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# PRIMARY-LIKE SUBMODULES AND A SCHEME OVER THE PRIMARY-LIKE SPECTRUM OF MODULES

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*Abstract.* Let $R$ be a commutative ring with identity and $M$ be a unitary $R$-module. In this paper, we obtain a scheme $(\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$ over the primary-like spectrum $\mathcal{X}(M)$ of $M$ and prove that $(\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$ is a Noetherian scheme when $R$ is a Noetherian ring.

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## 1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. For a submodule $N$ of an $R$-module $M$, $(N : M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and the annihilator of $M$, denoted by $Ann(M)$, is the ideal $(0 : M)$. An $R$-module $M$ is called faithful if $Ann(M) = (0)$. A submodule $P$ of an $R$-module $M$ is said to be $p$-prime if $P \neq M$ and for $p = (P : M)$, whenever $rm \in P$ (where $r \in R$ and $m \in M$) then $m \in P$ or $r \in p$ [7, 11]. The collection of all prime submodules of $M$ is denoted by $Spec(M)$. If $N$ is a submodule of $M$, then the radical of $N$, denoted by $\operatorname{rad} N$, is the intersection of all prime submodules of $M$ containing $N$, unless no such primes exist, in which case $\operatorname{rad} N = M$ [8].

A submodule $Q$ of $M$ is said to be primary-like if $Q \neq M$ and whenever $rm \in Q$ (where $r \in R$ and $m \in M$) implies $r \in (Q : M)$ or $m \in \operatorname{rad} Q$ [4]. An $R$-module $M$ is said to be primeful or $\psi$-module if either $M = (0)$ or $M \neq (0)$ and the map $\psi : Spec(M) \to Spec(R/Ann(M))$, defined by $\psi(P) = (P : M)/Ann(M)$ is surjective [10]. If $M/N$ is a $\psi$-module over $R$, then $\sqrt{(N : M)} = (\operatorname{rad} N : M)$ [10, Proposition 5.3]. It is easily seen that, if $Q$ is a primary-like submodule of $M$ such that $M/Q$ is a $\psi$-module over $R$, then $(Q : M)$ is a primary ideal of $R$ and so $p = \sqrt{(Q : M)}$ is a prime ideal of $R$ [4, Lemma 2.1], and in this case $Q$ is called a $p$-primary-like submodule of $M$. The primary-like spectrum of $M$ denoted by $\mathcal{X}(M)$ is defined to be the set of all primary-like submodules $Q$ of $M$, where $M/Q$ is a $\psi$-module.

An $R$-module $M$ is said to be a $\phi$-module if either $M = (0)$ or $M \neq (0)$ and the map $\phi : \mathcal{X}(M) \to Spec(R/Ann(M))$ defined by $\phi(Q) = \sqrt{(Q:M)}/Ann(M)$ is surjective. If $M$ is a $\phi$-module and $p$ is a prime ideal of $R$ containing $Ann(M)$, then there exists $Q \in \mathcal{X}(M)$ such that $\psi(S_p(Q + pM)) = \phi(Q) = p/Ann(M)$, where $S_p(Q + pM) = \{m \in M \mid cm \in Q + pM$ for some $c \in R \setminus p\}$ is the saturation of $Q + pM$ in $M$ with respect to $p$. Thus every $\phi$-module is a $\psi$-module; but the following example shows that the converse is not true.

*Example* 1 (cf. [10, Example 1]). Let $\Omega$ be the set of all prime integers, $M = \prod_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$ and $M' = \bigoplus_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$, where $p$ runs through $\Omega$. Hence $M$ is a faithful $\psi$-module over $\mathbb{Z}$ and $Spec(M) = \{M' = S_0(0)\} \cup \{pM : p \in \Omega\}$. Now if $\phi$ is surjective, then there exists $N \in \mathcal{X}(M)$ such that $\phi(N) = \sqrt{(N:M)} = 0$. It follows that $(N : M) = 0$. Since $M/N$ is a $\psi$-module, we have $N \subseteq \bigcap_{p\in\Omega} pM = 0$. But 0 is not prime and so is not primary-like because $\mathrm{rad}\, 0 = 0$. Hence $N \notin \mathcal{X}(M)$, a contradiction. Thus $M$ is not a $\phi$-module.

The Zariski topology on the spectrum of prime ideals of a commutative ring is one of the main tools in algebraic geometry. Recall that the spectrum $Spec(R)$ of a ring $R$ consists of all prime ideals of $R$ and is non-empty. For each ideal $I$ of $R$, we set $V(I)$ (or $V^R(I)) = \{p \in Spec(R) \mid p \supseteq I\}$). Then the sets $V(I)$, where $I$ is an ideal of $R$, satisfy the axioms for the closed sets of a topology on $Spec(R)$, called the Zariski topology (for example see [3]). It is well-known that for any ring $R$, there is a sheaf of rings on $Spec(R)$, called the structure sheaf, denoted by $\mathcal{O}_{Spec(R)}$, defined as follows: for each prime ideal $p$ of $R$, let $R_p$ be the localization of $R$ at $p$. For an open set $U \subseteq Spec(R)$ with respect to the Zariski topology, we define $\mathcal{O}_{Spec(R)}(U)$ to be the set of functions $r : U \to \coprod_{p\in U} R_p$, such that $r(p) \in R_p$, for each $p \in U$, and such that $r$ is a quotient of elements of $R$ locally: to be precise, we require that for each $p \in U$, there is a neighborhood $V$ of $p$, contained in $U$, and there are elements $a, s \in R$, such that for each $p' \in V$, $s \notin p'$ and $r(p') = \frac{a}{s}$ in $R_{p'}$ (see for example [5], for definition and basic properties of the sheaf $\mathcal{O}_{Spec(R)}$).

In the literature, there are many different generalizations of the Zariski topology for modules over commutative rings. For example, Lu has introduced a Zariski topology on $Spec(M)$ whose closed sets are $V(N) = \{P \in Spec(M) \mid (N : M) \subseteq (P : M)\}$ for any submodule $N$ of $M$ [9]. This topological space has been investigated from several point of views (see for example [1, 2, 6, 12]).

As a new generalization of the Zariski topology, the Zariski topology $\mathcal{T}$ on $\mathcal{X}(M)$ is a topology in which closed sets are of the form $\nu(N) = \{Q \in \mathcal{X}(M) \mid \sqrt{(N:M)} \subseteq \sqrt{(Q:M)}\}$ (Lemma 1). There are various generalizations of sheaves from rings to modules in which the sheaf on $Spec(M)$ is the set of all functions $r : Spec(M) \to \coprod_{p\in U} M_p$ with the property similar to that for $Spec(R)$ (some of these types of sheaves have been given and studied in [6, 12]). In parallel, we introduce a sheaf $\mathbb{O}_{\mathcal{X}(M)}$ over $\mathcal{X}(M)$.

