for some nonnegative constant \(H\), then \((s_n)\) converges to \(s\).

A generalization of Theorem 3 was given by Szász [16].

**Theorem 4.** If \((s_n)\) is Abel summable to \(s\) and \(\tau^n_n \geq -H\) for some nonnegative constant \(H\), then \((s_n)\) is \((C, 1)\) summable to \(s\).

Pati [14] have recently obtained more general Tauberian theorems generalizing the classical results for the product of the Abel and \((C, \alpha)\) summability methods.
Theorem 5. If \((s_n)\) is \((A)(C, \alpha)\) summable to \(s\), where \(\alpha > 0\), and \(r_n^\alpha \geq -H\) for some nonnegative constant \(H\), then \((s_n)\) is \((C, \alpha)\) summable to \(s\).

Theorem 6. The necessary and sufficient condition that the \((A)(C, \alpha + 1)\) summability of \((s_n)\) to \(s\), where \(\alpha > -1\), implies the \((C, \alpha)\) summability of \((s_n)\) to \(s\), is that \(\tau_n^\alpha + 1 = o(1)\).

Tauberian theorems in the sense of Pati were generalized by Çakn et al. [6], Çakn and Erdem [5] and Erdem and Çakn [7]. Çakn et al. [6] proved that if \((s_n)\) is \((A)(C, \alpha)\) summable to \(s\) and \((n \Delta)_m r_n^{\alpha + m} = o(1)\) for \(m = 1, 2\), then \((s_n)\) is convergent to \(s\). Later, Erdem and Çakn [7] proved the main result in Çakn et al. [6] for all integers \(m \geq 1\). Recently, Çakn and Erdem [5] have recovered convergence, \((C, \alpha)\) convergence, and \((C, \alpha)\) slow oscillation of \((s_n)\) depending on the conditions given in terms of \((n \Delta)_m r_n^{\alpha + m}\) for some special cases of \(m\).

In this paper, we recover \((C, \alpha)\) convergence of \((s_n)\) from its \((A)(C, \alpha + 1)\) summability under the one-sided boundedness of \((n \Delta)_m r_n^{\alpha + m}\), where \(m \geq 1\) and \(\alpha > -1\).

3. Main result

Our result is based on Theorem 1 and Theorem 3.

Theorem 7. If \((s_n)\) is \((A)(C, \alpha + 1)\), where \(\alpha > -1\), summable to \(s\), and for some integer \(m \geq 0\),

\[(n \Delta)_m r_n^{\alpha + m} \geq -H\] (3.1)

then \((s_n)\) is \((C, \alpha)\) summable to \(s\).

From Theorem 7, we deduce the following corollary:

Corollary 1. If \((s_n)\) is \((A)(C, \alpha + 1)\), where \(\alpha > -1\), summable to \(s\), and for some integer \(m \geq 0\),

\[(n \Delta)_m r_n^m \geq -H\] (3.2)

then \((s_n)\) converges to \(s\).

4. Auxiliary results

We need the following lemmas for the proof of Theorem 7.

