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COMPLETENESS OF ONE TWO-INTERVAL BOUNDARY VALUE PROBLEM WITH TRANSMISSION CONDITIONS

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Abstract. This paper presents a new approach to the two-interval Sturm-Liouville eigenfunction expansions, based essentially on the method of integral equations. We consider the Sturm-Liouville problem together with two supplementary transmission conditions at one interior point. Further we develop Green's function method for spectral analysis of the considered problem in modified Hilbert space.

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1. INTRODUCTION

The development of classical, rather than the operatoric, Sturm-Liouville theory in the years after 1950 can be found in various sources; in particular in the texts of Atkinson [2], Coddington and Levinson [5], Levitan and Sargsjan [9]. The operator theoretic development is given in the texts by Naimark [12] and Akhiezer and Glazman [1]. The subject of eigenfunction expansions is as old as operator theory. The Sturm-Liouville problem is also important because the solutions to a homogeneous boundary value problem (BVP) with homogeneous boundary conditions(BCs) produce a set of orthogonal functions. Such functions can be used to represent functions in Fourier series expansions. The completeness of classical systems of eigenfunction expansions was originally related to mechanical problems and boundary value problems for differential operators. Later the study of eigenfunctions expansions has gained an independent and abstract status. The expansion has an integral operator form whose kernel is a spectral function, the representation of which is the Green function of the operator. For the method of treating such problems see [8]. The method of Sturm expansions is widely used in calculations of the spectroscopic characteristics of atoms and molecules [13, 14]. The advantages of the method are both the possibility of adequate choice of the nature of the spectrum of an unperturbed problem in the calculation of quantum mechanical systems using the perturbation theory and the possibility of providing best convergence of corresponding expansions in basis functions. In this study we shall investigate one two-interval boundary value problem ,namely, the Sturm-Liouville equation

$$-p(x)y''(x)+q(x)y(x)=\lambda y(x) \tag{1.1}$$

to hold in disjoint intervals $(-\pi,0)$ and $(0,\pi)$ under boundary conditions

$$\cos\alpha y(-\pi)+\sin\alpha y'(-\pi)=0 \tag{1.2}$$

$$\cos\beta y(\pi)+\sin\beta y'(\pi)=0. \tag{1.3}$$

where singularity of the solution $y=y(x,\lambda)$ prescribed by transmission conditions

$$\beta_{11}^{-}y'(0-)+\beta_{10}^{-}y(0-)+\beta_{11}^{+}y'(0+)+\beta_{10}^{+}y(0+)=0 \tag{1.4}$$

$$\beta_{21}^{-}y'(0-)+\beta_{20}^{-}y(0-)+\beta_{21}^{+}y'(0+)+\beta_{20}^{+}y(0+)=0 \tag{1.5}$$

where $p(x)=p_1>0$ for $x\in[-\pi,0)$, $p(x)=p_2>0$ for $x\in(0,\pi]$, the potential $q(x)$ is given real-valued function which continuous in each of the intervals $[-\pi,0)$ and $(0,\pi]$, and has a finite limits $q(\mp 0)$, λ is a complex eigenparameter, $\beta_{ij}^{\pm}$, $(i=1,2$ and $j=0,1)$ are real numbers. In this study we introduce a new approach to the two-interval Sturm-Liouville eigenfunction expansions, based essentially on the method of integral equations. Note that in physics many problems arise in the form of boundary value problems involving second order ordinary differential equations. This derivation based on that in [7], however, is more thorough than that in most elementary physics texts; while most parameters such as density and other thermal properties are treated as constant in such treatments, the following allows fundamental properties of the bar to vary as a function of the bar's length, which will lead to a Sturm-Liouville problem of a more general nature. Detailed studies on spectral problems for ordinary differential operators depending on the parameter and/or with transmission conditions can be found in various publications, see e.g. [3,4,10,11,15], where further references and links to applications can be found.

