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ON THE GENERALIZED k–FIBONACCI NUMBERS

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Abstract. In this paper, new types of k–Fibonacci numbers are described, with respect to the definition of the distance between numbers using a recurrence relation. However, these sequences differ not only by the value of the natural number k but also according to the value of a new parameter r which is used in defining this distance. Furthermore, various properties of these new numbers are discussed.

In the second part of this paper, we apply the binomial transform to these generalized k–Fibonacci sequences.

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1. Introduction

Classical Fibonacci numbers have been generalized in different ways [8–11]. One such generalization which has recently increased interest among researchers mathematical terms pertains to the k–Fibonacci numbers [4, 5].

k–Fibonacci numbers are defined by the recurrence relation $F_{k,n+1} = k\,F_{k,n} + F_{k,n-1}$ for $n \geq 1$ with the initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$. If $F_k = \{F_{k,n}\}$, then F_1 is the classical Fibonacci sequence while F_2 is the Pell sequence.

2. Generalized k–Fibonacci numbers

In this section we apply the definition of r–distance to k–Fibonacci numbers in a generalized approach to previous results [2, 12]. The formulas used to calculate the general term of the sequences generated by the above definition are very interesting and they allow us to find the sum of n first terms.

Definition 1. With respect to natural numbers $k \geq 1,\ n \geq 0,\ r \geq 1$, we define the generalized (k,r)–Fibonacci numbers $F_{k,n}(r)$ by the recurrence relation

$$F_{k,n}(r) = k\,F_{k,n-r}(r) + F_{k,n-2}(r) \quad \text{for} \quad n \geq r, \tag{2.1}$$

with initial conditions $F_{k,n}(r) = 1,\ n = 0, 1, 2, \ldots r-1$, except $F_{k,1}(1) = k$.

The following proposition shows the formulae used to calculate the general term of the sequence $F_k(r)$, where $r \geq 2$ is odd or even (for $r = 1$ see [4,5]).

Theorem 1 (Main formula)**.** (1) *If r is even, $r = 2p$:*

$$F_{k,2n}(2p) = F_{k,2n+1}(2p) = \sum_{j=0}^{n/p} \binom{n-(p-1)j}{j} k^j. \tag{2.2}$$

(2) *If r is odd, $r = 2p+1 \geq 3$:*

$$F_{k,2n}(2p+1) = \sum_{j=0} \left[\binom{n-(2p-1)j}{2j} k^{2j} + \binom{n-p-(2p-1)j}{2j+1} k^{2j+1} \right], \tag{2.3}$$

$$F_{k,2n+1}(2p+1) = \sum_{j=0} \left[\binom{n-(2p-1)j}{2j} k^{2j} + \binom{n-(p-1)-(2p-1)j}{2j+1} k^{2j+1} \right]. \tag{2.4}$$

Proof. By induction.

Formula (2.2). Let $r = 2p$.
For $n = 0$, by definition, let $F_{k,0}(2p) = 1$, and the right hand side (RHS) of (2.2) is

$$F_{k,0}(2p) = \sum_{0}^{0} \binom{(1-p)j}{j} k^j = 1.$$

For $n = 1$, by definition, let $F_{k,2}(2) = 1+k$, and $F_{k,2}(r) = 1$ for $r > 2$. In formula (2.2) we have

$$F_{k,2}(2p) = \sum_{0}^{2} \binom{1-(p-1)j}{j} k^j = 1 + \binom{1-(p-1)j}{1} k.$$

