Miskolc Mathematical Notes
Vol. 15 (2014), No 2, pp. 717-724

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2014.1217

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HU e-ISSN 1787-2413

ON IDEALS WITH SKEW DERIVATIONS OF PRIME RINGS

NADEEM UR REHMAN AND MOHD ARIF RAZA

Received 24 April, 2014

Abstract. Let R be a prime ring and set $[x,y]_1 = [x,y] = xy - yx$ for all $x,y \in R$ and inductively $[x,y]_k = [[x,y]_{k-1}, y]$ for $k > 1$. We apply the theory of generalized polynomial identities with automorphism and skew derivations to obtain the following result: Let R be a prime ring and I a nonzero ideal of R. Suppose that (δ,φ) is a skew derivation of R such that $\delta([x,y]) = [x,y]_n$ for all $x,y \in I$, then R is commutative.

2010 *Mathematics Subject Classification:* 16N20; 16W25; 16N55; 16N60

Keywords: skew derivation, automorphism, generalized polynomial identity (GPI), prime ring, ideal

1. Introduction, Notation and Statements of the Results

Throughout this paper, unless specifically stated, R is always an associative prime ring with center $Z(R)$, Q its Martindale quotient ring. Note that Q is also prime and the center C of Q, which is called the extended centroid of R, is field (we refer the reader to [1] for the definitions and related properties of these objects). For any $x,y \in R$, the symbol $[x,y]$ stands for the commutator $xy - yx$. Recall that a ring R is called prime if for any $x,y \in R$, $xRy = \{0\}$ implies that either $x = 0$ or $y = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x,y \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x,y \in R$, denoted by (F,d). Hence, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier.

Given any automorphism φ of R, an additive mapping $\delta : R \to R$ satisfying $\delta(xy) = \delta(x)y + \varphi(x)\delta(y)$ for all $x,y \in R$ is called a φ-derivation of R, or a skew derivation of R with respect to φ, denoted by (δ,φ). It is easy to see if $\varphi = 1_R$, the identity map of R, then a φ-derivation is merely an ordinary derivation. And if $\varphi \neq 1_R$, then $\varphi - 1_R$ is a skew derivation. Thus the concept of skew derivations can be regard as a generalization of both derivations and automorphism. When $\delta(x) = \varphi(x)b - bx$ for some $b \in Q$, then (δ,φ) is called an inner skew derivation, and otherwise it is outer. Any skew derivation (δ,φ) extends uniquely to a skew derivation of Q [12] via extensions of each map to Q. Thus we may assume that any skew derivation of

R is the restriction of a skew derivation of Q. Recall that φ is called an inner automorphism if when acting on Q, $\varphi(q) = uqu^{-1}$ for some invertible $u \in Q$. When φ is not inner, then it is called an outer automorphism. The skew derivations have been extensively studied by many researchers from various views (see for instance [5] and [12], where further references can be found).

Let $Q_{*C}C\{X\}$ be the free product of Q and the free algebra $C\{X\}$ over C on an infinite set X, of indeterminate. Elements of $Q_{*C}C\{X\}$ are called generalized polynomials and a typical element in $Q_{*C}C\{X\}$ is a finite sum of monomials of the form $\alpha a_{i_0}x_{j_1}a_{i_1}x_{j_2}\cdots x_{j_n}a_{i_n}$ where $\alpha \in C$, $a_{ik} \in Q$ and $x_{jk} \in X$. We say that R satisfies a nontrivial generalized polynomial identity (abbreviated as GPI) if there exists a nonzero polynomial $\phi(x_i) \in Q_{*C}C\{X\}$ such that $\phi(r_i) = 0$ for all $r_i \in R$. By a generalized polynomial identity with automorphisms and skew derivations, we mean an identity of R expressed as the form $\phi(\varphi_j(x_i), \delta_k(x_i))$, where each φ_j is an automorphism, each δ_k is a skew derivation of R and $\phi(y_{ij}, z_{ik})$ is a generalized polynomial in distinct indeterminates y_{ij}, z_{ik}.

We need some well-known facts which will be used in the sequel.