2. Construction of the resolvent by means of Green's function

Let $u=\phi_1(x,\lambda)$ be solution of the equation (1.1) on left interval $[-\pi,0)$ satisfying the conditions

$$u(-\pi)=\sin\alpha,\ \ u'(-\pi)=-\cos\alpha$$

and $u=\phi_2(x,\lambda)$ be solution of same equation on right interval $(0,\pi]$ satisfying the conditions

$$u(0+,\lambda)=\frac{1}{\Delta_{12}}(\Delta_{23}\phi_1(0,\lambda)+\Delta_{24}\frac{\partial\phi_1(0,\lambda)}{\partial x})$$

$$u'(0+,\lambda)=\frac{-1}{\Delta_{12}}(\Delta_{13}\phi_1(0,\lambda)+\Delta_{14}\frac{\partial\phi_1(0,\lambda)}{\partial x})$$

Similarly we define by $u = \chi_2(x,\lambda)$ the solution of the equation (1.1) on the right interval $(0,\pi]$ by initial conditions

$$u(\pi) = -\sin\beta, \quad u'(\pi) = \cos\beta$$

and by $u = \chi_1(x,\lambda)$ the solution of the equation (1.1) on the left interval $[-\pi,0)$ by initial conditions

$$u(0-,\lambda) = \frac{-1}{\Delta_{34}}(\Delta_{14}\chi_2(0,\lambda) + \Delta_{24}\frac{\partial\chi_2(0,\lambda)}{\partial x})$$

$$u'(0-,\lambda) = \frac{1}{\Delta_{34}}(\Delta_{13}\chi_2(0,\lambda) + \Delta_{23}\frac{\partial\chi_2(0,\lambda)}{\partial x})$$

respectively, where Δ_{ij} $(1 \le i < j \le 4)$ denotes the determinant of the i-th and j-th columns of the matrix

$$T = \begin{bmatrix} \beta_{10}^{+} & \beta_{11}^{+} & \beta_{10}^{-} & \beta_{11}^{-} \\ \beta_{20}^{+} & \beta_{21}^{+} & \beta_{20}^{-} & \beta_{21}^{-} \end{bmatrix}.$$

By applying the method of [10] we can show that each of these solutions are entire functions of complex parameter λ. It is clear that each of the Wronskians

$$\omega_i(\lambda) = W(\phi_i(x,\lambda), \chi_i(x,\lambda)), \ \ i = 1,2$$

are independent of variable x. By using the definitions of solutions $\phi_i(x,\lambda)$ and $\chi_i(x,\lambda)$ we have

$$\begin{aligned} w_2(\lambda) &= \phi_2(0+,\lambda)\frac{\partial\chi_2(0+,\lambda)}{\partial x} - \frac{\partial\phi_2(0+,\lambda)}{\partial x}\chi_2(0+,\lambda) \\ &= \frac{\Delta_{34}}{\Delta_{12}}(\phi_1(0-,\lambda)\frac{\partial\chi_1(0-,\lambda)}{\partial x} - \frac{\partial\phi_1(0-,\lambda)}{\partial x}\chi_1(0-,\lambda)) \\ &= \frac{\Delta_{34}}{\Delta_{12}}w_1(\lambda). \end{aligned}$$

Introduce to the consideration the Green's function (see, [3]) $G(x,\xi,\lambda)$ of the problem (1.1) − (1.5) as

$$G(x,\xi;\lambda)=\begin{cases}\frac{\phi_1(x,\lambda)\chi_1(\xi,\lambda)}{\Delta_{34}p_1\omega_1(\lambda)}, & if\ x\in[-\pi,0),\ \xi\in[-\pi,x)\\ \frac{\chi_1(x,\lambda)\phi_1(\xi,\lambda)}{\Delta_{34}p_1\omega_1(\lambda)}, & if\ x\in[-\pi,0),\ \xi\in[x,0)\\ \frac{\chi_1(x,\lambda)\phi_2(\xi,\lambda)}{\Delta_{34}p_1\omega_1(\lambda)}, & if\ x\in[-\pi,0),\ \xi\in(0,\pi]\\ \frac{\phi_2(x,\lambda)\chi_2(\xi,\lambda)}{\Delta_{12}p_2\omega_2(\lambda)}, & if\ x\in(0,\pi],\ \xi\in(0,x]\\ \frac{\chi_2(x,\lambda)\phi_2(\xi,\lambda)}{\Delta_{12}p_2\omega_2(\lambda)}, & if\ x\in(0,\pi],\ \xi\in[x,\pi]\\ \frac{\phi_2(x,\lambda)\chi_1(\xi,\lambda)}{\Delta_{12}p_2\omega_2(\lambda)}, & if\ x\in(0,\pi],\ \xi\in[-\pi,0)\end{cases}\tag{2.1}$$