**Lemma 1 ([10, 11])**. For \(\alpha > -1\), \(r_n^\alpha = n \Delta s_n^\alpha = n(s_n^\alpha - s_{n-1}^\alpha)\).

**Lemma 2 ([8, 11])**. For \(\alpha > -1\), \(r_n^{\alpha + 1} = (\alpha + 1)(s_n^\alpha - s_{n-1}^{\alpha + 1})\).

**Lemma 3 ([6])**. For \(\alpha > -1\), \(n \Delta r_n^{\alpha + 1} = (\alpha + 1)(r_n^\alpha - r_{n-1}^{\alpha + 1})\).

**Lemma 4 ([13])**. For \(-1 < \alpha < \beta\), \((A)(C, \alpha) \subset (A)(C, \beta)\).
Lemma 5 ([7]). Let $\alpha > -1$. For any integer $m \geq 2$,
\[
(n \Delta)_m \tau_n^{\alpha+m} = \sum_{j=1}^{m} (-1)^{j+1} A_m^{(j)}(\alpha) n \Delta \tau_n^{(\alpha+j)},
\]
where
\[
A_m^{(j)}(\alpha) = a_m^{(j-1)}(\alpha) + a_m^{(j)}(\alpha), \quad a_m^{(0)}(\alpha) = 0,
\]
and
\[
a_m^{(j)}(\alpha) = \prod_{k=j+1}^{m} (\alpha + k) \left[ \sum_{j_1 \leq t_1, j_2, \ldots, j_{j-1} \leq m, \ r < s \Rightarrow t_r \leq t_s} (\alpha + t_1)(\alpha + t_2) \ldots (\alpha + t_{j-1}) \right]
\]
where $j = 1, 2, 3, \ldots, m$.

Lemma 6 ([7]). Let $\alpha > -1$. For any integer $m \geq 2$,
\[
(\alpha + j) A_m^{(j-1)}(\alpha + 1) + (\alpha + j + 1) A_m^{(j)}(\alpha + 1) = A_{m+1}^{(j)}(\alpha),
\]
where $A_m^{(j)}(\alpha)$ is as in Lemma 5.

Lemma 7 ([15]). For $\alpha > -1$, $\sigma_n(s^\alpha) = \frac{1}{\alpha + 1} s_n^{\alpha+1} + \left( 1 - \frac{1}{\alpha + 1} \right) \sigma_n(s^\alpha+1)$.

Lemma 8. i) For all $\lambda > 1$ and large enough $n$, that is, when $[\lambda n] > n$,
\[
s_n^{\alpha} - s_n^{\alpha+1} = \frac{\alpha}{\alpha + 1} \left[ \frac{[\lambda n] + 1}{[\lambda n] - n} \left( \sigma_{[\lambda n]}(s^{\alpha+1}) - \sigma_n(s^{\alpha+1}) \right) + (\sigma_n(s^{\alpha+1}) - s_n^{\alpha+1}) \right]
\]
\[
+ \frac{1}{\alpha} \frac{[\lambda n] + 1}{[\lambda n] - n} \left( s_{[\lambda n]}^{\alpha+1} - s_n^{\alpha+1} \right) - \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (s_k^{\alpha} - s_n^{\alpha}) \quad (4.1)
\]
ii) For all $0 < \lambda < 1$ and large enough $n$, that is, when $n > [\lambda n]$,
\[
s_n^{\alpha} - s_n^{\alpha+1} = \frac{\alpha}{\alpha + 1} \left[ \frac{[\lambda n] + 1}{n - [\lambda n]} \left( \sigma_n(s^{\alpha+1}) - \sigma_{[\lambda n]}(s^{\alpha+1}) \right) + (\sigma_n(s^{\alpha+1}) - s_n^{\alpha+1}) \right]
\]
\[
+ \frac{1}{\alpha} \frac{[\lambda n] + 1}{n - [\lambda n]} \left( s_{[\lambda n]}^{\alpha+1} - s_n^{\alpha+1} \right) - \frac{1}{n - [\lambda n]} \sum_{k=[\lambda n]+1}^{n} (s_k^{\alpha} - s_n^{\alpha}), \quad (4.2)
\]
where $[\lambda n]$ denotes the integer part of the product $\lambda n$.

Proof. Let $\tau_n^{\alpha, [\lambda n]} = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} s_k^{\alpha}$. Then we have,
\[
\tau_n^{\alpha, [\lambda n]} - s_n^{\alpha+1} = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} s_k^{\alpha} - s_n^{\alpha+1}
\]
By lemma 7, we have

\[
\tau_{n,[\lambda n]} = \frac{1}{[\lambda n] - n} - \frac{1}{\alpha + 1} \left( \frac{[\lambda n]}{\alpha + 1} \sigma_{n+1}^{\alpha} + \frac{1}{\alpha + 1} \sigma_{[\lambda n]}^{\alpha+1} \right) - s_n^{\alpha+1} - n + 1 \sigma_n^{\alpha+1} - (\lambda n) s_n^{\alpha+1}
\]

