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SOME INTERESTING CONGRUENCES FOR BALLOT NUMBERS

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Abstract. In this paper, we determine the sums $\sum_{k=0}^{n-1}\binom{2k+d}{k}/x^k$, $\sum_{k=0}^{n-1}k\binom{2k+d}{k}/x^k$ and some congruences can be obtained by using them. For example, for an odd prime $p\neq 5$,

$$\sum_{k=0}^{(p-1)/2}(-1)^k k\binom{2k+d}{k}\equiv\frac{(-1)^d}{5}\left(\frac{5}{p}\right)\left(F_{d+1-\left(\frac{5}{p}\right)}-(d+1)L_{d+1-\left(\frac{5}{p}\right)}\right)\pmod p,$$

and for an odd prime p,

$$\sum_{k=0}^{(p-1)/2}\frac{k}{(-4)^k}B(k,d)\equiv-(-2)^{d-1}\left(\frac{2}{p}\right)dP_{d+1-\left(\frac{2}{p}\right)}\pmod p,$$

where $d\in\{0,\dots,(p-1)/2\}$, F_n is the nth Fibonacci number, L_n is the nth Lucas number and P_n is the nth Pell number. $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

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1. INTRODUCTION

The some elementary combinatorial properties of the Catalan and Ballot numbers are given in [2], [4] and [3]. In [1], [7], É. Lucas and N.J. Fine gave how to compute binomial coefficients modulo a prime. Lucas Theorem is given as follows:

If p is a prime, n, m, n_0 and m_0 are non-negative integers, and n_0, m_0 are both less than p, then

$$\binom{np+n_0}{mp+m_0}\equiv\binom{n}{m}\binom{n_0}{m_0}\pmod p. \tag{1.1}$$

The Catalan numbers are given by

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n+1},\quad n\in\mathbb{N}.$$

In [2], the Catalan numbers are special cases of the Ballot numbers

$$B(n,k)=\frac{k}{2n+k}\binom{2n+k}{n}.$$

In [13], Z.W. Sun and R. Tauraso obtained $\sum_{k=0}^{p^a-1}\binom{2k}{k+d}/m^k$ and $\sum_{k=0}^{p-1}\binom{2k}{k+d}/km^{k-1}$ mod p for all $d=0,1,...,p^a$, where m is any integer not divisible by p. For example, they showed that if $p\neq 2,5$, then

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}\equiv -5\frac{F_{p-\left(\frac{p}{5}\right)}}{p}\pmod{p}.$$

In [8], Z.W. Sun determined $\sum_{k=0}^{p^a-1}\binom{2k}{k+d}/m^k\ mod\, p^2$ for $d=0,1$; for example,

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+d}/m^k\equiv\left(\frac{m^2-4m}{p^a}\right)+\left(\frac{m^2-4m}{p^{a-1}}\right)u_{p-\left(\frac{m^2-4m}{p}\right)}\pmod{p^2},$$

where an odd prime p and $a,m\in\mathbb{Z}$ with $a>0$, $p\nmid m$.

In [11], Z.W. Sun used Lucas quotients in order to obtain $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}/m^k$ modulo p^2 for any integer $m\not\equiv 0 \pmod{p}$; especially, he determined the following congruence:

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}\equiv\left(\frac{3}{p}\right)\pmod{p^2}.$$

In [9], Z.W. Sun gave the following congruence:

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)4^k}\equiv(-1)^{(p+1)/2}q_p(2)\pmod{p^2},$$

where an odd prime p and $q_p(2)$ is a Fermat quotient.

In [6], S. Koparal and N. Ömür presented congruences involving central binomial coefficients and harmonic numbers. For example, for an odd prime p,

$$\sum_{k=0}^{(p-1)/2}(-1)^k\binom{2k}{k}H_{k-1}\equiv\frac{2^p}{p}\left(2F_{p+1}-5^{(p-1)/2}-1\right)\pmod{p},$$

where H_n is the nth harmonic number.

In [5], K.H. Pilehrood et all gave that for a prime $p \neq 2,5$,

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} F_{2k+1}}{(2k+1)\,16^k} \equiv (-1)^{(p+1)/2} \frac{F_p - \left(\frac{p}{5}\right)}{p} \pmod{p^2}.$$

In this paper, we determine the sums $\sum\limits_{k=0}^{n-1} \binom{2k+d}{k}/x^k$, $\sum\limits_{k=0}^{n-1} k\binom{2k+d}{k}/x^k$ and some congruences can be obtained by using them.