Let $f(x)$ be any function continuous in $[-\pi,0)$ and $(0,\pi]$, which has finite limits $f(\mp 0)$. Show that the function

$$u(x,\lambda)=\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G(x,\xi;\lambda)f(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G(x,\xi;\lambda)f(\xi)d\xi\tag{2.2}$$

satisfies the nonhomogeneous equation

$$-p(x)u''+\{q(x)-\lambda\}u=f(x),\ x\in[-\pi,0)\cup(0,\pi]\tag{2.3}$$

and boundary-transmission conditions(BTCs) (1.2)-(1.5). Indeed, putting (2.1) in (2.2) and then differentiating twice we have

$$u''(x,\lambda)=\begin{cases}\frac{\chi_1''(x,\lambda)}{p_1\omega_1(\lambda)}\int_{-\pi}^{x}\phi_1(\xi,\lambda)f(\xi)d\xi+\frac{\phi_1''(x,\lambda)}{p_1\omega_1(\lambda)}\int_{x}^{0}\chi_1(\xi,\lambda)f(\xi)d\xi\\ +\frac{\phi_1''(x,\lambda)}{p_2\omega_2(\lambda)}\int_{0}^{\pi}\chi_2(\xi,\lambda)f(\xi)d\xi-\frac{f(x)p_1}{p_1\omega_1(\lambda)}W(\phi_1(x,\lambda),\chi_1(x,\lambda))\\ \qquad for\ x\in[-\pi,0)\\ \\ \frac{\chi_2''(x,\lambda)}{p_1\omega_1(\lambda)}\int_{-\pi}^{0}\phi_1(\xi,\lambda)f(\xi)d\xi+\frac{\chi_2''(x,\lambda)}{\omega_2(\lambda)}\int_{0}^{x}\phi_2(\xi,\lambda)f(\xi)d\xi\\ +\frac{\phi_2''(x,\lambda)}{p_2\omega_2(\lambda)}\int_{x}^{\pi}\chi_2(\xi,\lambda)f(\xi)d\xi-\frac{f(x)p_2}{p_2\omega_2(\lambda)}W(\phi_2(x,\lambda),\chi_2(x,\lambda))\\ \qquad for\ x\in(0,\pi]\end{cases}$$

$$=(q(x)-\lambda)\begin{cases}\frac{\chi_1(x,\lambda)}{p_1\omega_1(\lambda)}\int_{-\pi}^{x}\phi_1(\xi,\lambda)f(\xi)d\xi+\frac{\phi_1(x,\lambda)}{p_1\omega_1(\lambda)}\int_{x}^{0}\chi_1(\xi,\lambda)f(\xi)d\xi\\ +\frac{\phi_1(x,\lambda)}{p_2\omega_2(\lambda)}\int_{0}^{\pi}\chi_2(\xi,\lambda)f(\xi)d\xi-f(x)\qquad for\ x\in[-\pi,0)\\ \\ \frac{\chi_2(x,\lambda)}{p_1\omega_1(\lambda)}\int_{-\pi}^{0}\phi_1(\xi,\lambda)f(\xi)d\xi+\frac{\phi_2(x,\lambda)}{p_2\omega_2(\lambda)}\int_{x}^{\pi}\chi_2(\xi,\lambda)f(\xi)d\xi\\ +\frac{\chi_2(x,\lambda)}{p_2\omega_2(\lambda)}\int_{0}^{x}\phi_2(\xi,\lambda)f(\xi)d\xi-f(x)\qquad for\ x\in(0,\pi]\end{cases}$$

$$= (q(x) - \lambda)u(x,\lambda) - f(x)$$

i.e. the function $u(x,\lambda)$ satisfies the nonhomogeneous equation (2.3). The function $u(x,\lambda)$ also satisfies the transmission conditions (1.4)-(1.5) and both boundary conditions (1.2)-(1.3). Therefore $u(x,\lambda)$ forms an resolvent of the problem (1.1)-(1.5).