or

\[
s_n^{\alpha} - s_n^{\alpha+1} = s_n^{\alpha} - \tau_{n,[\lambda n]} + \tau_{n,[\lambda n]} - s_n^{\alpha+1}
\]

\[
= s_n^{\alpha} - \tau_{n,[\lambda n]} + \frac{1}{[\lambda n] - n} \left( \frac{[\lambda n]}{\alpha + 1} \sigma_{n+1}^{\alpha} + \frac{[\lambda n]}{\alpha + 1} \sigma_{[\lambda n]}^{\alpha+1} \right) - n + 1 \sigma_n^{\alpha+1} - (\lambda n) s_n^{\alpha+1}
\]

\[
= s_n^{\alpha} - \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} s_k^{\alpha} + \frac{1}{[\lambda n] - n} \left( \frac{[\lambda n]}{\alpha + 1} \sigma_{[\lambda n]}^{\alpha+1} + \frac{[\lambda n]}{\alpha + 1} \sigma_{[\lambda n]}^{\alpha+1} \right) - n + 1 \sigma_n^{\alpha+1} - (\lambda n) s_n^{\alpha+1}
\]

We finally have

\[
s_n^{\alpha} - s_n^{\alpha+1} = \frac{1}{[\lambda n] - n} \left( \frac{[\lambda n]}{\alpha + 1} \sigma_{[\lambda n]}^{\alpha+1} + \frac{[\lambda n]}{\alpha + 1} \sigma_{[\lambda n]}^{\alpha+1} \right) - \frac{\alpha(n+1)}{\alpha + 1} \sigma_n^{\alpha+1} - (\lambda n) s_n^{\alpha+1}
\]
\[
- \frac{1}{[\lambda n] - n} \left( \sum_{k=n+1}^{[\lambda n]} s_k^\alpha - ([\lambda n] - n) s_n^\alpha \right)
= \frac{1}{[\lambda n] - n} \left( \frac{\alpha([\lambda n] + 1)}{\alpha + 1} \sigma_n(s_n^\alpha + 1) - \frac{\alpha(n + 1)}{\alpha + 1} \sigma_n(s_n^\alpha + 1) \right)
+ \frac{\alpha([\lambda n] + 1)}{\alpha + 1} \sigma_n(s_n^\alpha + 1) - \frac{\alpha([\lambda n] + 1)}{\alpha + 1} \sigma_n(s_n^\alpha + 1) \right)
+ \frac{[\lambda n] + 1}{\alpha + 1} s_n^\alpha - \frac{\alpha[\lambda n] - \alpha n + [\lambda n] + 1}{\alpha + 1} s_n^\alpha \right)
- \frac{1}{[\lambda n] - n} \left( \sum_{k=n+1}^{[\lambda n]} s_k^\alpha - \sum_{k=n+1}^{[\lambda n]} s_n^\alpha \right)
= \frac{1}{[\lambda n] - n} \left( \frac{\alpha([\lambda n] + 1)}{\alpha + 1} \sigma_n(s_n^\alpha + 1) - \frac{\alpha(n - [\lambda n])}{\alpha + 1} \sigma_n(s_n^\alpha + 1) + \frac{[\lambda n] + 1}{\alpha + 1} s_n^\alpha + \frac{\alpha(n - [\lambda n])}{\alpha + 1} s_n^\alpha \right)
- \frac{1}{[\lambda n] - n} \left( \sum_{k=n+1}^{[\lambda n]} (s_k^\alpha - s_n^\alpha) \right)
= \frac{\alpha}{\alpha + 1} \left( \frac{\alpha([\lambda n] + 1)}{[\lambda n] - n} \sigma_n(s_n^\alpha + 1) - \sigma_n(s_n^\alpha + 1) \right)
+ \sigma_n(s_n^\alpha + 1) - s_n^\alpha + 1) \right) \right) + \frac{1}{[\lambda n] - n} \left( s_n^\alpha + 1) \right)
- \frac{1}{[\lambda n] - n} \left( \sum_{k=n+1}^{[\lambda n]} (s_k^\alpha - s_n^\alpha) \right).
\]

This completes the proof. \(\square\)