Then, $F_{k,2}(2) = 1+k$ and $F_{k,2}(2p) = 1$ for $2p = 4,6,8,\ldots$
Suppose this formula holds for n. Then

$$\begin{aligned}
F_{k,2n+2}(2p) &= k F_{k,2n+2-2p}(2p) + F_{k,2n}(2p) \\
&= \sum_{j=0} \binom{n-(p-1)j}{j} k^j + k F_{k,2(n+1-p)}(2p) \\
&= \sum_{j=0} \binom{n-(p-1)j}{j} k^j + k \sum_{j=0} \binom{n+1-p-(p-1)j}{j} k^j \\
&= 1 + \sum_{j=1} \binom{n-(p-1)j}{j} k^j + \sum_{j=0} \binom{n-(p-1)(j+1)}{j} k^{j+1} \\
&= 1 + \sum_{j=0} \left[\binom{n-(p-1)(j+1)}{j+1} k^{j+1} + \binom{n-(p-1)(j+1)}{j} k^{j+1} \right] \\
&= 1 + \sum_{j=0} \binom{n-(p-1)(j+1)+1}{j+1} k^{j+1}
\end{aligned}$$

$$= \sum_{j=0} \binom{n+1-(p-1)j}{j} k^j = F_{k,2n+2}(2p)$$

as we proposed to prove.

We will prove formulae (2.3) and (2.4) together.

For $n=0$ we have $F_{k,0}(2p+1)=1$ and the RHS of (2.3) is

$$\sum_0^0 \left[\binom{-(2p-1)j}{2j} k^{2j} + \binom{-p-(2p-1)j}{2j+1} k^{2j+1} \right] = 1.$$

In the same manner, we have $F_{k,1}(2p+1)=1$ and the RHS of (2.4) is

$$\sum_0^0 \left[\binom{-(2p-1)j}{2j} k^{2j} + \binom{1-p-(2p-1)j}{2j+1} k^{2j+1} \right] = 1 \text{ because } 1-p<0.$$

For $n=1$ we have $F_{k,2}(2p+1)=1$ and the RHS of (2.3) is

$$\sum_0^0 \left[\binom{1-(2p-1)j}{2j} k^{2j} + \binom{1-p-(2p-1)j}{2j+1} k^{2j+1} \right] = 1,$$

because the condition $p \geq 1$ involves $1-p-(2p-1)j<0$.

For formula (2.4), if $p=1$, the left hand side (LHS) is $F_{k,3}(2p+1)=1+k$, while the RHS is

$$\sum_0^0 \left[\binom{1-j}{2j} k^{2j} + \binom{1-j}{2j+1} k^{2j+1} \right] = 1+k.$$

If $p>1$, the LHS and the RHS of (2.4) are equal to 1.

Suppose the formula holds for $2n$ and $2n+1$. Then

$$\begin{aligned}
F_{k,2n+2}(2p+1) &= kF_{k,2n+1-2p}(2p+1) + F_{k,2n}(2p+1) \\
&= kF_{k,2(n-p)+1}(2p+1) + F_{k,2n}(2p+1) \\
&= k\sum_{j=0} \left[\binom{n-p-(2p-1)j}{2j} k^{2j} + \binom{n-(2p-1)-(2p-1)j}{2j+1} k^{2j+1} \right] \\
&\quad + \sum_{j=0} \left[\binom{n-(2p-1)j}{2j} k^{2j} + \binom{n-p-(2p-1)j}{2j+1} k^{2j+1} \right] \\
&= \sum_{j=0} \left[\binom{n-p-(2p-1)j}{2j} + \binom{n-p-(2p-1)j}{2j+1} \right] k^{2j+1} \\
&\quad + \sum_{j=0} \binom{n-(2p-1)(j+1)}{2j+1} k^{2j+2} + 1 + \sum_{j=1} \binom{n-(2p-1)j}{2j} k^{2j} \\
&= \sum_{j=0} \binom{n+1-p-(2p-1)j}{2j+1} k^{2j+1}
\end{aligned}$$

$$+\sum_{j=1}\binom{n-(2p-1)j}{2j-1}k^{2j}+1+\sum_{j=1}\binom{n-(2p-1)j}{2j}k^{2j}$$
$$=\sum_{j=0}\binom{n+1-p-(2p-1)j}{2j+1}k^{2j+1}+\sum_{j=1}\binom{n+1-(2p-1)j}{2j}k^{2j}+1$$
$$=\sum_{j=0}\binom{n+1-p-(2p-1)j}{2j+1}k^{2j+1}+\sum_{j=0}\binom{n+1-(2p-1)j}{2j}k^{2j}$$
$$=F_{k,2n+2}(2p+1).$$

A similar development shows the formula for $F_{k,2n+3}(2p+1)$. □

In [3] the following formulas are proven:

(1) Sum: $S_{k,n}(r)=\frac{1}{k}\Big(F_{k,n+r-1}(r)+F_{k,n+r}(r)-2\Big)$.