Two sequences $\{u_n(x)\}$ and $\{v_n(x)\}$ of polynomials are defined by for $n > 0$

$$u_{n+1}(x) = xu_n(x) - u_{n-1}(x) \quad \text{and} \quad v_{n+1}(x) = xv_n(x) - v_{n-1}(x),$$

where $u_0(x) = 0$, $u_1(x) = 1$ and $v_0(x) = 2$, $v_1(x) = x$, respectively. The characteristic equation $y^2 - xy + 1 = 0$ of the sequences $\{u_n(x)\}$ and $\{v_n(x)\}$ has two roots

$$\alpha(x) = \frac{x + \sqrt{\Delta}}{2} \text{ and } \beta(x) = \frac{x - \sqrt{\Delta}}{2},$$

where $\Delta = x^2 - 4$. The Binet formulas of the sequences $\{u_n(x)\}$ and $\{v_n(x)\}$ are as follows:

$$u_n(x) = \frac{(\alpha(x))^n - (\beta(x))^n}{\alpha(x) - \beta(x)} \text{ and } v_n(x) = (\alpha(x))^n + (\beta(x))^n,$$

respectively. Clearly, for any $n \in \mathbb{N}$,

$$xu_n(x) + v_n(x) = 2u_{n+1}(x), \tag{1.2}$$

$$u_n(x) + \frac{x+2}{\Delta} v_n(x) = \frac{2}{\Delta}\left(v_{n+1}(x) + v_n(x)\right), \tag{1.3}$$

$$\Delta u_n(x) = v_{n+1}(x) - v_{n-1}(x). \tag{1.4}$$

It is seen that

$$(-1)^{n-1} u_n(-3) = F_{2n} \text{ and } (-1)^n v_n(-3) = L_{2n}, \tag{1.5}$$

$$(-1)^{n-1} u_n(-6) = \frac{1}{2} P_{2n} \text{ and } (-1)^n v_n(-6) = Q_{2n},$$

where F_n and L_n are the nth Fibonacci number and nth Lucas number, and P_n and Q_n are the nth Pell number and the nth Pell-Lucas number, respectively.

2. Some congruences related to Ballot numbers

In this section, we will investigate some congruences with the combinatorial identities. Now, we give the following lemmas for further use.

Lemma 1. *Let $r,s \in \mathbb{Z}$ and $D = r^2 - 4s$. Suppose that p is an odd prime with $p \nmid sD$. Then*

$$\left(\frac{r \pm \sqrt{D}}{2}\right)^{p-\left(\frac{D}{p}\right)} \equiv s^{\left(1-\left(\frac{D}{p}\right)\right)/2} \pmod{p}, \tag{2.1}$$

where $\frac{r \pm \sqrt{D}}{2}$ are roots of the equation $y^2 - ry + s = 0$ [10].

Lemma 2. *For any $n \in \mathbb{N}$, we have*

$$\frac{d}{dx}(u_n(x-2)) = \frac{1}{\Delta}((n+1)v_n(x-2) - 2u_{n+1}(x-2)),$$

where Δ as before.

Proof. By differentiating both sides of $\alpha(x-2) = \frac{x-2+\sqrt{\Delta}}{2}$, we write

$$\frac{d}{dx}(\alpha(x-2)) = \frac{x-2+\sqrt{\Delta}}{2\sqrt{\Delta}} = \frac{\alpha(x-2)}{\sqrt{\Delta}}.$$

Similarly, it is clearly seen that

$$\frac{d}{dx}(\beta(x-2)) = \frac{-\beta(x-2)}{\sqrt{\Delta}} \text{ and } \frac{d}{dx}\left(\sqrt{\Delta}\right) = \frac{2\alpha(x-2)}{\sqrt{\Delta}} - 1 = \frac{2\beta(x-2)}{\sqrt{\Delta}} + 1.$$

Thus

$$\begin{aligned}
&\frac{d}{dx}(u_n(x-2)) = \\
&= \frac{d}{dx}\left(\frac{\alpha^n(x-2) - \beta^n(x-2)}{\sqrt{\Delta}}\right) \\
&= \frac{1}{\Delta}\left(\left(n\frac{\alpha^n(x-2)}{\sqrt{\Delta}} + n\frac{\beta^n(x-2)}{\sqrt{\Delta}}\right)\sqrt{\Delta}\right. \\
&\quad \left. -\left(\frac{2\alpha(x-2)}{\sqrt{\Delta}} - 1\right)\alpha^n(x-2) + \left(\frac{2\beta(x-2)}{\sqrt{\Delta}} + 1\right)\beta^n(x-2)\right) \\
&= \frac{1}{\Delta}\left(n\alpha^n(x-2) + n\beta^n(x-2) - 2\left(\frac{\alpha^{n+1}(x-2) - \beta^{n+1}(x-2)}{\sqrt{\Delta}}\right)\right. \\
&\quad \left. +\alpha^n(x-2) + \beta^n(x-2)\right) \\
&= \frac{1}{\Delta}(nv_n(x-2) - 2u_{n+1}(x-2) + v_n(x-2)) \\
&= \frac{1}{\Delta}((n+1)v_n(x-2) - 2u_{n+1}(x-2)).
\end{aligned}$$

This concludes the proof. □

Theorem 1. *For any $n,\, d \in \mathbb{Z}^+$, we have*

$$\sum_{k=0}^{n-1} \binom{2k+d}{k} x^{n-1-k} = \sum_{k=0}^{n+d-1} \binom{2n+d}{k} u_{n+d-k}(x-2) \tag{2.2}$$
$$- x^{n+\lfloor d/2 \rfloor} \left(\varepsilon u_{\lfloor d/2 \rfloor +1}(x-2) + u_{\lfloor d/2 \rfloor}(x-2) \right),$$

where $\varepsilon = \left(1-(-1)^d\right)/2$.