Fact 1 ([5, Theorem 1]). Let R be a prime ring with an automorphism φ. Suppose that (δ, φ) is a Q-outer derivation of R. Then any generalized polynomial identity of R in the form $\phi(x_i, \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i) = 0$ of R, where x_i, y_i are distinct indeterminates.

Fact 2 ([5, Theorem 1]). Let R be a prime ring with an automorphism φ. Suppose that (δ, φ) is a Q-outer derivation of R. Then any generalized polynomial identity of R in the form $\phi(x_i, \varphi(x_i), \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i, z_i) = 0$ of R, where x_i, y_i, z_i are distinct indeterminates.

Fact 3 ([14, Proposition]). Let R be a prime algebra over an infinite field k and let K be a field extension over k. Then R and $R \otimes_k K$ satisfy the same generalized polynomial identities with coefficients in R.

The next result is a slight generalization of [13, Lemma 2] and can be obtained directly by the proof of [13, Lemma 2] and Fact 3.

Fact 4. Let R be a non-commutative simple algebra, finite dimensional over its center Z. Then $R \subseteq M_n(F)$ with $n > 1$ for some field F and R and $M_n(F)$ satisfy the same generalized polynomial identities with coefficients in R.

In 1992, Daif and Bell [6, Theorem 3], showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. If R is a prime ring, this implies that R is commutative. Later in 2011, Huang [8, Theorem 2.1], prove that if R is a prime ring, I a nonzero ideal of R and d a derivation of R such that $d([x, y])^m = [x, y]_n$ for all $x, y \in I$, then R is commutative. At this point the natural question is what happens in case the derivation is replaced by a generalized derivation. In [16], Quadri et. al., generalize

Daif and Bell result for generalized derivation, they showed that if R is a prime ring, I a nonzero ideal of R and (F,d) a generalized derivation with $d \neq 0$ such that $F([x,y]) = [x,y]$ for all $x,y \in I$, then R is commutative. In 2013, Huang and Davvaz [9], generalized Quadri et. al., results, more precisely they proved that if R be a prime ring, m,n are fixed positive integers, and (F,d) a generalized derivation with $d \neq 0$ such that $(F([x,y]))^m = [x,y]^n$ for all $x,y \in R$, then R is commutative.

Here we will continue the study of analogue problems on ideals of a prime ring involving skew derivations. The goal of this paper is to extend Daif and Bell theorem [6], and Huang theorem [8], in a systematic way by using the theory of generalized polynomial identities with automorphisms and skew derivations as developed by Kharchenko [11], Chuang [3,4] and recently by Chuang and Lee [5].

Explicitly we shall prove the following theorem.

Theorem 1. *Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. Suppose that (δ,φ) is a skew derivation of R such that $\delta([x,y]) = [x,y]_n$ for all $x,y \in I$, then R is commutative.*

When $\delta = \varphi - 1_R$, we obtain the following

Corollary 1. *Let R be a prime ring, I a nonzero ideal of R, and n a fixed positive integer. If φ is a non-identity automorphism of R such that $\varphi([x,y]) = [x,y]_n$ for all $x,y \in I$, then R is commutative.*

Let R be a unital ring. For a unit $u \in R$, the map $\varphi_u : x \to uxu^{-1}$ defines an automorphism of R. If d is a derivation of R, then it is easy to see that the map $ud : x \to ud(x)$ defines a φ_u-derivation of R. So we have

Corollary 2. *Let R be a prime unital ring, u a unit in R, I a nonzero ideal of R, and n a fixed positive integer. Suppose that φ_u is a derivation of R such that $\varphi_u([x,y]) = [x,y]_n$ for all $x,y \in I$, then R is commutative.*

2. Main Result

Now, we are in a position to prove the main result:

Theorem 2. *Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. Suppose that (δ,φ) is a skew derivation of R such that $\delta([x,y]) = [x,y]_n$ for all $x,y \in I$, then R is commutative.*

Proof. If $\delta = 0$, then $[x,y]_n = 0$ for all $x,y \in I$, which can be rewritten as

$$[x,y]_n = 0 = [I_x(y),y]_{n-1} \text{ for all } x,y \in I.$$

By Lanski [13, Theorem 1], either R is commutative or $I_x = 0$, i.e., $I \subseteq Z(R)$ in which case R is also commutative by Mayne [15, Lemma 3].