(2) Generating function: $f_k(r,x)=\dfrac{1+x}{1-x^2-k\,x^r}$.

3. Binomial Transform of the Generalized k–Fibonacci Numbers

In this section we will apply the binomial transform to the preceding sequences and will obtain new integer sequences [7].

Definition 2. Binomial transform of the generalized k–Fibonacci sequence is defined in the classical form as

$$BF_{k,n}(r)=\sum_{j=0}^{n}\binom{n}{j}F_{k,j}(r). \tag{3.1}$$

For $r=1$, see [6].

So, for $r=2,3,\ldots$ the sequences obtained by applying this transformation are:

$$BF_k(2)=\{1,2,4+k,8+4k,16+12k+k^2,32+32k+6k^2,$$
$$64+80k+24k^2+k^3,128+192k+80k^2+8k^3,\ldots\}$$
$$BF_k(3)=\{1,2,4,8+k,16+5k,32+17k,64+49k+k^2,128+129k+8k^2,256+321k$$
$$+39k^2,512+769k+150k^2+k^3,1024+1793k+501k^2+11k^3,\ldots\}$$
$$BF_k(4)=\{1,2,4,8,16+k,32+6k,64+23k,128+72k,256+201k+k^2,$$
$$512+522k+10k^2,1024+1291k+58k^2,2048+3084k+256k^2,\ldots\}$$

Curiously, $BF_1(2)$ is the classical Pell sequence.

3.1. *Generating Function of the Sequences $BF_k(r)$*

[1] shows that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then $S(x)=\dfrac{1}{1-x}A\left(\dfrac{x}{1-x}\right)$ is the generating function of the sequence $\{b_n\}$ with $b_n=\sum_j\binom{n}{j}a_n$.

So, we can deduce the generating function of the $F_k(r)$ sequence, for $r = 2,3,\dots$ is $g_r(x) = \frac{1}{1-x} f_r\left(\frac{x}{1-x}\right)$, that is:

$$g_r(x) = \frac{(1-x)^{r-2}}{(1-2x)(1-x)^{r-2} - kx^r}. \tag{3.2}$$

3.2. *Recurrence Relation of the Sequences $BF_k(r)$*

Taking into account $g_r(x)$ is the generating function of the sequences $BF_k(r)$, the coefficients of the denominator of this function shows the recurrence relation of the sequences $BF_k(r)$.

For clarity, we indicate as $b_n(r)$ the elements of the sequence $BF_k(r)$. The coefficients of the polynomial $D(x) = (1-2x)(1-x)^{r-2} - kx^r$, for $r = 2,3\dots$, show the recurrence relation of the sequences $\{b_n(r)\}$, and consequently:

$$r = 2 \to \{1,-2,-k\} \to b_n(2) = 2b_{n-1}(2) + k\,b_{n-2}(2) \tag{3.3}$$

$$r = 3 \to \{1,-3,2,-k\} \to b_n(3) = 3b_{n-1}(3) - 2b_{n-2}(3) + k\,b_{n-3}(3) \tag{3.4}$$

$$\begin{aligned} r = 4 \to \{1,-4,5,-2,-k\} &\to b_n(4) \\ &= 4b_{n-1}(4) - 5b_{n-2}(4) + 2b_{n-3}(4) + k\,b_{n-4}(4) \end{aligned} \tag{3.5}$$

$$\begin{aligned} r = 5 \to \{1,-5,9,-7,2,-k\} &\to b_n(5) \\ &= 5b_{n-1}(5) - 9b_{n-2}(5) + 7b_{n-3}(5) - 2b_{n-4}(5) + k\,b_{n-5}(5) \end{aligned} \tag{3.6}$$

with the initial conditions $b_n(r) = 2^n$ for $n = 0,1,2\dots r-1$.