We show that the set $\mathcal{B} = \{\mathcal{X}_r : r \in R\}$, where $\mathcal{X}_r = \mathcal{X}(M) - \nu(rM)$ is a basis for the Zariski topology over $\mathcal{X}(M)$ (Lemma 5). In particular, if $M$ is a $\phi$-module, then the elements $\mathcal{X}_r$ of $\mathcal{B}$ are quasi-compact (Corollary 3). This basis is used to show that $\mathbb{O}_{\mathcal{X}(M)}(\mathcal{X}_s) \cong R_s$ for each $s \in R$, where $M$ is a faithful $\phi$-module and $R_s = \{\frac{a}{s^n} : a \in R, n \in \mathbb{N}\}$ (Theorem 4). Finally we show that if $M$ is a $\phi$-module over a Noetherian ring $R$ and $\mathcal{X}(M)$ is a $T_0$-space, then $(\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$ is a Noetherian scheme (Theorem 5).

## 2. The Zariski topology on $\mathcal{X}(M)$

We begin with a lemma to see that the sets $\nu(N) = \{Q \in \mathcal{X}(M) \mid \sqrt{(N : M)} \subseteq \sqrt{(Q : M)}\}$ satisfy the axioms of closed sets for a topology.

**Lemma 1.** *Let $M$ be an $R$-module. Then*

(1) $\nu(0) = \mathcal{X}(M)$ *and* $\nu(M) = \varnothing$.
(2) $\bigcap_{i \in I} \nu(N_i) = \nu(\sum_{i \in I}(N_i : M)M)$, *for each family* $\{N_i \mid i \in I\}$ *of submodules of* $M$.
(3) $\nu(N) \cup \nu(N') = \nu(N \cap N')$, *for each pair* $N$, $N'$ *of submodules of* $M$.

*Proof.* (1) and (3) are trivial.
(2) Since $M/Q$ is a $\psi$-module, $(\operatorname{rad} Q : M) = \sqrt{(Q : M)}$ [10, Proposition 5.3]. Also it is easily verified that $((\operatorname{rad} Q : M)M : M) = (\operatorname{rad} Q : M))$. Using these facts we have the following implications.

$$
\begin{aligned}
Q \in \bigcap_{i \in I} \nu(N_i) &\Rightarrow \sqrt{(Q : M)} \supseteq \sum_{i \in I}(N_i : M) \\
&\Rightarrow \sqrt{(Q : M)}M \supseteq (\sum_{i \in I}(N_i : M))M \\
&\Rightarrow (\sqrt{(Q : M)}M : M) \supseteq ((\sum_{i \in I}(N_i : M))M : M) \\
&\Rightarrow ((\operatorname{rad} Q : M)M : M) \supseteq ((\sum_{i \in I}(N_i : M))M : M) \\
&\Rightarrow (\operatorname{rad} Q : M) \supseteq ((\sum_{i \in I}(N_i : M))M : M) \\
&\Rightarrow \sqrt{(Q : M)} \supseteq \sqrt{((\sum_{i \in I}(N_i : M))M : M)} \\
&\Rightarrow Q \in \nu((\sum_{i \in I}(N_i : M))M).
\end{aligned}
$$

For the reverse inclusion we have

$$\begin{aligned} Q \in \nu(\sum_{i\in I}(N_i : M)M) &\Rightarrow \sqrt{(Q:M)} \supseteq ((\sum_{i\in I}(N_i:M))M : M) \\ &\Rightarrow \sqrt{(Q:M)} \supseteq ((N_i:M)M:M) \ \ \forall i \in I \\ &\Rightarrow \sqrt{(Q:M)} \supseteq (N_i:M) \ \ \forall i \in I \\ &\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{(N_i:M)} \ \ \forall i \in I \\ &\Rightarrow Q \in \bigcap_{i\in I} \nu(N_i) \end{aligned}$$

□

We will use $\overline{R}$ and $X^{\overline{R}}$ to represent $R/Ann(M)$ and $Spec(R/Ann(M))$ respectively.

**Proposition 1.** *Let $M$ be an $R$-module. Then $\phi^{-1}(V^{\overline{R}}(\overline{I})) = \nu(IM)$, for every ideal $I \in V(Ann(M))$. Therefore the map $\phi$ is continuous with respect to the Zariski topology on $\mathcal{X}(M)$.*

*Proof.* Suppose $Q \in \phi^{-1}(V^{\overline{R}}(\overline{I}))$. Then $\phi(Q) \in V^{\overline{R}}(\overline{I})$ and so $\sqrt{(Q:M)} \supseteq I$. Hence $\sqrt{(Q:M)} \supseteq \sqrt{(IM:M)}$. Thus $Q \in \nu(IM)$. The argument is reversible and so $\phi$ is continuous. □

**Theorem 1.** *Let $M$ be a $\phi$-module over a ring $R$. Then $\phi(\nu(N)) = V^{\overline{R}}(\overline{(N:M)})$ and $\phi(\mathcal{X}(M) - \nu(N)) = X^{\overline{R}} - V^{\overline{R}}(\overline{(N:M)})$ for every submodule $N$ of $M$, i.e., $\phi$ is both closed and open.*

*Proof.* As we have seen in Proposition 1, $\phi^{-1}(V^{\overline{R}}(\overline{I})) = \nu(IM)$, for every ideal $I \in V(Ann(M))$. Hence for every submodule $N$ of $M$, $\phi^{-1}(V^{\overline{R}}(\overline{(N:M)})) = \nu((N:M)M) = \nu(N)$. So $\phi(\nu(N)) = \phi \circ \phi^{-1}(V^{\overline{R}}(\overline{(N:M)})) = V^{\overline{R}}(\overline{(N:M)})$ as $\phi$ is surjective. Thus

$$\phi(\mathcal{X}(M) - \nu(N)) = \phi(\phi^{-1}(X^{\overline{R}}) - \phi^{-1}(V^{\overline{R}}(\overline{(N:M)}))) = X^{\overline{R}} - V^{\overline{R}}(\overline{(N:M)})$$

□

**Corollary 1.** *Let $M$ be an $R$-module. Then $\phi$ is a bijection if and only if $\phi$ is a homeomorphism.*

**Proposition 2.** *Let $M$ be an $R$-module and $Q$, $Q' \in \mathcal{X}(M)$. Then the following statements are equivalent.*

(1) *If $\nu(Q) = \nu(Q')$, then $Q = Q'$;*
(2) *For each $p \in Spec(R)$, the set $\{Q \in \mathcal{X}(M) : \sqrt{(Q:M)} = p\}$ is empty or a singleton set;*
(3) *$\phi$ is injective.*

*Proof.* (1) $\Rightarrow$ (2) Let $Q$, $Q' \in \mathcal{X}(M)$ and $\sqrt{(Q:M)} = \sqrt{(Q':M)} = p$. Then $\nu(Q) = \nu(Q')$. Thus $Q = Q'$ by (1).
(2) $\Rightarrow$ (3) Suppose $Q$, $Q' \in \mathcal{X}(M)$ and $\phi(Q) = \phi(Q')$. Hence $\sqrt{(Q:M)} = \sqrt{(Q':M)} = p$. Thus $Q = Q'$ by (2).
(3) $\Rightarrow$ (1) is clear. □

Let $\mathcal{Y}$ be a subset of $\mathcal{X}(M)$ for a module $M$. We will denote the closure of $\mathcal{Y}$ in $\mathcal{X}(M)$ by $\overline{\mathcal{Y}}$.