3. Expansion results for Green's function

Through in below we assume that the homogeneous equation

$$p(x)u'' - q(x)u = 0$$

under the same BTCs (1.2)-(1.5) has only the trivial solution $u = 0$. This amounts to the assumption that $\lambda = 0$ is not an eigenvalue of the considered BVTP (1.1)-(1.5). There is no less of generality in this assumption, since otherwise we may consider a new equation

$$-p(x)u'' + \widetilde{q}(x)u = \lambda u$$

for $\widetilde{q}(x) = q(x) - \widetilde{\lambda}$ under the same BTCs (1.2)-(1.5). Obviously this problem has the same eigenfunctions as for the considered BVTP (1.1)-(1.5), all eigenvalues are shifted through $\widetilde{\lambda}$ to the right and therefore, we can chose $\widetilde{\lambda}$ such that $\lambda = 0$ is not eigenvalue of the new problem. Now defining $G_0(x,\xi) = G(x,\xi;0)$ we see that the function

$$u_0(x) = \frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} G_0(x,\xi)f(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} G_0(x,\xi)f(\xi)d\xi$$

solves the nonhomogeneous equation $p(x)u'' - q(x)u = f(x)$ and satisfies all boundary-transmission conditions (1.2)-(1.5). Rewritting (2.3) in the form

$$-p(x)u'' - q(x)u = \lambda u - f(x)$$

We see that this equation under the same BTC's (1.2)-(1.5) reduces to the integral equation

$$u(x) + \lambda\{\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} G_0(x,\xi)u(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} G_0(x,\xi)u(\xi)d\xi\}$$
$$= \frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} G_0(x,\xi)f(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} G_0(x,\xi)f(\xi)d\xi$$

Consequently the considered BVTP (1.1)-(1.5) can be written in equivalent integral equation form as

$$u(x) = -\lambda\{\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} G_0(x,\xi)u(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} G_0(x,\xi)u(\xi)d\xi\} \tag{3.1}$$

By applying the same method as in [10] we can prove that the BVTP (1.1)-(1.5) has precisely denumerable many eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$, with following asymptotic

behaviour as $n \to \infty$.
(i) If $\sin\beta \neq 0$ and $\sin\alpha \neq 0$, then

$$s_n = (\frac{n-1}{2}) + O\left(\frac{1}{n}\right)$$

(ii) If $\sin\beta \neq 0$ and $\sin\alpha = 0$, then

$$s_n = \frac{n}{2} + O\left(\frac{1}{n}\right),$$

(iii) If $\sin\beta = 0$ and $\sin\alpha \neq 0$, then

$$s_n = \frac{n}{2} + O\left(\frac{1}{n}\right),$$

(iv) If $\sin\beta = 0$ and $\sin\alpha = 0$, then

$$s_n = \frac{n}{2} + O\left(\frac{1}{n}\right),$$

as $n \to \infty$ where $\lambda_n = s_n^2$. It is evident that the functions $\phi_n(x) = \phi(x,\lambda_n)$ are eigenfunctions corresponding to the eigenvalues λ_n. Let

$$\varphi_n(x) = (\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} \phi_n^2(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} \phi_n^2(x)dx)^{-\frac{1}{2}}\Phi_n(x)$$

It is obviously seen that the sequence $(\varphi_n(x))$ forms an orthonormal set of eigenfunctions in the sense of

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} \varphi_n(x)\varphi_m(x)dx + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} \varphi_n(x)\varphi_m(x)dx = \delta_{nm}, \tag{3.2}$$

where δ_{nm} is the Kronecker delta.

Theorem 1. *The Green's function $G_0(x,\xi)$ can be expanded into an eigenfunction series*

$$G_0(x,\xi) = -\sum_{n=0}^{\infty} \lambda_n^{-1}\varphi_n(x)\varphi_n(\xi) \tag{3.3}$$

which converges absolutely and uniformly on $([-\pi,0) \cup (0,\pi])^2$.

Proof. By using the definition of eigenfunctions $\varphi_n(x)$ and asymptotic behaviour of eigenvalues it is not difficult to show that the series in (3.3) converges absolutely and uniformly and therefore represents a continuous function there. To prove the equality (3.3), suppose, it possible that, the function

$$\widetilde{G_0}(x,\xi) = G_0(x,\xi) + \sum_{n=0}^{\infty} \lambda_n^{-1}\varphi_n(x)\varphi_n(\xi)$$

is not identically zero. Taking in view that the Kernel $\widetilde{G_0}(x,\xi)$ is symmetric slightly modifying the method of proving of familiar theorem in the theory of integral equations (see, for example, [6]) which assert that any symmetric kernel which is not identically zero has at least one eigenfunction, we can prove that there is a real number $\mu_0 \neq 0$ and real valued function $\psi_0 \neq 0$ such that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\widetilde{G}_0(x,\xi)\psi_0(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\widetilde{G}_0(x,\xi)\psi_0(\xi)d\xi=\mu_0\psi_0(x) \tag{3.4}$$