Proof. To prove (2.2), we shall apply induction method on n.

For $n = 1$, we must show that for $d \in \mathbb{Z}^+$,

$$\sum_{k=0}^{d} \binom{d+2}{k} u_{d+1-k}(x-2) = \tag{2.3}$$
$$= 1 + x^{\lfloor d/2 \rfloor +1} \left(\varepsilon u_{\lfloor d/2 \rfloor +1}(x-2) + u_{\lfloor d/2 \rfloor}(x-2) \right).$$

We have

$$\sum_{k=0}^{d} \binom{d+2}{k} u_{d+1-k}(x-2) =$$
$$= 1 + \sum_{k=0}^{d+2} \binom{d+2}{k} u_{d+1-k}(x-2) = 1 + \sum_{k=0}^{d+2} \binom{d+2}{k} u_{k-1}(x-2)$$
$$= 1 + \frac{1}{\alpha(x-2) - \beta(x-2)} \left\{ \frac{1}{\alpha(x-2)} \sum_{k=0}^{d+2} \binom{d+2}{k} (\alpha(x-2))^k \right.$$
$$\left. - \frac{1}{\beta(x-2)} \sum_{k=0}^{d+2} \binom{d+2}{k} (\beta(x-2))^k \right\}$$
$$= 1 + \frac{1}{\alpha(x-2) - \beta(x-2)} \left\{ \frac{1}{\alpha(x-2)} (1+\alpha(x-2))^{d+2} \right.$$
$$\left. - \frac{1}{\beta(x-2)} (1+\beta(x-2))^{d+2} \right\}.$$

Using the identity $\alpha(x-2)\,\beta(x-2) = 1$, we get

$$\sum_{k=0}^{d} \binom{d+2}{k} u_{d+1-k}(x-2) =$$
$$= 1 + \frac{1}{\alpha(x-2) - \beta(x-2)} \left\{ (\beta(x-2))(1+\alpha(x-2))^{d+2} \right.$$
$$\left. - (\alpha(x-2))(1+\beta(x-2))^{d+2} \right\}.$$

From $d = \lfloor d/2 \rfloor + \lfloor d/2 \rfloor + \varepsilon$, we rewrite

$$\sum_{k=0}^{d} \binom{d+2}{k} u_{d+1-k}(x-2) =$$
$$= 1 + \frac{1}{\alpha(x-2) - \beta(x-2)}$$
$$\times \Big\{ (\beta(x-2))(1+\alpha(x-2))^{\varepsilon}(1+\alpha(x-2))^{\lfloor d/2 \rfloor + 1}(1+\alpha(x-2))^{\lfloor d/2 \rfloor + 1}$$
$$-(\alpha(x-2))(1+\beta(x-2))^{\varepsilon}(1+\beta(x-2))^{\lfloor d/2 \rfloor + 1}(1+\beta(x-2))^{\lfloor d/2 \rfloor + 1} \Big\}.$$

Using the identity $(1+\alpha(x-2))(1+\beta(x-2)) = x$, we get

$$\sum_{k=0}^{d} \binom{d+2}{k} u_{d+1-k}(x-2) =$$
$$= 1 + \frac{1}{\alpha(x-2) - \beta(x-2)}$$
$$\times \Big\{ (\beta(x-2)+\varepsilon)(1+\alpha(x-2))^{\lfloor d/2 \rfloor + 1}(1+\alpha(x-2))^{\lfloor d/2 \rfloor + 1}$$
$$-(\alpha(x-2)+\varepsilon)(1+\beta(x-2))^{\lfloor d/2 \rfloor + 1}(1+\beta(x-2))^{\lfloor d/2 \rfloor + 1} \Big\}$$
$$= 1 + \frac{x^{\lfloor d/2 \rfloor + 1}}{\alpha(x-2) - \beta(x-2)} \Big\{ (\alpha(x-2))^{\lfloor d/2 \rfloor} - (\beta(x-2))^{\lfloor d/2 \rfloor}$$
$$+\varepsilon(\alpha(x-2))^{\lfloor d/2 \rfloor + 1} - \varepsilon(\beta(x-2))^{\lfloor d/2 \rfloor + 1} \Big\}$$
$$= 1 + x^{\lfloor d/2 \rfloor + 1}\left(u_{\lfloor d/2 \rfloor}(x-2) + \varepsilon u_{\lfloor d/2 \rfloor + 1}(x-2)\right).$$

So, (2.2) holds for $n = 1$.