The proof for ii) is similar to that of i).
5. Proof of Theorem 7

By hypothesis, \( s_n^{\alpha+1} \to s(A) \). By Lemma 4, we have \( s_n^{\alpha+2} \to s(A) \), \( \ldots \), \( s_n^{\alpha+m} \to s(A) \) where \( m \) is any positive integer. Hence, by Lemma 2, we get

\[
\begin{align*}
(\alpha + 2)(s_n^{\alpha+1} - s_n^{\alpha+2}) &= \tau_n^{\alpha+2} \\
(\alpha + 3)(s_n^{\alpha+2} - s_n^{\alpha+3}) &= \tau_n^{\alpha+3} \\
&\vdots \\
(\alpha + m + 1)(s_n^{\alpha+m} - s_n^{\alpha+m+1}) &= \tau_n^{\alpha+m+1}.
\end{align*}
\]

Since \( s_n^{\alpha+k} \to s(A) \) for \( k = 1, 2, \ldots, m + 1 \), we have

\[
\begin{align*}
\tau_n^{\alpha+2} &\to 0 \quad (A) \\
\tau_n^{\alpha+3} &\to 0 \quad (A) \\
&\vdots \\
\tau_n^{\alpha+m+1} &\to 0 \quad (A)
\end{align*}
\]

and by Lemma 3,

\[
\begin{align*}
(\alpha + 3)(\tau_n^{\alpha+2} - \tau_n^{\alpha+3}) &= n \Delta \tau_n^{\alpha+3} \to 0 \quad (A) \\
(\alpha + 4)(\tau_n^{\alpha+3} - \tau_n^{\alpha+4}) &= n \Delta \tau_n^{\alpha+4} \to 0 \quad (A) \\
&\vdots \\
(\alpha + m + 1)(\tau_n^{\alpha+m} - \tau_n^{\alpha+m+1}) &= n \Delta \tau_n^{\alpha+m+1} \to 0 \quad (A).
\end{align*}
\]

Since

\[
(n \Delta)_{m-1} \tau_n^{\alpha+m} \geq -H
\]

then

\[
(n \Delta)_{m-1} \tau_n^{\alpha+m+j} \geq -H_1
\]

for \( j = 1, \ldots, m - 1 \), by Lemma 5, we have

\[
(n \Delta)_{m-1} \tau_n^{\alpha+m+1} = \sum_{j=1}^{m-1} (-1)^{j+1} A_{m-1}^j (\alpha + 2)n \Delta \tau_n^{\alpha+2+j}
\]

\[
= A_{m-1}^1 (\alpha + 2)n \Delta \tau_n^{\alpha+3} - A_{m-1}^2 (\alpha + 2)n \Delta \tau_n^{\alpha+4} + \ldots
\]

\[
+ (-1)^m A_{m-1}^{m-1} (\alpha + 2)n \Delta \tau_n^{\alpha+m+1}.
\]

For \( j = 1, \ldots, m - 1 \) we have \( n \Delta \tau_n^{\alpha+2+j} \to 0 \quad (A) \). Hence, we get