Multiplying by $\varphi_n(x)$ and make the necessary calculations we have

$$\begin{aligned}&\mu_0\Big(\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\psi_0(x)\varphi_n(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\psi_0(x)\varphi_n(x)dx\Big)\\&=\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\psi_0(\xi)\Big(\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\widetilde{G_0}(x,\xi)\varphi_n(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\widetilde{G_0}(x,\xi)\varphi_n(x)dx\Big)d\xi\\&\quad+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\psi_0(\xi)\Big(\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\widetilde{G_0}(x,\xi)\varphi_n(x)dx\\&\quad+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\widetilde{G_0}(x,\xi)\varphi_n(x)dx\Big)d\xi\end{aligned} \tag{3.5}$$

Recalling that the set of eigenfunctions $(\varphi_n(x))$ is orthonormal in the sense of (3.2) it is easy to see that

$$\begin{aligned}&\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\widetilde{G_0}(x,\xi)\varphi_n(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\widetilde{G_0}(x,\xi)\varphi_n(x)dx\\&=\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G(x,\xi)\varphi_n(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G(x,\xi)\varphi_n(x)dx+\frac{\varphi_n(\xi)}{\lambda_n}\end{aligned} \tag{3.6}$$

Substituting (3.6) in (3.5) and taking in view the fact that the eigenfunction $\varphi_n(x)$ satisfy the integral equation (3.1) for $\lambda=\lambda_n$ gives

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\psi_0(x)\varphi_n(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\psi_0(x)\varphi_n(x)dx=0 \tag{3.7}$$

for all $n=0,1,2...$, i.e. the function $\psi_0(x)$ is orthogonal to all eigenfunctions. On the other hand, from (3.4) and (3.7) it follows that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G_0(x,\xi)\psi_0(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G_0(x,\xi)\psi_0(\xi)d\xi=\mu_0\psi_0(x),$$

so $\psi_0(x)$ is also an eigenfunction of BVTP (1.1)-(1.5) corresponding to eigenvalue $-\frac{1}{\mu_0}$. Since it is orthogonal to all eigenfunctions, it is orthogonal to itself, i.e.

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\psi_0^2(s)ds+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\psi_0^2(s)ds=0$$

and hence $\psi_0(s)$ is identically zero. We get a contradiction, which complete the proof. □

4. Completeness of the eigenfunctions

To prove the completeness (φ_n) in square integrable function space at first we shall prove the next theorem.

Theorem 2. *Let $f(x)$ be any function on $[-\pi,0)\cup(0,\pi]$ satisfying the following conditions*
i) *f' and f'' are exist and continuous in both interval $[-\pi,0)$ and $(0,\pi]$*
ii) *There exist a finite one-hand side limits $f(\pm 0), f'(\pm 0)$ and $f''(\pm 0)$*
iii) *f satisfies the BTC's (1.2)-(1.5).*
Then $f(x)$ can be expanded into an absolutely and uniformly convergent series of eigenfunctions (φ_n), namely

$$f(x)=\sum_{n=0}^{\infty}\{\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} f(\xi)\varphi_n(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi} f(\xi)\varphi_n(\xi)d\xi\}\varphi_n(x)$$

Proof. Let f be satisfied all conditions of Theorem. Then by virtue of (3.1) the equality

$$\begin{aligned} f(x)=&\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} G_0(x,\xi)(f''(\xi)-q(\xi)f(\xi))d\xi \\ &+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi} G_0(x,\xi)(f''(\xi)-q(\xi)f(\xi))d\xi \end{aligned}$$

is hold. Substituting (3.3) in the right hand of this equality and integrating by parts twice yields the needed equality

$$f(x)=\sum_{n=0}^{\infty}\{\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} f(\xi)\varphi_n(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi} f(\xi)\varphi_n(\xi)dx\}\varphi_n(x)$$

The proof is complete. □

Corollary 1. *Let f be as in previous Theorem. Then the modified parseval equality*

$$\begin{aligned} \frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} f^2(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi} f^2(x)dx=&\sum_{n=0}^{\infty}(\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} f(\xi)\varphi_n(\xi)d\xi \\ &+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi} f(\xi)\varphi_n(\xi)dx\}\varphi_n(\xi))^2 \end{aligned}$$

is hold. This is also called the completeness relation.