We assume that the result is true for some integer $n \geq 1$.

We must show that for $n+1$, (2.2) holds. By the induction hypothesis, for any $d \in \mathbb{Z}^{+}$, we write

$$\sum_{k=0}^{n} \binom{2k+d}{k} x^{n-k} =$$
$$= \binom{2n+d}{n} + x \sum_{k=0}^{n-1} \binom{2k+d}{k} x^{n-1-k}$$
$$= \binom{2n+d}{n} + x \sum_{k=0}^{n+d-1} \binom{2n+d}{k} u_{n+d-k}(x-2)$$
$$-x^{n+1+\lfloor d/2 \rfloor}\left(\varepsilon u_{\lfloor d/2 \rfloor + 1}(x-2) + u_{\lfloor d/2 \rfloor}(x-2)\right)$$

$$
\begin{aligned}
&= \binom{2n+d}{n} + x\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2) \\
&\quad -2\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2) + 2\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2) \\
&\quad -\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-1-k}(x-2) + \sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-1-k}(x-2) \\
&\quad -x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2) + u_{\lfloor d/2\rfloor}(x-2)\right) \\
&= (x-2)\sum_{k=0}^{n+d}\binom{2n+d}{k}u_{n+d-k}(x-2) - \sum_{k=0}^{n+d}\binom{2n+d}{k}u_{n+d-1-k}(x-2) \\
&\quad +2\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2) + \sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-1-k}(x-2) \\
&\quad -x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2) + u_{\lfloor d/2\rfloor}(x-2)\right).
\end{aligned}
$$

Since $(x-2)\,u_n(x-2) = u_{n+1}(x-2) + u_{n-1}(x-2)$, we have

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{2k+d}{k}x^{n-k} = \\
&= \sum_{k=0}^{n+d}\binom{2n+d}{k}u_{n+d+1-k}(x-2) + 2\sum_{k=-1}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2) \\
&\quad +\sum_{k=-2}^{n+d-1}\binom{2n+d}{k}u_{n+d-1-k}(x-2) \\
&\quad -x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2) + u_{\lfloor d/2\rfloor}(x-2)\right) \\
&= \sum_{k=0}^{n+d}\binom{2n+d}{k}u_{n+d+1-k}(x-2) + 2\sum_{k=0}^{n+d}\binom{2n+d}{k-1}u_{n+d+1-k}(x-2) \\
&\quad +\sum_{k=0}^{n+d+1}\binom{2n+d}{k-2}u_{n+d+1-k}(x-2) \\
&\quad -x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2) + u_{\lfloor d/2\rfloor}(x-2)\right),
\end{aligned}
$$

and hence

$$\sum_{k=0}^{n}\binom{2k+d}{k}x^{n-k}=$$
$$=\sum_{k=0}^{n+d}\left(\binom{2n+d}{k}+2\binom{2n+d}{k-1}+\binom{2n+d}{k-2}\right)u_{n+d+1-k}(x-2)$$
$$-x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right).$$

By the binomial identity $\binom{2n+d}{k}+2\binom{2n+d}{k-1}+\binom{2n+d}{k-2}=\binom{2n+d+2}{k}$, we get

$$\sum_{k=0}^{n}\binom{2k+d}{k}x^{n-k}=\sum_{k=0}^{n+d}\binom{2(n+1)+d}{k}u_{n+d+1-k}(x-2)$$
$$-x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right).$$

Hence the result is true for all integers $n\geq 0$. □

As a result of Theorem 1, we may give the following congruence.

Corollary 1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d}{k}\equiv m^{\lfloor d/2\rfloor-(p-1)/2} \tag{2.4}$$
$$\times\left\{(1-\varepsilon)u_{(p-1)/2+\lfloor d/2\rfloor}(m-2)+((m-1)\varepsilon+1)u_{(p+1)/2+\lfloor d/2\rfloor}(m-2)\right\}$$
$$-m^{\lfloor d/2\rfloor+1}(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2))\pmod{p},$$

where ε, Δ as before, $d\in\{0,1,\dots,(p-1)/2\}$ and $m\in\mathbb{Z}$ with $p\nmid m\Delta$.

Proof. Substituting $n=(p+1)/2$ and $x=m$ in (2.2), we write

$$\sum_{k=0}^{(p-1)/2}\binom{2k+d}{k}m^{(p-1)/2-k}=\sum_{k=0}^{(p-1)/2+d}\binom{p+d+1}{k}u_{(p+1)/2+d-k}(m-2)$$
$$-m^{(p+1)/2+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right).$$