**Proposition 3.** *Let $M$ be an $R$-module and let $\mathcal{Y} = \{Q_1, Q_2, ..., Q_n\}$ be a finite subset of $\mathcal{X}(M)$. Then $\overline{\mathcal{Y}} = \nu(Q_1) \cup ... \cup \nu(Q_n)$.*

*Proof.* Clearly, $\mathcal{Y} \subseteq \nu(Q_1) \cup ... \cup \nu(Q_n)$. Assume that $\mathcal{F}$ is any closed subset of $\mathcal{X}(M)$ such that $\mathcal{Y} \subseteq \mathcal{F}$. Hence $\mathcal{F} = \nu(N)$ for the submodule $N$ of $M$. Let $Q \in \nu(Q_1) \cup ... \cup \nu(Q_n)$. Then there exists $i$ $(1 \leq i \leq n)$ such that $Q \in \nu(Q_i)$ and so $\sqrt{(Q_i:M)} \subseteq \sqrt{(Q:M)}$. Since $Q_i \in \mathcal{F}$, $\sqrt{(N:M)} \subseteq \sqrt{(Q_i:M)} \subseteq \sqrt{(Q:M)}$ and hence $Q \in \mathcal{F}$. Hence $\nu(Q_1) \cup ... \cup \nu(Q_n) \subseteq \mathcal{F}$. Thus $\overline{\mathcal{Y}} = \nu(Q_1) \cup ... \cup \nu(Q_n)$. □

The above proposition immediately yields that the following result.

**Corollary 2.** *Let $M$ be an $R$-module. Then*

(1) *$\overline{Q} = \nu(Q)$ for every $Q \in \mathcal{X}(M)$.*
(2) *$Q' \in \overline{Q}$ if and only if $\sqrt{(Q':M)} \supseteq \sqrt{(Q:M)}$ if and only if $\nu(Q') \subseteq \nu(Q)$.*

*Proof.* By Proposition 3 is clear. □

A topological space $\mathbb{X}$ is a $T_0$-space if and only if for any two distinct points in $\mathbb{X}$ there exists an open subset of $\mathbb{X}$ which contains one of the points but not the other. We know that, for any ring $R$, $Spec(R)$ is always a $T_0$-space for the usual Zariski topology. In [9, P. 429], it has been shown that if $M$ is a vector space, then $(Spec(M),)$ is not a $T_0$-space. This example can be used again to show that $(\mathcal{X}(M), \mathcal{T})$ is not also a $T_0$-space. In fact $\nu(N) = \mathcal{X}(M)$ for every proper vector subspace $N$ of $M$ so that the Zariski topology on $\mathcal{X}(M)$ is the trivial topology even when $|\mathcal{X}(M)| > 1$.

**Theorem 2.** *Let $M$ be an $R$-module. Then $\mathcal{X}(M)$ is a $T_0$-space if and only if one of the statements $(1)-(3)$ in Proposition 2 holds.*

*Proof.* First suppose $\mathcal{X}(M)$ is a $T_0$-space. We prove the item(1) of proposition 2. For this assume $\nu(Q) = \nu(Q')$ and $Q \neq Q'$. Since $\mathcal{X}(M)$ is a $T_0$-space, $\overline{Q} \neq \overline{Q'}$. Thus by Corollary 2 we have $\nu(Q) \neq \nu(Q')$, a contradiction. Conversely, suppose that $Q \neq Q' \in \mathcal{X}(M)$ and $\nu(Q) \neq \nu(Q')$. Therefore by Corollary 2, $\overline{Q} \neq \overline{Q'}$. Thus $\mathcal{X}(M)$ is a $T_0$-space. □

For each $r \in R$, we set $\mathcal{X}_r = \mathcal{X}(M) - \nu(rM)$ and $D_{\overline{r}} = X^{\overline{R}} - V(\overline{R}\overline{r})$. It is easily seen that $\mathcal{X}_{0_R} = \varnothing$, $\mathcal{X}_{1_R} = \mathcal{X}(M)$.

**Lemma 2.** *Let $M$ be an $R$-module. Then $\phi(\mathcal{X}_r) \subseteq D_{\overline{r}}$; the equality holds if $M$ is a $\phi$-module.*

*Proof.* By Proposition 1, $\phi^{-1}(D_{\overline{r}}) = \phi^{-1}(X^{\overline{R}} - V(\overline{R}\overline{r})) = \mathcal{X}(M) - \nu(rM) = \mathcal{X}_r$. The equality follows form Theorem 1. □

**Lemma 3.** *Let $r, s \in R$. Then the following hold.*

(1) $\mathcal{X}_{rs} = \mathcal{X}_r \cap \mathcal{X}_s$.
(2) $\mathcal{X}_{r^n} = \mathcal{X}_r$ *for all* $n \in \mathbb{N}$.
(3) *If $r$ is nilpotent, then* $\mathcal{X}_r = \varnothing$.

*Proof.* (1) By Proposition 1, $\mathcal{X}_{rs} = \phi^{-1}(D_{\overline{rs}})$. Hence $\mathcal{X}_{rs} = \phi^{-1}(D_{\overline{r}}) \cap \phi^{-1}(D_{\overline{s}})$ $= \mathcal{X}_r \cap \mathcal{X}_s$.
(2) follows from (1).
(3) Since $r$ is nilpotent, $r^n = 0$ for some $n \in \mathbb{N}$. Hence by (2), $\mathcal{X}_r = \mathcal{X}_{r^n} = \mathcal{X}_0 = \varnothing$. □

**Lemma 4.** *Let $r, s \in R$ and $M$ be a faithful $\phi$-module over $R$. If $\mathcal{X}_s \subseteq \mathcal{X}_r$, then $s \in \sqrt{Rr}$.*

*Proof.* Suppose $\mathcal{X}_s \subseteq \mathcal{X}_r$. Hence $\phi(\mathcal{X}_s) \subseteq \phi(\mathcal{X}_r)$. Since $M$ is a $\phi$-module, $D_{\overline{s}} \subseteq D_{\overline{r}}$ by Lemma 2. Now since $M$ is faithful, $D_s \subseteq D_r$. Thus we have $s \in \sqrt{Rr}$. □

**Lemma 5.** *Let $M$ be an $R$-module. Then the set $\mathcal{B} = \{\mathcal{X}_r : r \in R\}$ forms a basis for the Zariski topology on $\mathcal{X}(M)$.*