We already know that the resolvent $u(x,\lambda)$ which determined by (2.2) for all λ are not eigenvalues, satisfies all conditions of the Theorem. By using this fact, denoting $\ell u(x) = p(x)u''(x) - q(x)u(x)$ and integrating by parts twice we have

$$\begin{aligned} c_n(\ell u(.,\lambda)) &= \frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} \ell u(\xi,\lambda)\varphi_n(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} \ell u(\xi,\lambda)\varphi_n(\xi)d\xi \\ &= \frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} u(\xi,\lambda)\ell\varphi_n(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} u(\xi,\lambda)\ell\varphi_n(\xi)d\xi \\ &= -\lambda_n c_n(u(.,\lambda)) \end{aligned}$$

where the Fourier coefficients c_n are defined as

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} f(\xi)\varphi_n(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} f(\xi)\varphi_n(\xi)d\xi.$$

Since the resolvent $u(x,\lambda)$ satisfies the equation $\ell u(x,\lambda) + \lambda u(x,\lambda) = f(x)$ we have

$$\begin{aligned} c_n(f) &= c_n(\ell u(.,\lambda) + \lambda u(.,\lambda)) = c_n(\ell u(.,\lambda)) + \lambda c_n(u(.,\lambda)) \\ &= -\lambda_n c_n(u(.,\lambda)) + \lambda c_n(u(.,\lambda)) = (\lambda - \lambda_n)c_n(u(.,\lambda)). \end{aligned}$$

Thus if we know the expansion of given function $f(x)$ into an eigenfunctions, then the solution of homogeneous equation (2.3) satisfying BTC's (1.2)-(1.4) can be obtain by formula

$$u(x,\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda - \lambda_n}\left(\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} f(\xi)\varphi_n(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} f(\xi)\varphi_n(\xi)d\xi\right)\varphi_n(x))$$

for all λ are not eigenvalues.

5. Expansion of mean-square integrable functions into a series of eigenfunction

The expansion theorem will now be extended to the square integrable functions.

Theorem 3. *Let $f(x)$ be any square integrable function on $[-\pi,0)$ and $(0,\pi]$. Then f can be expanded into Fourier series of eigenfunctions in the sense of mean-square convergeness, namely the formula*

$$\begin{aligned} \lim_{n\to\infty}\{&\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} (f(x) - \sum_{k=0}^{n} c_k(f)\varphi_k(x))^2 dx \\ &+ \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} (f(x) - \sum_{k=0}^{n} c_k(f)\varphi_k(x))^2 dx\} = 0 \end{aligned} \tag{5.1}$$

is hold.

Proof. Given any $\epsilon > 0$, there exist an infinitely differentiable function $g(x)$ which vanish in the neighborhoods $x = -\pi,\ x = 0$ and $x = \pi$ such that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}(f(x)-g(x))^2dx + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi}(f(x)-g(x))^2dx < \epsilon \tag{5.2}$$

Denote, for shorting,

$$F_n(x) = \sum_{k=0}^{n} c_k(f)\varphi_k(x) \text{ and } G_n(x) = \sum_{k=0}^{n} c_k(g)\varphi_k(x)$$

By virtue of Theorem 2 there exist an integer N_1, depending on ϵ, such that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}(g(x)-G_n(x))^2dx + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi}(g(x)-G_n(x))^2dx < \epsilon \tag{5.3}$$

for all $n \geq N_1$. By the well known Bessel inequality it can be shown easily that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}(G_n(x)-F_n(x))^2dx + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi}(G_n(x)-F_n(x))^2dx < \epsilon \tag{5.4}$$

for all $\varepsilon > 0$. Finally, writing $f(x) - F_n(x)$ in the form $f(x) - F_n(x) = (f(x) - g(x)) + (g(x) - G_n(x)) + (G_n(x) - F_n(x))$ and using well-known Minkowski inequality from (5.2), (5.3) and (5.4) we can derive that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}(f(x)-F_n(x))^2dx + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi}(f(x)-F_n(x))^2dx < 3\epsilon$$

for $n \geq N_1$ which proving the formula (5.1). □

Corollary 2. *If f is as in previous theorem(i.e. $f \in L_2[-\pi,0) \oplus L_2(0,\pi]$) then the modified Parseval equality*

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} f^2(x)dx + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} f^2(x)dx = \sum_{n=0}^{\infty} c_n^2(f).$$

is hold.

This is also called the completeness relation.

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