By the congruence in (1.1), it is easily seen that

$$m^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k+d}{k}}{m^k}\equiv$$
$$\equiv\sum_{k=0}^{d+1}\binom{d+1}{k}u_{(p+1)/2+d-k}(m-2)$$
$$-m^{\lfloor d/2\rfloor+(p+1)/2}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right)$$

$$= \sum_{k=0}^{d+1} \binom{d+1}{k} u_{(p-1)/2+k}(m-2)$$
$$-m^{\lfloor d/2 \rfloor + (p+1)/2} \left(\varepsilon u_{\lfloor d/2 \rfloor +1}(m-2) + u_{\lfloor d/2 \rfloor}(m-2) \right)$$
$$= \frac{(\alpha(m-2))^{(p-1)/2}}{\alpha(m-2)-\beta(m-2)} \sum_{k=0}^{d+1} \binom{d+1}{k} (\alpha(m-2))^k$$
$$- \frac{(\beta(m-2))^{(p-1)/2}}{\alpha(m-2)-\beta(m-2)} \sum_{k=0}^{d+1} \binom{d+1}{k} (\beta(m-2))^k$$
$$-m^{(p+1)/2+\lfloor d/2 \rfloor} \left(\varepsilon u_{\lfloor d/2 \rfloor +1}(m-2) + u_{\lfloor d/2 \rfloor}(m-2) \right) \pmod{p}.$$

By Binomial theorem, we have

$$m^{(p-1)/2} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k+d}{k}}{m^k} \equiv$$
$$\equiv \frac{(\alpha(m-2))^{(p-1)/2}(1+\alpha(m-2))^{d+1}}{\alpha(m-2)-\beta(m-2)} - \frac{(\beta(m-2))^{(p-1)/2}(1+\beta(m-2))^{d+1}}{\alpha(m-2)-\beta(m-2)}$$
$$-m^{(p+1)/2+\lfloor d/2 \rfloor} \left(\varepsilon u_{\lfloor d/2 \rfloor +1}(m-2) + u_{\lfloor d/2 \rfloor}(m-2) \right) \pmod{p}.$$

From $d = \lfloor d/2 \rfloor + \lfloor d/2 \rfloor + \varepsilon$, then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k+d}{k}}{m^k} \equiv$$
$$\equiv \frac{1}{\alpha(m-2)-\beta(m-2)} \times$$
$$\times \left\{ (\alpha(m-2))^{(p-1)/2}(1+\alpha(m-2))^{\lfloor d/2 \rfloor+\varepsilon+1}(1+\alpha(m-2))^{\lfloor d/2 \rfloor} \right.$$
$$\left. -(\beta(m-2))^{(p-1)/2}(1+\beta(m-2))^{\lfloor d/2 \rfloor+\varepsilon+1}(1+\beta(m-2))^{\lfloor d/2 \rfloor} \right\}$$
$$-m^{(p+1)/2+\lfloor d/2 \rfloor} \left(\varepsilon u_{\lfloor d/2 \rfloor +1}(m-2) + u_{\lfloor d/2 \rfloor}(m-2) \right) \pmod{p}.$$

Using the equalities of $\alpha(m-2)\beta(m-2) = 1$ and $(1+\alpha(m-2))(1+\beta(m-2)) = m$, we have the proof. □

For example, when $d = 0$ and $m \in \mathbb{Z}$ with $p \nmid m$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m(m-4)}{p} \right) \pmod{p} \text{ [12]},$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{1}{(-4)^k} \binom{2k+d}{k} \equiv (-2)^d \left(\frac{2}{p}\right) P_{d-\left(\frac{2}{p}\right)} \pmod{p},$$

where $d \in \{0, 1, ..., (p-1)/2\}$.

Theorem 2. *For any n, $d \in \mathbb{Z}^+$, we have*

$$\sum_{k=0}^{n-1} k \binom{2k+d}{k} x^{n-1-k} = \sum_{k=0}^{n+d-1} \binom{2n+d}{k} \Big((n-1) u_{n+d-k}(x-2) \tag{2.5}$$
$$-\frac{x}{\Delta} \left((n+d+1-k) v_{n+d-k}(x-2) - 2u_{n+d+1-k}(x-2) \right) \Big)$$
$$+ (\lfloor d/2 \rfloor + 1) x^{n+\lfloor d/2 \rfloor} \left(\varepsilon u_{\lfloor d/2 \rfloor+1}(x-2) + u_{\lfloor d/2 \rfloor}(x-2) \right)$$
$$+ \frac{x^{n+\lfloor d/2 \rfloor+1}}{\Delta} \Big(\varepsilon \left((\lfloor d/2 \rfloor + 2) v_{\lfloor d/2 \rfloor+1}(x-2) - 2u_{\lfloor d/2 \rfloor+2}(x-2) \right)$$
$$+ (\lfloor d/2 \rfloor + 1) v_{\lfloor d/2 \rfloor}(x-2) - 2u_{\lfloor d/2 \rfloor+1}(x-2) \Big),$$

where ε, Δ as before.