Now we assume that $\delta \neq 0$ and $\delta([x,y]) = [x,y]_n$ for all $x,y \in I$, which can be rewritten as

$$(\delta(x)y + \varphi(x)\delta(y)) - (\delta(y)x + \varphi(y)\delta(x)) = [x,y]_n. \tag{2.1}$$

In the light of Kharchenko's theory [11], we split the proof into two cases:
Case 1. Let δ is Q-outer, then I satisfies the polynomial identities

$$(sy+\varphi(x)t)-(tx+\varphi(y)s)=[x,y]_n, \text{ for all } x,y,s,t\in I. \tag{2.2}$$

Firstly, we assume that φ is not Q-inner, then for all $x,y,s,t,u,v\in I$, we have

$$(sy+ut)-(tx+vs)=[x,y]_n, \text{ for all } x,y,s,t,u,v\in I.$$

In particular $s=t=0$, then I satisfied the polynomial identity $[x,y]_n=0$, for all $x,y\in I$, so by Lanski [13, Theorem 1], R is commutative.

Secondly, if φ is Q-inner, then there exist an invertible element $T\in Q$, $\varphi(x)=TxT^{-1}$ for all $x\in R$. Thus from (2.2), we have

$$(sy+TxT^{-1}t)-(tx+TyT^{-1}s)=[x,y]_n \text{ for all } x,y,s,t\in I.$$

In particular $s=t=0$, and using the same argument presented as above, R is commutative.
Case 2. Let δ is Q-inner, then $\delta(x)=\varphi(x)q-qx$ for all $x\in R$, $q\in Q$. From (2.1), we have

$$\begin{aligned}(\varphi(x)q-qx)y+\varphi(x)(\varphi(y)q-qy)-(\varphi(y)q-qy)x-\varphi(y)(\varphi(x)q-qx)&\\ =[x,y]_n \quad \text{for all } x,y\in I.&\end{aligned} \tag{2.3}$$

If φ is not Q-inner, then I satisfies the polynomial identity

$$\begin{aligned}(uq-qx)y+u(vq-qy)-(vq-qy)x-v(uq-qx)&\\ =[x,y]_n \quad \text{for all } x,y,u,v\in I.&\end{aligned}$$

In particular $u=v=0$, then I satisfied the following polynomial identity

$$(-qxy+qyx)=[x,y]_n, \text{ for all } x,y\in I.$$

By Chuang [5, Theorem 1 and Theorem 2], shows that Q satisfies this polynomial identity and hence R as well. Note that this is a polynomial identity and hence there exist a field $\mathbb{F}$ such that $R\subseteq M_k(\mathbb{F})$, the ring of $k\times k$ matrices over a field $\mathbb{F}$, where $k\geq 1$. Moreover, R and $M_k(\mathbb{F})$ satisfy the same polynomial identity[2], that is $M_k(\mathbb{F})$ satisfy

$$(qyx-qxy)=[x,y]_n.$$

Denote e_{ij} the usual matrix unit with 1 in (i,j)-entry and zero elsewhere. By choosing $x=e_{12}$, $y=e_{22}$, $q=e_{12}$, we see that

$$\begin{aligned}0=(q[y,x])-[x,y]_n&=(e_{12}[e_{22},e_{12}])-[e_{12},e_{22}]_n\\ &=-e_{12}\neq 0, \text{ a contradiction.}\end{aligned}$$

Now consider, if φ is Q-inner, then there exist an invertible element $T\in Q$, $\varphi(x)=TxT^{-1}$ for all $x\in R$. From (2.3) we can write,

$$\begin{aligned}(TxT^{-1}q-qx)y+TxT^{-1}(TyT^{-1}q-qy)-(TyT^{-1}q-qy)x&\\ -TyT^{-1}(TxT^{-1}q-qx)=[x,y]_n \text{ for all } x,y\in I.&\end{aligned}$$

We can see easily that if $T^{-1}q \in C$, then

$\delta(x) = TxT^{-1}q - qx = T(xT^{-1}q - T^{-1}qx) = T[x, T^{-1}q] = 0$, a contradiction.