*Proof.* If $\mathcal{X}(M) = \varnothing$, then $\mathcal{B} = \varnothing$ and the proposition is trivially true. Hence we assume that $\mathcal{X}(M) \neq \varnothing$ and let $\mathcal{U}$ be any open set in $\mathcal{X}(M)$. Hence $\mathcal{U} = \mathcal{X}(M) - \nu(IM)$ for some ideal $I$ of $R$. Note that $\nu(IM) = \nu(\sum_{a_i \in I} a_i M) = \nu(\sum_{a_i \in I}(a_i M : M)M) = \cap_{a_i \in I} \nu(a_i M)$. Hence $\mathcal{U} = \mathcal{X}(M) - \cap_{a_i \in I} \nu(a_i M) = \cup_{a_i \in I} \mathcal{X}_{a_i}$. This proves that $\mathcal{B}$ is a basis for the Zariski topology on $\mathcal{X}(M)$. □

**Theorem 3.** *Let $M$ be a $\phi$-module over a ring $R$. Then $\mathcal{X}_r$ and $\mathcal{X}_{r_1} \cap \ldots \cap \mathcal{X}_{r_n}$ are quasi-compact subsets of $\mathcal{X}(M)$.*

*Proof.* For any open covering of $\mathcal{X}_r$, there is a family $\{r_\lambda \in R : \lambda \in \Lambda\}$ of elements of $R$ such that $\mathcal{X}_r \subseteq \cup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}$ by Lemma 5. $D_{\overline{r}} = \phi(\mathcal{X}_r) \subseteq \bigcup_{\lambda \in \Lambda} \phi(\mathcal{X}_{r_\lambda}) = \cup_{\lambda \in \Lambda} D_{\overline{r}_\lambda}$ by Proposition 2. It follows that there exists a finite subset $\Lambda'$ of $\Lambda$ such that $D_{\overline{r}} \subseteq \cup_{\lambda \in \Lambda} D_{\overline{r}_\lambda}$ as $D_{\overline{r}}$ is quasi-compact, whence by Proposition 2, $\mathcal{X}_r = \phi^{-1}(D_{\overline{r}}) \subseteq \cup_{\lambda \in \Lambda'} \mathcal{X}_{r_\lambda}$. Thus $\mathcal{X}_r$ is quasi-compact. For the other part, it suffices to show that the intersection $\mathcal{X}_{r_1} \cap \mathcal{X}_{r_2}$ is a quasi-compact set. Let $\Omega$ be any open covering of $\mathcal{X}_{r_1} \cap \mathcal{X}_{r_2}$. Then $\Omega$ also covers each $\mathcal{X}_{r_i}$ $(i = 1, 2)$ which is quasi-compact. Hence each $\mathcal{X}_{r_i}$ has a finite subcover and so $\mathcal{X}_{r_1} \cap \mathcal{X}_{r_2}$ has a finite subcover. □

**Corollary 3.** *Let $M$ be a $\phi$-module over a ring $R$. Then $\mathcal{X}(M)$ is quasi-compact and has a basis of quasi-compact open subsets.*

## 3. SHEAF, LOCALLY RINGED SPACE AND SCHEME

Let $M$ be an $R$-module. For every open subset $\mathcal{U}$ of $\mathcal{X}(M)$ we define $\mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$ to be a subring of $\prod\{R_p \mid p = \sqrt{(Q:M)}, Q \in \mathcal{U}\}$, the ring of functions $r : \mathcal{U} \to \coprod\{R_p \mid p = \sqrt{(Q:M)}, Q \in \mathcal{U}\}$, where $r(Q) \in R_p$, for each $Q \in \mathcal{U}$ and $p = \sqrt{(Q:M)}$ such that for each $Q \in \mathcal{U}$, there is a neighborhood $\mathcal{V}$ of $Q$ , contained in $\mathcal{U}$ , and elements $s, t \in R$, such that for each $Q' \in \mathcal{V}$, $t \notin p' = \sqrt{(Q':M)}$, and $r(Q') = \frac{s}{t}$ in $R_{p'}$. It is easy to check that $\mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$ is a commutative ring with identity. Furthermore, for open sets $\mathcal{V} \subseteq \mathcal{U}$ we define $\vartheta_{\mathcal{U},\mathcal{V}} : \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U}) \to \mathbb{O}_{\mathcal{X}(M)}(\mathcal{V})$ given by $\{r_p\}_{Q\in\mathcal{U}} \mapsto \{r'_{p'}\}_{Q'\in\mathcal{V}}$, where $p = \sqrt{(Q:M)}$ and $p' = \sqrt{(Q':M)}$. It is easy to check that $\mathbb{O}_{\mathcal{X}(M)}$ is a sheaf of rings.
For any sheaf $\mathbb{O}$ on a topological space $\mathbb{X}$ and for any $x \in \mathbb{X}$, the stalk of $\mathbb{O}$ at $x$, denoted by $\mathbb{O}_x$, is $\mathbb{O}_x = \{m \mid$ there exists a neighborhood $\mathbb{U}$ of $x$ and $r \in \mathbb{O}_{\mathbb{X}}(\mathbb{U})$ such that $m$ is the germ of $r$ at the point $x\}$. We say that $m$ is the germ of $r$ at the point $x$ if there exists a neighborhood $\mathbb{V}$ containing $x$ such that $\mathbb{V} \subseteq \mathbb{U}$ and $\vartheta_{\mathcal{U},\mathcal{V}}(r) = m$. Two such pairs $< \mathbb{U}, r >$ and $< \mathbb{V}, s >$ define the same element for $m$ of $\mathbb{O}_x$ if and only if there is an open neighborhood $\mathbb{W}$ of $x$ with $\mathbb{W} \subseteq \mathbb{U} \cap \mathbb{V}$ such that $x \in \mathbb{W}$ $r|_{\mathbb{W}} = s|_{\mathbb{W}}$.