\[
(n \Delta)_{m-1} \tau_n^{\alpha+m+1} \to 0 \quad (A)
\]
It follows from (5.3) that
\[(n \Delta)_{m-1} r_n^{\alpha+m+1} = o(1)\] (5.5)
by Theorem 3. By Lemma 3, we obtain
\[(\alpha + m + 1)(n \Delta)_{m-1} r_n^{\alpha+m} - (n \Delta)_{m-1} r_n^{\alpha+m+1}) = (n \Delta)_{m-1} r_n^{\alpha+m+1}.\]
(5.6)
Substituting (5.3) and (5.5) into (5.6), we have
\[(n \Delta)_{m-1} r_n^{\alpha+m} \geq -H_2.\]
(5.7)
Since
\[(n \Delta)_{m-2} r_n^{\alpha+m} = \sum_{j=1}^{m-2} (-1)^{j+1} A_{m-2}^j (\alpha + 2)n \Delta t_n^{\alpha+2+j} + (-1)^{m-2} A_{m-2}^{m-2} (\alpha + 2)n \Delta t_n^{\alpha+m}
+ (-1)^{m-2} A_{m-2}^{m-2} (\alpha + 2)n \Delta t_n^{\alpha+m}
\]
by Lemma 5, we have
\[(n \Delta)_{m-2} r_n^{\alpha+m} \to 0 (A).\]
(5.8)
From (5.7) and (5.8), we obtain, by Theorem 3,
\[(n \Delta)_{m-2} r_n^{\alpha+m} = o(1).\]
(5.9)
By Lemma 3, we obtain
\[(\alpha + m)(n \Delta)_{m-2} r_n^{\alpha+m-1} - (n \Delta)_{m-2} r_n^{\alpha+m} = (n \Delta)_{m-1} r_n^{\alpha+m}.\]
(5.10)
Substituting (5.7) and (5.9) into (5.10), we have
\[(n \Delta)_{m-2} r_n^{\alpha+m-1} \geq -H_3.\]
(5.11)
Since
\[(n \Delta)_{m-3} r_n^{\alpha+m-1} = \sum_{j=1}^{m-3} (-1)^{j+1} A_{m-3}^j (\alpha + 2)n \Delta t_n^{\alpha+2+j} + (-1)^{m-3} A_{m-3}^{m-2} (\alpha + 2)n \Delta t_n^{\alpha+m-2}
\]
by Lemma 3, we have
\[(n \Delta)_{m-3} r_n^{\alpha+m-1} \to 0 (A).\]
(5.12)
From (5.11) and (5.12), we have, by Theorem 3,
\[(n \Delta)_{m-3} r_n^{\alpha+m-1} = o(1).\]
(5.13)
By Lemma 3, we obtain
\[(\alpha + m - 1)(n \Delta)_{m-3} r_n^{\alpha+m-2} - (n \Delta)_{m-3} r_n^{\alpha+m-1} = (n \Delta)_{m-2} r_n^{\alpha+m-1}.\]
(5.14)
Substituting (5.11) and (5.13) into (5.14), we have
\[(n \Delta)_{n-3} \tau_n^{a+m-2} \geq -H_4.\]  
(5.15)

Continuing in this way, we obtain
\[(n \Delta)_2 \tau_n^{a+4} \to 0 \quad (A).\]  
(5.16)

and
\[(n \Delta)_3 \tau_n^{a+4} \geq -H_5.\]  
(5.17)

From (5.16) and (5.17), we have, by Theorem 3,
\[(n \Delta)_2 \tau_n^{a+4} = o(1).\]  
(5.18)

By Lemma 3, we obtain
\[(\alpha + 4)((n \Delta)_2 \tau_n^{a+3} - (n \Delta)_2 \tau_n^{a+4}) = (n \Delta)_3 \tau_n^{a+4}.\]  
(5.19)

Substituting (5.17) and (5.18) into (5.19), we have
\[(n \Delta)_2 \tau_n^{a+3} \geq -H_6.\]  
(5.20)

From \(n \Delta \tau_n^{a+3} \to 0 \quad (A)\) and (5.20), we have, by Theorem 3,
\[n \Delta \tau_n^{a+3} = o(1).\]  
(5.21)

By Lemma 3, we obtain
\[(\alpha + 3)(n \Delta \tau_n^{a+2} - n \Delta \tau_n^{a+3}) = (n \Delta)_2 \tau_n^{a+3}.\]  
(5.22)

Substituting (5.20) and (5.21) into (5.22), we have
\[n \Delta \tau_n^{a+2} \geq -H_7.\]  
(5.23)

It follows from (5.1) and (5.23) by Theorem 1, we get
\[\tau_n^{a+2} = o(1).\]  
(5.24)

Substituting (5.23) and (5.24) into
\[(\alpha + 2)(\tau_n^{a+1} - \tau_n^{a+2}) = n \Delta \tau_n^{a+2}\]
we have
\[\tau_n^{a+1} \geq -H_8.\]