Thus $T^{-1}q \notin C$. with this,

$$\phi(x,y) = (TxT^{-1}q - qx)y + TxT^{-1}(TyT^{-1}q - qy) \\ - (TyT^{-1}q - qy)x - TyT^{-1}(TxT^{-1}q - qx) - [x,y]_n. \tag{2.4}$$

Since by [2] or [1, Theorem 6.4.4], I and Q satisfy the same generalized polynomial identities, with this we can see easily that $\phi(x,y) = 0$ is a nontrivial generalized polynomial identity of Q. Let $\mathcal{F}$ be the algebraic closure of C if C is infinite, otherwise let $\mathcal{F}$ be C. By Fact 3, $\phi(x,y)$ is also a generalized polynomial identity of $Q \otimes_C \mathcal{F}$. Moreover, in view of [7, Theorem 3.5], $Q \otimes_C \mathcal{F}$ is a prime ring with $\mathcal{F}$ as its extended centroid. Thus $Q \otimes_C \mathcal{F}$ is a prime ring satisfies a nontrivial generalized polynomial identity and its extended centroid $\mathcal{F}$ is either an algebraically closed field or a finite field. Since both Q and $Q \otimes_C \mathcal{F}$ are prime and centrally closed [7, Theorem 3.5], we may replace R by Q or $Q \otimes_C \mathcal{F}$. Thus we may assume that R is centrally closed and the field $\mathcal{F}$ which is either algebraically closed or finite and R satisfies generalized polynomial identity (2.4). By Martindale's theorem [1, Corollary 6.1.7], R is a primitive ring having nonzero socle with the field $\mathcal{D}$ as its associated division ring. By Jacobson theorem [10, p.75], R is isomorphic to a dense subring of the ring of linear transformations on a vector space V over $\mathcal{D}$(or $End(V_{\mathcal{D}})$ in brief), containing nonzero linear transformations of finite rank.

We assume that $dim(V_{\mathcal{D}}) \geq 2$, otherwise we are done.

Step 1. We want to show that w and $T^{-1}qw$ are linearly $\mathcal{D}$-dependent for all $w \in \mathcal{V}$. If $T^{-1}qw = 0$ then $\{w, T^{-1}qw\}$ is linearly $\mathcal{D}$-dependent. Suppose on contrary that w_0 and $T^{-1}qw_0$ are linearly $\mathcal{D}$-independent for some $w_0 \in \mathcal{D}$.

If $T^{-1}w_0 \notin Span_{\mathcal{D}}\{w_0, T^{-1}qw_0\}$ then $\{w_0, T^{-1}qw_0, T^{-1}w_0\}$ are linearly $\mathcal{D}$-independent. By the density of R there exist $x, y \in R$ such that

$$\begin{array}{lll} xw_0 = 0, & xT^{-1}qw_0 = T^{-1}w_0, & xT^{-1}w_0 = 0 \\ yw_0 = w_0, & yT^{-1}qw_0 = 0, & yT^{-1}w_0 = T^{-1}w_0. \end{array}$$

With all these, we obtained from (2.4),

$$-w_0 = \big((TxT^{-1}q - qx)y + TxT^{-1}(TyT^{-1}q - qy) - (TyT^{-1}q - qy)x \\ - TyT^{-1}(TxT^{-1}q - qx) - [x,y]_n\big)w_0, \text{ a contradiction.}$$

If $T^{-1}w_0 \in Span_{\mathcal{D}}\{w_0, T^{-1}qw_0\}$ then $T^{-1}w_0 = w_0\beta + T^{-1}qw_0\gamma$ for some $\beta, \gamma \in \mathcal{D}$ and $\beta \neq 0$. Since w_0 and $T^{-1}qw_0$ are linearly $\mathcal{D}$-independent, by the density of R there exist $x, y \in R$ such that

$$\begin{array}{ll} xw_0 = 0, & xT^{-1}qw_0 = w_0\beta + T^{-1}qw_0\gamma \\ yw_0 = w_0, & yT^{-1}qw_0 = 0. \end{array}$$

The application of (2.4) implies that

$$\begin{aligned} 0 = &\big((TxT^{-1}q - qx)y + TxT^{-1}(TyT^{-1}q - qy) - (TyT^{-1}q - qy)x \\ &- TyT^{-1}(TxT^{-1}q - qx) - [x,y]_n\big)w_0 = -Tw_0\beta = -w_0\beta \neq 0, \end{aligned}$$

and we arrive at a contradiction. So we conclude that $\{w_0, T^{-1}w_0\}$ are linearly $\mathcal{D}$-dependent, for all $w_0 \in \mathcal{V}$ as claimed.