**Lemma 6.** *Let $M$ be an $R$-module. Then for each $Q \in \mathcal{X}(M)$, the stalk $\mathbb{O}_Q$ of the sheaf $\mathbb{O}_{\mathcal{X}(M)}$ is isomorphic to $R_p$, where $p = \sqrt{(Q:M)}$.*

*Proof.* Assume $Q \in \mathcal{X}(M)$ and $m \in \mathbb{O}_Q$. Therefore there exists a neighborhood $\mathcal{U}$ of $Q$ and $r \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$ such that $m$ is the germ of $r$ at the point $Q$. For $p = \sqrt{(Q:M)}$ we define $\mu : \mathbb{O}_Q \to R_p$ given by $m \mapsto r(Q)$. It is easy to check that $\mu$ is a well-defined homomorphism. Suppose $\mathcal{V}$ is another neighborhood of $Q$ and $s \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{V})$ such that $m$ is the germ of $s$ at the point $Q$. Hence there is an open neighborhood $\mathcal{W}$ of $Q$ with $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ such that $r|_{\mathcal{W}} = s|_{\mathcal{W}}$. Since $Q \in \mathcal{W}$, then $r(Q) = s(Q)$. The map $\mu$ is surjective, because any element of $R_p$ can be represented as a quotient $\frac{a}{s}$ with $a \in R$ and $s \in R\backslash p$. Now we define $r(Q') = \frac{a}{s}$ in $R_{p'}$, where $p' = \sqrt{(Q':M)}$ for all $Q' \in \mathcal{X}_s$ . Then $r \in \mathbb{O}(\mathcal{X}_s)$. If $m$ is the equivalent class of $r$ in $\mathbb{O}_Q$, then $\mu(m) = \frac{a}{s}$. To show that $\mu$ is injective, let $\mathcal{U}$ be a neighborhood of $Q$, and let $r, r' \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$ be elements having the same value $r(Q) = r'(Q)$ at $Q$. By the definition of our sheaf, we may assume that $r = \frac{a}{s}$ and $r' = \frac{a'}{s'}$, where $a, a' \in R$ and $s, s' \in R\backslash p$. Since $\frac{a}{s}$ and $\frac{a'}{s'}$ have the same image in $R_p$, it follows from the definition of localization that there is an $s'' \in R\backslash p$ such that $s''(s'a - sa') = 0$ in $R$. Therefore $\frac{a}{s} = \frac{a'}{s'}$ in every local ring $R_{p'}$ such that $s, s', s'' \in R\backslash p'$. But the set of such $Q'$, where $p' = \sqrt{(Q':M)}$ is the open set $\mathcal{X}_s \cap \mathcal{X}_{s'} \cap \mathcal{X}_{s''}$, which contains $Q$. Hence $r = r'$ in a whole neighborhood of $Q$, so they have the same stalk at $Q$. □

A locally ringed space $(\mathbb{X}, \mathbb{O}_{\mathbb{X}})$ is a pair consisting of a topological space $\mathbb{X}$ and a sheaf of rings $\mathbb{O}_{\mathbb{X}}$ all of whose stalks are local rings.

**Corollary 4.** *Let $M$ be an $R$-module. Then $(\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$ is a locally ringed space.*

*Proof.* Use Lemma 6. □

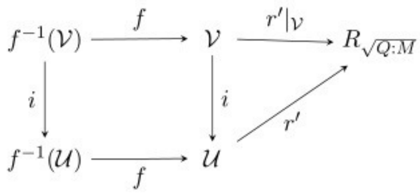
Let $(\mathbb{X}, \mathbb{O}_{\mathbb{X}})$ be a locally ringed space. The stalk $\mathbb{O}_{\mathbb{X},x}$ of $\mathbb{X}$ at $x$ is said to be the local ring of $\mathbb{X}$ at $x$. A morphism of ringed spaces $(f, f^{\sharp}) : (\mathbb{X}, \mathbb{O}_{\mathbb{X}}) \to (\mathbb{Y}, \mathbb{O}_{\mathbb{Y}})$ is given by a continuous map $f : \mathbb{X} \to \mathbb{Y}$ and an $f$-map of sheaves of rings $f^{\sharp} : \mathbb{O}_{\mathbb{Y}} \to \mathbb{O}_{\mathbb{X}}$. You can think of $f^{\sharp}$ as a map $\mathbb{O}_{\mathbb{Y}} \to f_*\mathbb{O}_{\mathbb{X}}$, where $f_*\mathbb{O}_{\mathbb{X}}$ is a sheaf over $\mathbb{X}$ defined by $f_*\mathbb{O}_{\mathbb{X}}(\mathbb{V}) = \mathbb{O}_{\mathbb{X}}(f^{-1}(\mathbb{V}))$ for any open subset $\mathbb{V} \subseteq \mathbb{Y}$. Moreover the restriction map on an inclusion of open sets of $\mathbb{Y}$ is defined naturally. A morphism of locally ringed spaces $(f, f^{\sharp}) : (\mathbb{X}, \mathbb{O}_{\mathbb{X}}) \to (\mathbb{Y}, \mathbb{O}_{\mathbb{Y}})$ is a morphism of ringed spaces such that for all $x \in \mathbb{X}$ the induced ring map $\mathbb{O}_{\mathbb{Y},f(x)} \to \mathbb{O}_{\mathbb{X},x}$ is a local ring map.

**Proposition 4.** *Let $M$ and $M'$ be $R$-modules and $\pi : M \to M'$ be an isomorphism of modules. Then $\pi$ induces a morphism of locally ringed spaces $(f, f^{\sharp}) : (\mathcal{X}(M'), \mathbb{O}_{\mathcal{X}(M')}) \to (\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$.*

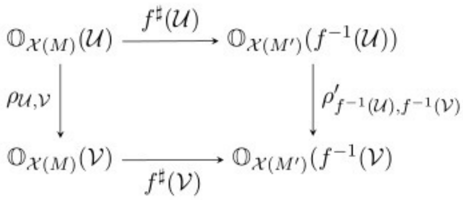
*Proof.* We define $f(Q') = \pi^{-1}(Q')$ For any $Q' \in \mathcal{X}(M')$. It is easily seen that $f$ is well-defined. In the following it is shown that $f^{-1}(\nu(N)) = \nu((N : M)M')$ for any closed set $\nu(N)$ of $\mathcal{X}(M)$ and so $f$ is continuous.

$$
\begin{aligned}
Q' \in f^{-1}(\nu(N)) &\Leftrightarrow f(Q') \in \nu(N) \\
&\Leftrightarrow \sqrt{(f(Q') : M)} \supseteq \sqrt{(N : M)} \\
&\Leftrightarrow \sqrt{(f(Q') : M)} \supseteq \sqrt{((N : M)M : M)} \\
&\Leftrightarrow \sqrt{(\pi^{-1}(Q') : M)} \supseteq \sqrt{((N : M)M : M)} \\
&\Leftrightarrow (rad(\pi^{-1}(Q')) : M)M \supseteq (N : M)M \\
&\Leftrightarrow \pi^{-1}(radQ') \supseteq (N : M)M \\
&\Leftrightarrow radQ' \supseteq (N : M)M' \\
&\Leftrightarrow Q' \in \nu((N : M)M').
\end{aligned}
$$

