On ideals with skew derivations of prime rings

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Abstract. Let $R$ be a prime ring and set $[x, y]_1 = [x, y] = xy - yx$ for all $x, y \in R$ and inductively $[x, y]_k = [x, y]_{k-1}$ for $k > 1$. We apply the theory of generalized polynomial identities with automorphism and skew derivations to obtain the following result: Let $R$ be a prime ring and $I$ a nonzero ideal of $R$. Suppose that $(\delta, \varphi)$ is a skew derivation of $R$ such that $\delta([x, y]) = [x, y]_n$ for all $x, y \in I$, then $R$ is commutative.

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1. Introduction, Notation and Statements of the Results

Throughout this paper, unless specifically stated, $R$ is always an associative prime ring with center $Z(R)$, $Q$ its Martindale quotient ring. Note that $Q$ is also prime and the center $C$ of $Q$, which is called the extended centroid of $R$, is field (we refer the reader to [1] for the definitions and related properties of these objects). For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. Recall that a ring $R$ is called prime if for any $x, y \in R$, $xRy = \{0\}$ implies that either $x = 0$ or $y = 0$. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, denoted by $(F, d)$. Hence, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier.

Given any automorphism $\varphi$ of $R$, an additive mapping $\delta : R \rightarrow R$ satisfying $\delta(xy) = \delta(x)y + \varphi(x)\delta(y)$ for all $x, y \in R$ is called a $\varphi$-derivation of $R$, or a skew derivation of $R$ with respect to $\varphi$, denoted by $(\delta, \varphi)$. It is easy to see if $\varphi = 1_R$, the identity map of $R$, then a $\varphi$-derivation is merely an ordinary derivation. And if $\varphi \neq 1_R$, then $\varphi - 1_R$ is a skew derivation. Thus the concept of skew derivations can be regard as a generalization of both derivations and automorphism. When $\delta(x) = \varphi(x)b - bx$ for some $b \in Q$, then $(\delta, \varphi)$ is called an inner skew derivation, and otherwise it is outer. Any skew derivation $(\delta, \varphi)$ extends uniquely to a skew derivation of $Q$ [12] via extensions of each map to $Q$. Thus we may assume that any skew derivation of
Let $Q$ be a prime ring with an automorphism $\varphi$. Suppose that $(\delta, \varphi)$ is a $Q$-outer derivation of $R$. Then any generalized polynomial identity of $R$ in the form $\phi(x_i, \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i) = 0$ of $R$, where $x_i, y_i$ are distinct indeterminates.

Fact 2 ([5, Theorem 1]). Let $R$ be a prime ring with an automorphism $\varphi$. Suppose that $(\delta, \varphi)$ is a $Q$-outer derivation of $R$. Then any generalized polynomial identity of $R$ in the form $\phi(x_i, \varphi(x_i)), \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i, z_i) = 0$ of $R$, where $x_i, y_i, z_i$ are distinct indeterminates.

Fact 3 ([14, Proposition]). Let $R$ be a prime algebra over an infinite field $k$ and let $K$ be a field extension over $k$. Then $R$ and $R \otimes_k K$ satisfy the same generalized polynomial identities with coefficients in $R$.

The next result is a slight generalization of [13, Lemma 2] and can be obtained directly by the proof of [13, Lemma 2] and Fact 3.

Fact 4. Let $R$ be a non-commutative simple algebra, finite dimensional over its center $Z$. Then $R \subseteq M_n(F)$ with $n > 1$ for some field $F$ and $R$ and $M_n(F)$ satisfy the same generalized polynomial identities with coefficients in $R$.

In 1992, Daif and Bell [6, Theorem 3], showed that if in a semiprime ring $R$ there exists a nonzero ideal $I$ of $R$ and a derivation $d$ such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. If $R$ is a prime ring, this implies that $R$ is commutative. Later in 2011, Huang [8, Theorem 2.1], prove that if $R$ is a prime ring, $I$ a nonzero ideal of $R$ and $d$ a derivation of $R$ such that $d([x, y])^m = [x, y]^m$ for all $x, y \in I$, then $R$ is commutative. At this point the natural question is what happens in case the derivation is replaced by a generalized derivation. In [16], Quadri et. al., generalize
Daif and Bell result for generalized derivation, they showed that if $R$ is a prime ring, $I$ a nonzero ideal of $R$ and $(F, d)$ a generalized derivation with $d \neq 0$ such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then $R$ is commutative. In 2013, Huang and Davvaz [9], generalized Quadri et. al., results, more precisely they proved that if $R$ be a prime ring, $m, n$ are fixed positive integers, and $(F, d)$ a generalized derivation with $d \neq 0$ such that $(F([x, y]))^m = [x, y]^n$ for all $x, y \in R$, then $R$ is commutative.

Here we will continue the study of analogue problems on ideals of a prime ring involving skew derivations. The goal of this paper is to extend Daif and Bell theorem [6], and Huang theorem [8], in a systematic way by using the theory of generalized polynomial identities with automorphisms and skew derivations as developed by Kharchenko [11], Chuang [3, 4] and recently by Chuang and Lee [5].

Explicitly we shall prove the following theorem.

**Theorem 1.** Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $n$ a fixed positive integer. Suppose that $(\delta, \varphi)$ is a skew derivation of $R$ such that $\delta([x, y]) = [x, y]^n$ for all $x, y \in I$, then $R$ is commutative.