Step 2. By using the arguments presented above, we prove that $T^{-1}qw_0 = w_0\mu(w)$, for all $w \in \mathcal{V}$, where $\mu(w) \in \mathcal{D}$ depends on $w \in \mathcal{V}$. In fact, it is easy to check that $\mu(w)$ is independent of choice $w \in \mathcal{V}$. Indeed, for any $w, z \in \mathcal{V}$, in view of above situation, there exist $\mu(w), \mu(z), \mu(w+z) \in \mathcal{D}$ such that

$$T^{-1}qw = w\mu(w),\ T^{-1}qz = z\mu(z),\ T^{-1}q(w+z) = (w+z)\mu(w+z)$$

and therefore,

$$w\mu(w) + z\mu(z) = T^{-1}q(w+z) = (w+z)\mu(w+z).$$

Hence,

$$w(\mu(w) - \mu(w+z)) + z(\mu(z) - \mu(w+z)) = 0.$$

Since w and z are $\mathcal{D}$-independent, then $\mu(w) = \mu(z) = \mu(w+z)$. Otherwise, w and z are $\mathcal{D}$-dependent, say $w = \lambda z$ for some $\lambda \in \mathcal{D}$. Thus,

$$w\mu(w) = T^{-1}qw = T^{-1}q\lambda z = \lambda T^{-1}qz = \lambda z\mu(z) = w\mu(z)$$

i.e., $\mathcal{V}(\mu(w) - \mu(z)) = 0$. Since $\mathcal{V}$ is faithful, we get $\mu(w) = \mu(z)$. Hence, we conclude that there exists $\chi \in \mathcal{D}$ such that $T^{-1}qw = w\chi$ for all $w \in \mathcal{V}$.

At last, we want to show that $\chi \in Z(\mathcal{D})$ (the center of $\mathcal{D}$). Indeed, for any $\eta \in \mathcal{D}$, we have

$$T^{-1}q(w\eta) = (w\eta)\chi = w(\eta\chi),$$

and on the other hand,

$$T^{-1}q(w\eta) = (T^{-1}qw)\eta = (w\chi)\eta = w(\chi\eta).$$

Therefore, $\mathcal{V}(\eta\chi - \chi\eta) = 0$ and thus, $\eta\chi = \chi\eta$, which implies that $\chi \in Z(\mathcal{D})$. Hence, $T^{-1}q \in C$, a contradiction. With this completes the proof of the theorem. □

The following example demonstrates that the hypothesis of primeness of R is essential in Theorem 1.

Example 1. Let S be the set of all integers. Consider

$R = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S\right\}$ and $I = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S\right\}$. Define maps $\varphi : R \to R$ by $\varphi\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & 0 \end{pmatrix}$ and $\delta : R \to R$ by $\delta\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -2b \\ 0 & 0 \end{pmatrix}$. The fact that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$ implies that R is not prime. It is easy to check that I is a nonzero ideal of R and (δ,φ) is a skew derivation of R such that $\delta([x,y]) = [x,y]_n$ for all $x,y \in I$. However, R is not commutative.

Remark 1. In view of the above result, it is an obvious question, what about the commutativity of R, if $\delta([x,y])^m = [x,y]_n$ for all $x,y \in I$ (or a Lie ideal L). Unfortunately, we are unable to solve it and leave as an open question whether or not this result can be prove.

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Authors' addresses

Nadeem ur Rehman
Department of Mathematics, Aligarh Muslim University, 202002, Aligarh, India
E-mail address: nu.rehman.mm@amu.ac.in

Mohd Arif Raza
Department of Mathematics, Aligarh Muslim University, 202002, Aligarh, India
E-mail address: arifraza03@gmail.com