Proof. To prove equality (2.5), we take the derivative of (2.2) with respect to x:

$$\sum_{k=0}^{n-1} \binom{2k+d}{k} (n-1-k) x^{n-2-k} =$$
$$= \sum_{k=0}^{n+d-1} \binom{2n+d}{k} \frac{1}{\Delta} \left((n+d+1-k) v_{n+d-k}(x-2) \; -2u_{n+d+1-k}(x-2) \right)$$
$$- (n + \lfloor d/2 \rfloor) x^{n+\lfloor d/2 \rfloor-1} \left(\varepsilon u_{\lfloor d/2 \rfloor+1}(x-2) + u_{\lfloor d/2 \rfloor}(x-2) \right)$$
$$- \frac{x^{n+\lfloor d/2 \rfloor}}{\Delta} \Big(\varepsilon \left((\lfloor d/2 \rfloor + 2) v_{\lfloor d/2 \rfloor+1}(x-2) - 2u_{\lfloor d/2 \rfloor+2}(x-2) \right)$$
$$+ (\lfloor d/2 \rfloor + 1) v_{\lfloor d/2 \rfloor}(x-2) - 2u_{\lfloor d/2 \rfloor+1}(x-2) \Big),$$

and then

$$\sum_{k=0}^{n-1} k \binom{2k+d}{k} x^{n-2-k} = \tag{2.6}$$
$$= \frac{n-1}{x} \sum_{k=0}^{n-1} \binom{2k+d}{k} x^{n-1-k}$$
$$- \frac{1}{\Delta} \sum_{k=0}^{n+d-1} \binom{2n+d}{k} \left((n+d+1-k) v_{n+d-k}(x-2) - 2u_{n+d+1-k}(x-2) \right)$$

$$+\left(n+\lfloor d/2\rfloor\right)x^{n+\lfloor d/2\rfloor-1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right)$$
$$+\frac{x^{n+\lfloor d/2\rfloor}}{\Delta}\left(\varepsilon\left(\left(\lfloor d/2\rfloor+2\right)v_{\lfloor d/2\rfloor+1}(x-2)-2u_{\lfloor d/2\rfloor+2}(x-2)\right)\right.$$
$$\left.\left(\lfloor d/2\rfloor+1\right)v_{\lfloor d/2\rfloor}(x-2)-2u_{\lfloor d/2\rfloor+1}(x-2)\right).$$

Multiplying both sides of (2.6) with x and using Lemma 1, the proof is the complete. □

Now, from Theorem 2, we have the following congruence:

Corollary 2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2} k\frac{\binom{2k+d}{k}}{m^k}\equiv \tag{2.7}$$
$$\equiv -\frac{m^{\lfloor d/2\rfloor+1-(p-1)/2}}{\Delta}\times \tag{2.8}$$
$$\times\left(\left(d+2\right)\left(v_{(p+1)/2+\lfloor d/2\rfloor}(m-2)+\varepsilon v_{(p+1)/2+\lfloor d/2\rfloor+1}(m-2)\right)\right.$$
$$\left.-2\left(\left(1-\varepsilon\right)u_{(p+1)/2+\lfloor d/2\rfloor}(m-2)+\left(\left(m-1\right)\varepsilon+1\right)u_{(p+1)/2+\lfloor d/2\rfloor+1}(m-2)\right)\right)$$
$$+\frac{m^{\lfloor d/2\rfloor+\varepsilon+1}}{\Delta}\left(2\left(\lfloor d/2\rfloor+1\right)\left(v_{\lfloor d/2\rfloor+1}(m-2)+\left(1-\varepsilon\right)v_{\lfloor d/2\rfloor}(m-2)\right)\right.$$
$$\left.-\left(2-\varepsilon\right)mu_{\lfloor d/2\rfloor+1}(m-2)\right)\pmod p,$$

where ε, Δ as before, $d\in\{0,1,\ldots,(p-1)/2\}$ and $m\in\mathbb{Z}$ with $p\nmid m\Delta$.

Proof. Substituting $n=(p+1)/2$ and $x=m$ in (2.5), we write

$$\sum_{k=0}^{(p-1)/2} k\binom{2k+d}{k}m^{(p-1)/2-k}=$$
$$=\sum_{k=0}^{(p-1)/2+d}\binom{p+d+1}{k}\left(\frac{p-1}{2}u_{(p+1)/2+d-k}(m-2)\right.$$
$$\left.-\frac{m}{\Delta}\left(\left(\frac{p+1}{2}+d+1-k\right)v_{(p+1)/2+d-k}(m-2)-2u_{(p+1)/2+d+1-k}(m-2)\right)\right)$$
$$+\left(\lfloor d/2\rfloor+1\right)m^{(p+1)/2+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right)$$
$$+\frac{m^{(p+1)/2+\lfloor d/2\rfloor+1}}{\Delta}\left(\varepsilon\left(\left(\lfloor d/2\rfloor+2\right)v_{\lfloor d/2\rfloor+1}(m-2)-2u_{\lfloor d/2\rfloor+2}(m-2)\right)\right.$$
$$\left.+\left(\lfloor d/2\rfloor+1\right)v_{\lfloor d/2\rfloor}(m-2)-2u_{\lfloor d/2\rfloor+1}(m-2)\right).$$