3.3. *Sums of the Sequences $BF_k(r)$*

In the sequel we will prove the formulas for the sums of the sequences $BF_k(2)$ and $BF_k(3)$ and show for $BF_k(4)$.
Let $b_n(2) = BF_{k,n}(2)$ be and we will indicate as b_n.
From (3.3) it is $b_n = 2b_{n-1} + k\,b_{n-2}$. Then

$$\begin{aligned} S_n(2) &= \sum_{j=0}^{n} b_j = b_0 + b_1 + \sum_{2}^{n} b_j = 1 + 2 + 2\sum_{2}^{n} b_{j-1} + k\sum_{2}^{n} b_{j-2} \\ &= 3 + 2(S_n - b_0 - b_n) + k(S_n - b_{n-1} - b_n) \\ &= 1 + 2S_n + kS_n - (2+k)b_n - k\,b_{n-1} \to \\ S_n(2) &= \frac{1}{1+k}\Big((2+k)b_n(2) + k\,b_{n-1}(2) - 1\Big). \end{aligned}$$

Let $b_n(3) = BF_{k,n}(3)$ be and we will idicate as b_n.
From (3.4) it is $b_n = 3b_{n-1} - 2b_{n-2} + k\,b_{n-3}$. Then

$$S_n(3) = \sum_{j=0}^{n} b_j = b_0 + b_1 + b_2 + \sum_{3}^{n} b_j$$

$$
\begin{aligned}
&= 1+2+4+3\sum_{3}^{n} b_{j-1} - 2\sum_{3}^{n} b_{j-2} + k\sum_{3}^{n} b_{j-3} \\
&= 7+3(S_n - b_0 - b_1 - b_n) - 2(S_n - b_0 - b_{n-1} - b_n) + k(S_n - b_{n-2} - b_{n-1} - b_n) \\
&= S_n + kS_n - (1+k)S_n - (k-2)b_{n-1} - kb_{n-2} \to
\end{aligned}
$$

$$
\begin{aligned}
S_n(3) &= \frac{1}{k}\left((1+k)b_n + (k-2)b_{n-1} + kb_{n-2}\right) \\
&= \frac{1}{k}(b_n - 2b_{n-1} + kb_{n-2} + (b_n + b_{n-1} + b_{n-2}) \\
&= \frac{1}{k}(b_{n+1} - 2b_n) + S_n - S_{n-3} \to S_{n-3}(3) = \frac{1}{k}(b_{n+1} - 2b_n) \to
\end{aligned}
$$

$$
S_n(3) = \frac{1}{k}\Big(b_{n+4}(3) - 2b_{n+3}(3)\Big).
$$

And so,

$$
\begin{aligned}
S_n(4) &= \frac{1}{k}\Big(b_{n+4}(4) - 3b_{n+3}(4) + 2b_{n+2}(4)\Big), \\
S_n(5) &= \frac{1}{k}\Big(b_{n+4}(5) - 4b_{n+3}(5) + 5b_{n+2}(5) - 2b_{n+1}(5)\Big).
\end{aligned}
$$

And we see that the coefficients of the sum $S_n(r)$ are the constant coefficients of the polynomial $D(x)$ for $r = n-1$. So, the following sum must be

$$
S_n(6) = \frac{1}{k}\Big(b_{n+4}(6) - 5B_{n+3}(6) + 9b_{n+2}(6) - 7b_{n+1}(6) + 2b_n(6)\Big).
$$

And rightly so!

Conclusions

We have generalized the r–distance Fibonacci numbers to the case of k–Fibonacci numbers, getting more general formulas that previously found.

These formulas include which allows to find the general term of a sequence of this type according to r is even or odd.

This paper also shows the formula to find the sum of the terms of the generalized (k,r)–Fibonacci sequences.

We apply the binomial transform to these sequences and find its generating function. Later, we found both the recurrence relation for the sequences of the binomial transforms of this new type of k–Fibonacci numbers and the formula for the sum of the terms of these sequences.

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