Since \(s_n^{a+1} \to s \quad (A)\) and
\[\tau_n^{a+1} = n \Delta s_n^{a+1} \geq -H_8,\]  
(5.25)

we obtain, by Theorem 3,
\[s_n^{a+1} \to s.\]  
(5.26)

Now we need to show that
\[\tau_n^a = n \Delta s_n^a \geq -C\]  
(5.27)

for some constant \(C\).
From (5.2), by Lemma 5, we have, for \( j = 1, \ldots, m - 1 \),

\[
(n \Delta)_{m-1}^{\alpha + m} = \sum_{j=1}^{m-1} (-1)^{j+1} A_{m-1}^j (\alpha + 1) n \Delta \tau_n^{\alpha + 1 + j} \\
= A_{m-1}^1 (\alpha + 1) n \Delta \tau_n^{\alpha + 2} - A_{m-1}^2 (\alpha + 1) n \Delta \tau_n^{\alpha + 3} + \ldots \\
+ (-1)^m A_{m-1}^{m-1} (\alpha + 1) n \Delta \tau_n^{\alpha + m}.
\]

We have \( n \Delta \tau_n^{\alpha + 1 + j} \to 0 (A) \) for \( j = 1, \ldots, m - 1 \). Hence, we get

\[
(n \Delta)_{m-1}^{\alpha + m} \to 0 (A). \tag{5.28}
\]

It follows from (5.2) that

\[
(n \Delta)_{m-1}^{\alpha + m} = o(1) \tag{5.29}
\]

by Theorem 3. By Lemma 3, we obtain

\[(\alpha + m)((n \Delta)_{m-1}^{\alpha + m-1} - (n \Delta)_{m-1}^{\alpha + m}) = (n \Delta)_{m-1}^{\alpha + m}. \tag{5.30}\]

Substituting (5.2) and (5.29) into (5.30), we have

\[
(n \Delta)_{m-1}^{\alpha + m-1} \geq -H_{10}. \tag{5.31}
\]

Since

\[
(n \Delta)_{m-2}^{\alpha + m-1} = \sum_{j=1}^{m-2} (-1)^{j+1} A_{m-2}^j (\alpha + 1) n \Delta \tau_n^{\alpha + 1 + j} \\
= A_{m-2}^1 (\alpha + 1) n \Delta \tau_n^{\alpha + 2} - A_{m-2}^2 (\alpha + 1) n \Delta \tau_n^{\alpha + 3} + \ldots \\
+ (-1)^{m-2} A_{m-2}^{m-2} (\alpha + 1) n \Delta \tau_n^{\alpha + m-1}
\]

by Lemma 5, we have

\[
(n \Delta)_{m-2}^{\alpha + m-1} \to 0 (A). \tag{5.32}
\]

From (5.31) and (5.32), we obtain, by Theorem 3,

\[
(n \Delta)_{m-2}^{\alpha + m-1} = o(1). \tag{5.33}
\]

By Lemma 3, we obtain

\[(\alpha + m - 1)((n \Delta)_{m-2}^{\alpha + m-2} - (n \Delta)_{m-2}^{\alpha + m-1}) = (n \Delta)_{m-1}^{\alpha + m-1}. \tag{5.34}\]

Substituting (5.31) and (5.33) into (5.34), we have

\[
(n \Delta)_{m-2}^{\alpha + m-2} \geq -H_{11}. \tag{5.35}
\]

Since

\[
(n \Delta)_{m-3}^{\alpha + m-2} = \sum_{j=1}^{m-3} (-1)^{j+1} A_{m-3}^j (\alpha + 1) n \Delta \tau_n^{\alpha + 1 + j} \\
= A_{m-3}^1 (\alpha + 1) n \Delta \tau_n^{\alpha + 2} - A_{m-3}^2 (\alpha + 1) n \Delta \tau_n^{\alpha + 3} + \ldots
\]
\[ + (-1)^{m-2} A_{m-3}^{m-3} (\alpha + 1) n \Delta r^{\alpha + m-3}_n \]

by Lemma 3, we have
\[ (n\Delta)_{m-3} r^{\alpha + m-2}_n \rightarrow 0 \text{ (A).} \quad (5.36) \]