By the congruence in (1.1), it is easily seen that

$$
\begin{aligned}
&m^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k+d}{k}}{m^k}\equiv\\
&\equiv\sum_{k=0}^{d+1}\binom{d+1}{k}\Bigg(-\frac{1}{2}u_{(p+1)/2+d-k}(m-2)\\
&\quad-\frac{m}{\Delta}\left(\left(\frac{3}{2}+d-k\right)v_{(p+1)/2+d-k}(m-2)-2u_{(p+1)/2+d+1-k}(m-2)\right)\Bigg)\\
&\quad+(\lfloor d/2\rfloor+1)\,m^{(p+1)/2+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right)\\
&\quad+\frac{m^{(p+1)/2+\lfloor d/2\rfloor+1}}{\Delta}\big(\varepsilon\left((\lfloor d/2\rfloor+2)\,v_{\lfloor d/2\rfloor+1}(m-2)-2u_{\lfloor d/2\rfloor+2}(m-2)\right)\\
&\quad+(\lfloor d/2\rfloor+1)\,v_{\lfloor d/2\rfloor}(m-2)-2u_{\lfloor d/2\rfloor+1}(m-2)\big)\\
&=-\frac{1}{2}\sum_{k=0}^{d+1}\binom{d+1}{k}u_{(p-1)/2+k}(m-2)+\frac{2m}{\Delta}\sum_{k=0}^{d+1}\binom{d+1}{k}u_{(p+1)/2+k}(m-2)\\
&\quad-\frac{m}{\Delta}\sum_{k=0}^{d+1}\binom{d+1}{k}\left(k+\frac{1}{2}\right)v_{(p-1)/2+k}(m-2)\\
&\quad+(\lfloor d/2\rfloor+1)\,m^{(p+1)/2+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right)\\
&\quad+\frac{m^{(p+1)/2+\lfloor d/2\rfloor+1}}{\Delta}\big(\varepsilon\left((\lfloor d/2\rfloor+2)\,v_{\lfloor d/2\rfloor+1}(m-2)-2u_{\lfloor d/2\rfloor+2}(m-2)\right)\\
&\quad+(\lfloor d/2\rfloor+1)\,v_{\lfloor d/2\rfloor}(m-2)-2u_{\lfloor d/2\rfloor+1}(m-2)\big)\pmod{p}.
\end{aligned}
$$

From the Binet formulae of the sequences $\{u_n(m-2)\}$ and $\{v_n(m-2)\}$ and Binomial theorem, we have

$$
\begin{aligned}
&m^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k+d}{k}}{m^k}\equiv\\
&\equiv-\frac{1}{2}m^{\lfloor d/2\rfloor}\left((1-\varepsilon)\,u_{(p-1)/2+\lfloor d/2\rfloor}(m-2)+((m-1)\,\varepsilon+1)\,u_{(p+1)/2+\lfloor d/2\rfloor}(m-2)\right)\\
&\quad+\frac{2}{\Delta}m^{\lfloor d/2\rfloor+1}\left((1-\varepsilon)\,u_{(p+1)/2+\lfloor d/2\rfloor}(m-2)+((m-1)\,\varepsilon+1)\,u_{(p+1)/2+\lfloor d/2\rfloor+1}(m-2)\right)\\
&\quad-\frac{m^{\lfloor d/2\rfloor+1}}{\Delta}(d+1)\left(v_{(p+1)/2+\lfloor d/2\rfloor}(m-2)+\varepsilon v_{(p+1)/2+\lfloor d/2\rfloor+1}(m-2)\right)\\
&\quad-\frac{m^{\lfloor d/2\rfloor+1}}{2\Delta}\left((1-\varepsilon)\,v_{(p-1)/2+\lfloor d/2\rfloor}(m-2)+((m-1)\,\varepsilon+1)\,v_{(p+1)/2+\lfloor d/2\rfloor}(m-2)\right)\\
&\quad+(\lfloor d/2\rfloor+1)\,m^{(p+1)/2+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right)
\end{aligned}
$$

$$+\frac{m^{(p+1)/2+\lfloor d/2\rfloor+1}}{\Delta}\big(\varepsilon\big((\lfloor d/2\rfloor+2)\,v_{\lfloor d/2\rfloor+1}(m-2)-2u_{\lfloor d/2\rfloor+2}(m-2)\big)$$
$$+(\lfloor d/2\rfloor+1)\,v_{\lfloor d/2\rfloor}(m-2)-2u_{\lfloor d/2\rfloor+1}(m-2)\big) \pmod p.$$