Assume $\mathcal{U}$ is an open subset of $\mathcal{X}(M)$ and $r \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$. Let $Q \in f^{-1}(\mathcal{U})$. Then $f(Q) = \pi^{-1}(Q) \in \mathcal{U}$. Assume that $\mathcal{W}$ is an open neighborhood of $\pi^{-1}(Q)$ with $\mathcal{W} \subseteq \mathcal{U}$ and $a, s \in R$ such that for each $Q' \in \mathcal{W}$ , $s \notin p' = \sqrt{(Q' : M)}$ and $r(Q') = \frac{a}{s}$ in $R_{p'}$. Since $\pi^{-1}(Q) \in \mathcal{W}$, then $Q \in f^{-1}(\mathcal{W})$. Since $f$ is continuous, $f^{-1}(\mathcal{W})$ is an open subset of $\mathcal{X}(M')$. We show that for each $Q'' \in f^{-1}(\mathcal{W})$ we have $s \notin \sqrt{(Q'' : M')}$. Suppose, on the contrary, $s \in \sqrt{(Q'' : M')}$ for some $Q'' \in f^{-1}(\mathcal{W})$. So $\pi^{-1}(Q'') = f(Q'') \in \mathcal{W}$. Since $\pi$ is an epimorphism, $\sqrt{(Q'' : M')} = \sqrt{(\pi^{-1}(Q'') : M)}$. Hence $s \in \sqrt{(\pi^{-1}(Q'') : M)}$, a contradiction. Therefore, we can define $f^{\sharp}(\mathcal{U}) : \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U}) \to \mathbb{O}_{\mathcal{X}(M')}(f^{-1}(\mathcal{U}))$ given by $f^{\sharp}(\mathcal{U})(r) = r \circ f$. Suppose $\mathcal{V} \subseteq \mathcal{U}$ and $Q \in f^{-1}(\mathcal{V})$. According to the commutativity of the following diagram:




We have $(r' \circ f)|_{f^{-1}(\mathcal{V})}(Q) = r'|_{\mathcal{V}} \circ f(Q)$. Now, we show that the following diagram commutes.




Suppose that $r' \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$. For each $Q \in \mathcal{U}$, we have

$$\rho'_{f^{-1}(\mathcal{U}),f^{-1}(\mathcal{V})} f^{\sharp}(\mathcal{U})(r')(Q) = \rho'_{f^{-1}(\mathcal{U}),f^{-1}(\mathcal{V})}(r' \circ f)(Q) =$$
$$(r' \circ f)|_{f^{-1}(\mathcal{V})}(Q) = r'|_{\mathcal{V}} \circ f(Q) = \rho_{\mathcal{U},\mathcal{V}}(r') \circ f(Q) = f^{\sharp}(\mathcal{V})\rho_{\mathcal{U},\mathcal{V}}(r')(Q).$$

It follows that $f^{\sharp} : \mathbb{O}_{\mathcal{X}(M)} \to f_* \mathbb{O}_{\mathcal{X}(M')}$ is a morphism of sheaves. By Lemma 6, the map $f^{\sharp}_Q : \mathbb{O}_{\mathcal{X}(M), f(Q)} \to \mathbb{O}_{\mathcal{X}(M'), Q}$ on stalks is clearly the map of local rings $R_{\sqrt{(f(Q):M)}} \to R_{\sqrt{(Q:M')}}$. Thus the proof is completed. □

**Proposition 5.** *Let $g : R \to R'$ be a ring homomorphism, $M'$ be an $R'$-module and $M$ be a $\phi$-module over $R$ such that $\mathcal{X}(M)$ is a $T_0$-space and $Ann_R(M) \subseteq Ann_R(M')$. Then $g$ induces a morphism of locally ringed spaces $(f, f^{\sharp})$: $(\mathcal{X}(M'), \mathbb{O}_{\mathcal{X}(M')}) \to (\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$.*

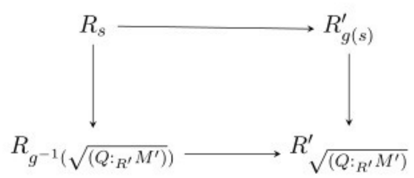
*Proof.* Since $Ann_R(M) \subseteq Ann_R(M')$, then $\overline{g} : \overline{R} \to \overline{R'}$ is induced by $g$. It is well known $h : Spec(R') \to Spec(R)$ given by $p \mapsto g^{-1}(p)$ and $\overline{h} : X^{\overline{R'}} \to X^{\overline{R}}$ given by $\overline{p} \mapsto \overline{g}^{-1}(\overline{p})$ are continuous maps. Also by Proposition 1, $\phi_{M'} : \mathcal{X}(M') \to X^{\overline{R'}}$ is a continuous map and by Corollary 1 and Theorem 2, $\phi_M : \mathcal{X}(M) \to X^{\overline{R}}$ is a homeomorphism. Therefore the map $f : \mathcal{X}(M') \to \mathcal{X}(M)$ given by $Q \mapsto \phi_M^{-1} \circ \overline{h} \circ \phi_{M'}(Q)$ is continuous. For each $Q \in \mathcal{X}(M')$, we get a local homomorphism

$$g_{\sqrt{(Q:_{R'}M')}} : R_{h(\sqrt{(Q:_{R'}M')})} \to R'_{\sqrt{(Q:_{R'}M')}}$$

given by $\frac{r}{s} \mapsto \frac{g(r)}{g(s)}$. This map is well-defined, because if $s \notin h(\sqrt{(Q :_{R'} M')}) = g^{-1}(\sqrt{(Q :_{R'} M')})$, then $g(s) \notin \sqrt{(Q :_{R'} M')}$. Let $\mathcal{U} \subseteq \mathcal{X}(M)$ be an open subset and $r \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$. Suppose $Q \in f^{-1}(\mathcal{U})$. Then $f(Q) \in \mathcal{U}$ and there exists a neighborhood $\mathcal{W}$ of $f(Q)$ with $\mathcal{W} \subseteq \mathcal{U}$ and elements $a, s \in R$ such that for each $Q' \in \mathcal{W}$, we have $s \notin \sqrt{(Q' :_R M)}$ and $r(Q') = \frac{a}{s} \in R_{\sqrt{(Q' :_R M)}}$. Hence $s \notin \sqrt{(f(Q) :_R M)}$. By definition of $f$, we have

$$\begin{aligned} f(Q) &= (\phi_M^{-1} \circ \bar{h} \circ \phi_{M'}^{-1})(Q) = (\phi_M^{-1} \circ \bar{h})(\overline{\sqrt{(Q :_{R'} M')}}) \\ &= \phi_M^{-1}(\bar{g}^{-1}(\overline{(\sqrt{(Q :_{R'} M')})})) = \phi_M^{-1}(\overline{g^{-1}(\sqrt{(Q :_{R'} M')})}) \\ &= K, \end{aligned}$$

for some $K \in \mathcal{X}(M)$. Now since $M$ is a $\phi$-module, $\overline{\sqrt{(K :_R M)}} = \phi_M(K) = \overline{g^{-1}(\sqrt{(Q :_{R'} M')})}$ and hence $\sqrt{(f(Q) :_R M)} = \sqrt{(K :_R M)} = g^{-1}(\sqrt{(Q :_{R'} M')})$. Therefore $s \notin \sqrt{(f(Q) :_R M)}$ follows that $g(s) \notin \sqrt{(Q :_{R'} M')}$. Thus $g_{\sqrt{(Q:_{R'}M')}}(\frac{a}{s})$ define a section on $\mathbb{O}_{\mathcal{X}(M')}(f^{-1}(\mathcal{W}))$. Since