From (5.35) and (5.36), we have, by Theorem 3,
\[ (n\Delta)_{m-3} r^{\alpha + m-2}_n = o(1). \quad (5.37) \]

By Lemma 3, we obtain
\[ (\alpha + m - 2)((n\Delta)_{m-3} r^{\alpha + m-3}_n - (n\Delta)_{m-3} r^{\alpha + m-2}_n) = (n\Delta)_{m-2} r^{\alpha + m-2}_n. \quad (5.38) \]

Substituting (5.35) and (5.37) into (5.38), we have
\[ (n\Delta)_{m-3} r^{\alpha + m-3}_n \geq -H_{12}. \quad (5.39) \]

Continuing in this way, we obtain
\[ (n\Delta)_{2} r^{\alpha + 3}_n \rightarrow 0 \text{ (A).} \quad (5.40) \]

and
\[ (n\Delta)_{3} r^{\alpha + 3}_n \geq -H_{13}. \quad (5.41) \]

From (5.40) and (5.41), we have, by Theorem 3,
\[ (n\Delta)_{2} r^{\alpha + 3}_n = o(1). \quad (5.42) \]

By Lemma 3, we obtain
\[ (\alpha + 3)((n\Delta)_{2} r^{\alpha + 2}_n - (n\Delta)_{2} r^{\alpha + 3}_n) = (n\Delta)_{3} r^{\alpha + 3}_n. \quad (5.43) \]

Substituting (5.41) and (5.42) into (5.43), we have
\[ (n\Delta)_{2} r^{\alpha + 2}_n \geq -H_{14}. \quad (5.44) \]

From \( n\Delta r^{\alpha + 2}_n \rightarrow 0 \text{ (A) and (5.44), we have, by Theorem 3,} \]
\[ n\Delta r^{\alpha + 2}_n = o(1). \quad (5.45) \]

By Lemma 3, we obtain
\[ (\alpha + 2)((n\Delta)_{2} r^{\alpha + 1}_n - n\Delta r^{\alpha + 2}_n) = (n\Delta)_{2} r^{\alpha + 2}_n. \quad (5.46) \]

Substituting (5.44) and (5.45) into (5.46), we have
\[ n\Delta r^{\alpha + 1}_n \geq -H_{15}. \quad (5.47) \]

Substituting (5.47), (5.25) and (5.27) into
\[ (\alpha + 1)(r^{\alpha}_n - r^{\alpha + 1}_n) = n\Delta r^{\alpha + 1}_n \]

we have
\[ r^{\alpha}_n = n\Delta s^{\alpha}_n \geq -H_{16}. \quad (5.48) \]

By Lemma 8 i) and (5.48), we have
\[ s^{\alpha}_n - s^{\alpha + 1}_n = \frac{\alpha}{\alpha + 1} \left[ \frac{[\lambda n] + 1}{[\lambda n] - n} (\sigma_{[\lambda n]}(s^{\alpha + 1}_n) - \sigma_n(s^{\alpha + 1}_n)) + (\sigma_n(s^{\alpha + 1}_n) - s^{\alpha + 1}_n) \right] \]
By Lemma 8 ii) and (5.48), we have

\[
\frac{1}{\alpha} \left[ \frac{\lambda n}{\alpha} + 1 \right] \left( s_{\lambda n}^{\alpha} - s_n^{\alpha+1} \right) - \frac{1}{\alpha} \left[ \frac{\lambda n}{\alpha} - n \right] \sum_{k=1}^{n} \sum_{j=1}^{k} \Delta s_{j}^\alpha
\]

\[
\leq \frac{\alpha}{\alpha + 1} \left[ \frac{\lambda n}{\alpha} + 1 \right] \left( \sigma_{\lambda n}(s_{\lambda n}^{\alpha} - s_n^{\alpha+1}) + \sigma_n(s_{\lambda n}^{\alpha+1} - s_n^{\alpha+1}) \right) + \frac{1}{\alpha} \left[ \frac{\lambda n}{\alpha} - n \right] \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{H}{j}
\]