From (1.2), (1.3) and (1.4), we obtained the desired result. □

For example, for an odd prime $p\neq 5$

$$\sum_{k=0}^{(p-1)/2}(-1)^k k\binom{2k+d}{k}\equiv$$
$$\equiv\frac{(-1)^d}{5}\left(\frac{5}{p}\right)\left(F_{d+1-\left(\frac{5}{p}\right)}-(d+1)\,L_{d+1-\left(\frac{5}{p}\right)}\right) \pmod p,$$

and for an odd prime p,

$$\sum_{k=0}^{(p-1)/2}\frac{k}{(-4)^k}\binom{2k+d}{k}\equiv$$
$$\equiv(-2)^{d-2}\left(\frac{2}{p}\right)\left(P_{d+1-\left(\frac{2}{p}\right)}-\frac{d+1}{2}Q_{d+1-\left(\frac{2}{p}\right)}\right) \pmod p,$$

where $d\in\{0,1,\dots,(p-1)/2\}$.

From Corollary 1 and Corollary 2, clearly the congruences are given as follows:

Corollary 3. *Let p be an odd prime and $d\in\{0,1,2,\dots,(p-1)/2\}$. For $m\in\mathbb{Z}$ with $p\nmid m\Delta$, then*

$$\sum_{k=0}^{(p-1)/2}\frac{B(k,d)}{m^k}\equiv$$
$$\equiv m^{\lfloor(d+1)/2\rfloor-(p-1)/2-\varepsilon}\left\{u_{(p-1)/2+\lfloor(d+1)/2\rfloor-\varepsilon}(m-2)-u_{(p+1)/2+\lfloor(d+1)/2\rfloor}(m-2)\right\}$$
$$-m^{\lfloor(d+1)/2\rfloor}(u_{\lfloor(d-1)/2\rfloor}(m-2)-u_{\lfloor(d-1)/2\rfloor+2-\varepsilon}(m-2))-\delta_{0,d} \pmod p,$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{k}{m^k}B(k,d)\equiv$$
$$\equiv\frac{m^{\lfloor(d+1)/2\rfloor}}{\Delta}d\left\{m^{1-\varepsilon}\left(v_{\lfloor(d+1)/2\rfloor-\varepsilon}(m-2)-v_{\lfloor(d+1)/2\rfloor+1}(m-2)\right)\right.$$
$$\left.-m^{-(p-1)/2}\left(v_{(p+1)/2+\lfloor(d-1)/2\rfloor}(m-2)-v_{(p+1)/2+\lfloor(d+1)/2\rfloor+1-\varepsilon}(m-2)\right)\right\} \pmod p,$$

where ε,Δ as before and $\delta_{i,j}$ is the Kronecker delta.

Proof. Using the binomial identities $\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}$ and $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, we get

$$\sum_{k=0}^{(p-1)/2}\frac{B(k,d)}{m^k}=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\frac{d}{2k+d}\binom{2k+d}{k}=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\frac{d}{k}\binom{2k+d-1}{k-1}.$$

For $k,d\in\mathbb{Z}^+$, it is known that

$$\binom{2k+d-1}{k}=\binom{2k+d-1}{k-1}+\frac{d}{k}\binom{2k+d-1}{k-1}.$$

So, we have

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2}\frac{B(k,d)}{m^k}\\
&=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\left(\binom{2k+d-1}{k}-\binom{2k+d-1}{k-1}\right)\\
&=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k}-\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k-1}\\
&=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k}-\sum_{k=-1}^{(p-3)/2}\frac{1}{m^{k+1}}\binom{2k+d+1}{k}\\
&=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k}-\frac{1}{m}\sum_{k=0}^{(p-3)/2}\frac{1}{m^k}\binom{2k+d+1}{k}-\binom{d-1}{-1}\\
&=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k}-\frac{1}{m}\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d+1}{k}-\delta_{0,d}\\
&\quad+\frac{1}{m^{(p+1)/2}}\binom{p+d}{(p-1)/2}.
\end{aligned}$$

Since $\binom{p+d}{(p-1)/2}\equiv 0 \pmod p$ for $d\in\{0,1,2,...,(p-1)/2\}$, from (1.1), we get

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2}\frac{B(k,d)}{m^k}\equiv\\
&\equiv\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k}-\frac{1}{m}\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d+1}{k}-\delta_{0,d}\pmod p.
\end{aligned}$$

So, taking $d-1$ and $d+1$ instead of d in (2.4), respectively, this concludes the proof. □

For example, for an odd prime $p \neq 5$

$$\sum_{k=0}^{(p-1)/2} (-1)^k B(k,d) \equiv (-1)^{d+1} \left(\frac{5}{p}\right) L_{d-\left(\frac{5}{p}\right)} - \delta_{0,d} \pmod{p},$$

and for an odd prime p,

$$\sum_{k=0}^{(p-1)/2} \frac{k}{(-4)^k} B(k,d) \equiv -(-2)^{d-1} \left(\frac{2}{p}\right) dP_{d+1-\left(\frac{2}{p}\right)} \pmod{p},$$

where $d \in \{0,1,2,\ldots,(p-1)/2\}$.

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