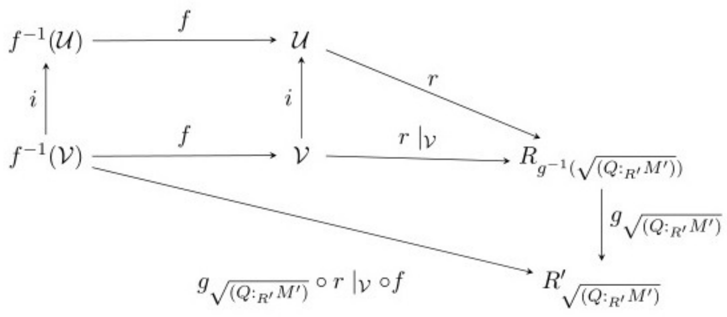


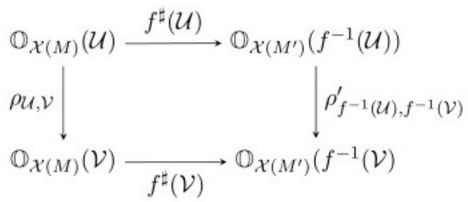

is a commutative diagram of natural maps, we define

$$f^{\sharp}(\mathcal{U}) : \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U}) \to f_*\mathbb{O}_{\mathcal{X}(M')}(\mathcal{U}) = \mathbb{O}_{\mathcal{X}(M')}(f^{-1}(\mathcal{U}))$$

which is given by $f^{\sharp}(\mathcal{U})(r)(Q) = g_{\sqrt{(Q:_{R'}M')}}(r(f(Q)))$ for each $r \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$ and $Q \in f^{-1}(\mathcal{U})$. Suppose $\mathcal{V} \subseteq \mathcal{U}$ and $Q \in f^{-1}(\mathcal{V})$. According to the following commutative diagram
We have $g_{\sqrt{(Q:_{R'}M')}} \circ r|_{\mathcal{V}} \circ f(Q) = (g_{\sqrt{(Q:_{R'}M')}} \circ r \circ f)|_{f^{-1}(\mathcal{V})}(Q)$. Considering the diagram
It is easy to check that

$$\begin{gathered} \rho'_{f^{-1}(\mathcal{U}), f^{-1}(\mathcal{V})} f^{\sharp}(\mathcal{U})(r)(Q) = \rho'_{f^{-1}(\mathcal{U}), f^{-1}(\mathcal{V})} g_{\sqrt{(Q:_{R'}M')}} r \circ f(Q) = \\ (g_{\sqrt{(Q:_{R'}M')}} r \circ f)|_{f^{-1}(\mathcal{V})}(Q) = g_{\sqrt{(Q:_{R'}M')}} r|_{\mathcal{V}} \circ f(Q) = f^{\sharp}(\mathcal{V})(r|_{\mathcal{V}})(Q) = \\ f^{\sharp}(\mathcal{V}) \rho_{\mathcal{U},\mathcal{V}}(r)(Q). \end{gathered}$$




Thus the diagram is commutative and it follows that $f^{\sharp} : \mathbb{O}_{\mathcal{X}(M)} \to f_*\mathbb{O}_{\mathcal{X}(M')}$ is a morphism of sheaves. By Lemma 6, the map $f_Q^{\sharp} : \mathbb{O}_{\mathcal{X}(M),f(Q)} \to \mathbb{O}_{\mathcal{X}(M'),Q}$ on stalks is clearly $R_{h(\sqrt{(Q:_{R'}M')})} \to R'_{\sqrt{(Q:_{R'}M')}}$. Thus the proof is completed. □

**Theorem 4.** *Let $s \in R$ and $M$ be a faithful $\phi$-module over a ring $R$. Then $\mathbb{O}_{\mathcal{X}(M)}(\mathcal{X}_s) \cong R_s$.*

*Proof.* Suppose $\mu : R_s \to \mathbb{O}_{\mathcal{X}(M)}(\mathcal{X}_s)$ given by $\frac{a}{s^n} \mapsto (r : Q \mapsto \frac{a}{s^n} \in R_{\sqrt{(Q:M)}})$. Indeed $\mu$ sends that $\frac{a}{s^n}$ to the section $r \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{X}_s)$ which assigns to each $Q$ the image of $\frac{a}{s^n} \in R_{\sqrt{(Q:M)}}$. It is clear that $\mu(\frac{a}{s^n})$ is unique, since the range of $r$ is $\frac{a}{s^n}$. Therefore to show that $\mu$ is well-defined, it suffices to verify that $s^n \notin \sqrt{(Q:M)}$. Since $Q \in \mathcal{X}_s = \mathcal{X}(M) - \nu(sM)$, we have $\sqrt{(sM:M)} \not\subseteq \sqrt{(Q:M)}$. Now if $s^n \in \sqrt{(Q:M)}$ (or equivalently $s \in \sqrt{(Q:M)}$), we have

$$\begin{aligned} r \in \sqrt{(sM:M)} &\Rightarrow r^n M \subseteq sM \subseteq \sqrt{(Q:M)}M \quad \text{for some } n > 0 \\ &\Rightarrow r^n \in (\sqrt{(Q:M)}M : M) = ((\mathrm{rad}(Q):M)M : M) \\ &\Rightarrow r^n \in (\mathrm{rad}(Q):M) = \sqrt{(Q:M)} \\ &\Rightarrow r \in \sqrt{(Q:M)} \end{aligned}$$

which gives the contradiction $\sqrt{(sM:M)} \subseteq \sqrt{(Q:M)}$. Moreover $\mu$ is a homomorphism, since $\mathbb{O}_{\mathcal{X}(M)}(\mathcal{X}_s)$ is a ring with the operations $(r_1+r_2)(Q) = r_1(Q)+r_2(Q)$ and $(r_1r_2)(Q) = r_1(Q)r_2(Q)$. Now we are going to show that $\mu$ is injective. Let $\mu(\frac{a}{s^n}) = \mu(\frac{a'}{s^m})$, then for every $Q \in \mathcal{X}_s$, $\frac{a}{s^n}$ and $\frac{a'}{s^m}$ have the same image in