Taking the lim sup of both sides, we get

\[
\limsup_{n \to \infty} (s_n^{\alpha} - s_n^{\alpha+1}) \leq \limsup_{n \to \infty} \left\{ \frac{\alpha}{\alpha + 1} \left[ \frac{\lambda n}{\alpha} + 1 \right] \left( \sigma_{\lambda n}(s_{\lambda n}^{\alpha} - s_n^{\alpha+1}) + \sigma_n(s_{\lambda n}^{\alpha+1} - s_n^{\alpha+1}) \right) + \frac{1}{\alpha} \left[ \frac{\lambda n}{\alpha} - n \right] \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{H}{j} \right\} + H \log \frac{\lambda n}{n}
\]

where \( H > 0 \). Since

\[
\frac{\lambda n + 1}{\lambda n - n} \leq \frac{2\lambda}{\lambda - 1}
\]

for \( \lambda > 1 \) and sufficiently large \( n \) and \( s_{\lambda n}^{\alpha+1} \rightarrow s \), we have

\[
\lim_{\lambda \to 1^+} \limsup_{n \to \infty} (s_n^{\alpha} - s_n^{\alpha+1}) \leq 0
\]

By Lemma 8 ii) and (5.48), we have

\[
s_n^{\alpha} - s_n^{\alpha+1} = \frac{\alpha}{\alpha + 1} \left[ \frac{\lambda n}{\alpha} + 1 \right] \left( \sigma_n(s_{\lambda n}^{\alpha} - s_n^{\alpha+1}) + \sigma_{\lambda n}(s_{\lambda n}^{\alpha} - s_n^{\alpha+1}) \right) + \frac{1}{\alpha} \left[ \frac{\lambda n}{\alpha} - n \right] \sum_{k=1}^{n} \sum_{j=1}^{k} \Delta s_{j}^\alpha
\]

\[
\geq \frac{\alpha}{\alpha + 1} \left[ \frac{\lambda n}{\alpha} + 1 \right] \left( \sigma_n(s_{\lambda n}^{\alpha} - s_n^{\alpha+1}) + \sigma_{\lambda n}(s_{\lambda n}^{\alpha} - s_n^{\alpha+1}) \right) + \frac{1}{\alpha} \left[ \frac{\lambda n}{\alpha} - n \right] \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{H}{j}
\]

\[
\geq \frac{\alpha}{\alpha + 1} \left[ \frac{\lambda n}{\alpha} + 1 \right] \left( \sigma_n(s_{\lambda n}^{\alpha} - s_n^{\alpha+1}) + \sigma_{\lambda n}(s_{\lambda n}^{\alpha} - s_n^{\alpha+1}) \right)
\]
Taking the liminf of both sides, we get
\[
\liminf_{n \to \infty} \left( s_n^\alpha - s_n^{\alpha+1} \right) \geq \liminf_{n \to \infty} \left\{ \frac{\alpha}{\alpha + 1} \left[ \frac{[\lambda n]}{n} + \frac{1}{n - [\lambda n]} \left( \sigma_n(s_n^{\alpha+1}) - \sigma_{[\lambda n]}(s_n^{\alpha+1}) \right) \right] + \frac{1}{\alpha n - [\lambda n]} \left( s_n^{\alpha+1} - s_n^\alpha \right) \right\} + H \log \lambda
\]
where \( H > 0 \). Since
\[
\frac{[\lambda n]}{n} \leq \frac{2\lambda}{1-\lambda}
\]
for \( 0 < \lambda < 1 \) and sufficiently large \( n \) and \( s_n^{\alpha+1} \to s \), we have
\[
\lim_{\lambda \to 1^+} \liminf_{n \to \infty} (s_n^\alpha - s_n^{\alpha+1}) \geq 0.
\]
Combining (5.50) and (5.52) provides
\[
\lim_{\lambda \to 1^+} s_n^\alpha = \lim_{\lambda \to 1^+} s_n^{\alpha+1}.
\]
This completes the proof.

We like to note that we used \( H \) to denote a constant, possibly different at each occurrence above.

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