$R_p$, where $p = \sqrt{(Q:M)}$. Thus there exists $t \in R \backslash p$ such that $t(s^m a - s^n a') = 0$. Let $I = Ann(s^m a - s^n a')$. Then $t \in I$ and $t \notin p$, so $I \not\subseteq p$. This happens for any $Q \in \mathcal{X}_s$. Hence we conclude that $V(I) \cap \{\sqrt{(Q:M)} \mid Q \in \mathcal{X}_s\} = \emptyset$ and so $\{\sqrt{(Q:M)} \mid Q \in \mathcal{X}_f\} \subseteq Spec(R) - V(I)$. Since $M$ is a $\phi$-module, by Lemma 2 we have

$$D_s = \{\sqrt{(Q:M)} \mid Q \in \mathcal{X}_s\} \subseteq D(I).$$

Therefore $s \in \sqrt{I}$ and so $s^l \in I$ for some positive integer $l$. Now we have $s^l(s^m a - s^n a') = 0$ which shows that $\frac{a}{s^n} = \frac{a'}{s^m}$ in $R_p$. Thus $\mu$ is injective. Now we show that $\mu$ surjective. Assume $r \in \mathcal{O}_{\mathcal{X}(M)}(\mathcal{X}_s)$. Then we can cover $\mathcal{X}_s$ with open subset $\mathcal{V}_i$, on which $s$ is represented by $\frac{a_i}{b_i}$ , with $b_i \notin \sqrt{(Q:M)}$ for all $Q \in \mathcal{V}_i$ and so $\mathcal{V}_i \subseteq \mathcal{X}_{b_i}$. By Lemma 5, the open sets of the form $\mathcal{X}_k$ form a basis for the Zariski topology. So, we may assume that $\mathcal{V}_i = \mathcal{X}_{k_i}$ for some $k_i \in R$. Since $\mathcal{X}_{k_i} \subseteq \mathcal{X}_{b_i}$, by Lemma 4, $k_i \in \sqrt{Rb_i}$. Thus $k_i^n \in Rb_i$ for some $n \in \mathbb{N}$. So $k_i^n = cb_i$ and $\frac{a_i}{b_i} = \frac{ca_i}{cb_i} = \frac{ca_i}{k_i^n}$. We see that $r$ is represented by $\frac{a_i'}{h_i}$, $(a_i' = ca_i, h_i = k_i^n)$ on $\mathcal{X}_{h_i}$ and (since $\mathcal{X}_{k_i} = \mathcal{X}_{k_i^n}$) the $\mathcal{X}_{h_i}$ cover $\mathcal{X}_s$. The open cover $\mathcal{X}_s = \cup \mathcal{X}_{h_i}$ has a finite subcover by Theorem 3. Assume $\mathcal{X}_s \subseteq \mathcal{X}_{h_1} \cup \cdots \cup \mathcal{X}_{h_n}$. For $1 \le i,j \le n$, $\frac{a_i'}{h_i}$ and $\frac{a_j'}{h_j}$ both represent $r$ on $\mathcal{X}_{h_i} \cap \mathcal{X}_{h_j}$ . By Lemma 3 $\mathcal{X}_{h_i} \cap \mathcal{X}_{h_j} = \mathcal{X}_{h_i h_j}$ and by injectivity of $\mu$, we get $\frac{a_i'}{h_i} = \frac{a_j'}{h_j}$ in $R_{h_i h_j}$ . Hence for some $n_{ij}$, we have $(h_i h_j)^{n_{ij}}(h_j a_i' - h_i a_j') = 0$. Let $m = max\{n_{ij} \mid 1 \le i,j \le n\}$. Then

$$h_j^{m+1}(h_i a_i') - h_i^{m+1}(h_j a_j') = 0.$$

By replacing each $h_i$ by $h_i^{m+1}$, and $a_i'$ by $h_i a_i'$, we still see that $r$ is represented on $\mathcal{X}_{h_i}$ by $\frac{a_i'}{h_i}$, and furthermore, we have $h_j a_i' = h_i a_j'$ for all $i, j$. Since $\mathcal{X}_s \subseteq \mathcal{X}_{h_1} \cup \cdots \cup \mathcal{X}_{h_n}$, by Lemma 2 we have

$$D_s = \phi(\mathcal{X}_s) \subseteq \cup_{i=1}^n \phi(\mathcal{X}_{h_i}) = \cup_{i=1}^n D_{h_i}.$$

Hence there are $c_1, \cdots, c_n \in R$ and $n' \in \mathbb{N}$, such that $s^{n'} = \sum_i c_i h_i$. Let $a = \sum_i c_i a_i'$. Then for each $j$ we have

$$h_j a = \sum_i c_i a_i' h_j = \sum_i c_i h_i a_j' = a_j' s^{n'}.$$

It follows that $\frac{a}{s^{n'}} = \frac{a_j'}{h_j}$ on $\mathcal{X}_{h_j}$. So $\mu(\frac{a}{s^{n'}}) = r$ everywhere, which shows that $\mu$ is surjective. □

**Corollary 5.** *Let $M$ be a faithful $\phi$-module over a ring $R$. Then $\mathcal{O}_{\mathcal{X}(M)}(\mathcal{X}(M)) \cong R$.*

*Proof.* Use Theorem 4. □

An affine scheme is a locally ringed space isomorphic as a locally ringed space to $Spec(R)$ for some ring $R$. A scheme is a locally ringed space with the property that every point has an open neighborhood which is an affine scheme. A scheme is locally Noetherian if it can be covered by open affine subsets $Spec(R_i)$, where each $R_i$ is a Noetherian ring. A scheme is Noetherian if it is locally Noetherian and quasi-compact [5].

**Theorem 5.** *Let $M$ be a $\phi$-module over a ring $R$ such that $\mathcal{X}(M)$ is a $T_0$-space. Then $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a scheme. Moreover, if $R$ is Noetherian, then $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a Noetherian scheme.*

*Proof.* Suppose $r \in R$. Therefore by Proposition 1, $\phi|_{\mathcal{X}_r}$ is continuous. Also by Theorem 2, $\phi|_{\mathcal{X}_r}$ is a bijection. Let $\mathcal{F}$ be a closed subset of $\mathcal{X}_r$. Then $\mathcal{F} = \mathcal{X}_r \cap \nu(N)$ for some submodule $N$ of $M$. Hence $\phi(\mathcal{F}) = \phi(\mathcal{X}_r) \cap V(\sqrt{(N:M)})$ is a closed subset of $\phi(\mathcal{X}_r)$. Thus $\phi|_{\mathcal{X}_r}$ is a homeomorphism. Assume that $\mathcal{X}(M) = \cup_{i \in I} \mathcal{X}_{r_i}$. Since $\phi$ is a bijection, then for $i \in I$ we have $\mathcal{X}_{r_i} \cong \phi(\mathcal{X}_{r_i}) = \{\sqrt{(Q:M)} \mid Q \in \mathcal{X}_{r_i}\} = D_{r_i} \cong Spec(R_{r_i})$. Thus by Theorem 4, $\mathcal{X}_{r_i}$ is an affine scheme. So it implies that $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a scheme. For the last statement, since $R$ is Noetherian, so is $R_{r_i}$ for each $i \in I$ . Hence $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a locally Noetherian scheme. By Corollary 3, $\mathcal{X}(M)$ is quasi-compact. Thus $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a Noetherian scheme. □

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