When $\delta = \varphi - 1_R$, we obtain the following

**Corollary 1.** Let $R$ be a prime ring, $I$ a nonzero ideal of $R$, and $n$ a fixed positive integer. If $\varphi$ is a non-identity automorphism of $R$ such that $\varphi([x, y]) = [x, y]^n$ for all $x, y \in I$, then $R$ is commutative.

Let $R$ be a unital ring. For a unit $u \in R$, the map $\varphi_u : x \to uxu^{-1}$ defines an automorphism of $R$. If $d$ is a derivation of $R$, then it is easy to see that the map $ud : x \to ud(x)$ defines a $\varphi_u$-derivation of $R$. So we have

**Corollary 2.** Let $R$ be a prime unital ring, $u$ a unit in $R$, $I$ a nonzero ideal of $R$, and $n$ a fixed positive integer. Suppose that $\varphi_u$ is a derivation of $R$ such that $\varphi_u([x, y]) = [x, y]^n$ for all $x, y \in I$, then $R$ is commutative.

2. **Main Result**

Now, we are in a position to prove the main result:

**Theorem 2.** Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $n$ a fixed positive integer. Suppose that $(\delta, \varphi)$ is a skew derivation of $R$ such that $\delta([x, y]) = [x, y]^n$ for all $x, y \in I$, then $R$ is commutative.

**Proof.** If $\delta = 0$, then $[x, y]^n = 0$ for all $x, y \in I$, which can be rewritten as $[x, y]_n = 0 = [I_x(y), y]_{n-1}$ for all $x, y \in I$.

By Lanski [13, Theorem 1], either $R$ is commutative or $I_x = 0$, i.e., $I \subseteq Z(R)$ in which case $R$ is also commutative by Mayne [15, Lemma 3].

Now we assume that $\delta \neq 0$ and $\delta([x, y]) = [x, y]^n$ for all $x, y \in I$, which can be rewritten as

$$
(\delta(x)y + \varphi(x)\delta(y)) - (\delta(y)x + \varphi(y)\delta(x)) = [x, y]_n. \tag{2.1}
$$
In the light of Kharchenko’s theory [11], we split the proof into two cases:

**Case 1.** Let \( \delta \) is \( Q \)-outer, then \( I \) satisfies the polynomial identities

\[
(sy + \varphi(x)t) - (tx + \varphi(y)s) = [x, y]_n, \quad \text{for all} \ x, y, s, t \in I. \tag{2.2}
\]

Firstly, we assume that \( \varphi \) is not \( Q \)-inner, then for all \( x, y, s, t, u, v \in I \), we have

\[
(sy + ut) - (tx + vs) = [x, y]_n, \quad \text{for all} \ x, y, s, t, u, v \in I.
\]

In particular \( s = t = 0 \), then \( I \) satisfied the polynomial identity \( [x, y]_n = 0 \), for all \( x, y \in I \), so by Lanski [13, Theorem 1], \( R \) is commutative.

Secondly, if \( \varphi \) is \( Q \)-inner, then there exist an invertible element \( T \in Q \), \( \varphi(x) = TxT^{-1} \) for all \( x \in R \). Thus from (2.2), we have

\[
(sy + TxT^{-1}t) - (tx + TyT^{-1}s) = [x, y]_n \quad \text{for all} \ x, y, s, t \in I.
\]

In particular \( s = t = 0 \), and using the same argument presented as above, \( R \) is commutative.

**Case 2.** Let \( \delta \) is \( Q \)-inner, then \( \delta(x) = \varphi(x)q - qx \) for all \( x \in R \), \( q \in Q \). From (2.1), we have

\[
(\varphi(x)q - qx)y + \varphi(x)(\varphi(y)q - qy) - (\varphi(y)q - qy)x - \varphi(y)(\varphi(x)q - qx) = [x, y]_n \quad \text{for all} \ x, y \in I. \tag{2.3}
\]

If \( \varphi \) is not \( Q \)-inner, then \( I \) satisfies the polynomial identity

\[
(uq - qx)y + u(vq - qy) - (vq - qy)x - v(uq - qx) = [x, y]_n \quad \text{for all} \ x, y, u, v \in I.
\]

In particular \( u = v = 0 \), then \( I \) satisfied the following polynomial identity

\[
(-qx y + qyx) = [x, y]_n \quad \text{for all} \ x, y \in I.
\]

By Chuang [5, Theorem 1 and Theorem 2], shows that \( Q \) satisfies this polynomial identity and hence \( R \) as well. Note that this is a polynomial identity and hence there exist a field \( \mathbb{F} \) such that \( R \subseteq M_k(\mathbb{F}) \), the ring of \( k \times k \) matrices over a field \( \mathbb{F} \), where \( k \geq 1 \). Moreover, \( R \) and \( M_k(\mathbb{F}) \) satisfy the same polynomial identity[2], that is \( M_k(\mathbb{F}) \) satisfy

\[
(qyx - qxy) = [x, y]_n.
\]

Denote \( e_{ij} \) the usual matrix unit with 1 in \((i, j)\)-entry and zero elsewhere. By choosing \( x = e_{12}, y = e_{22}, q = e_{12} \), we see that

\[
0 = (q[y, x]) - [x, y]_n = (e_{12}[e_{22}, e_{12}]) - [e_{12}, e_{22}]_n
= -e_{12} \neq 0, \quad \text{a contradiction.}
\]

Now consider, if \( \varphi \) is \( Q \)-inner, then there exist an invertible element \( T \in Q \), \( \varphi(x) = TxT^{-1} \) for all \( x \in R \). From (2.3) we can write,

\[
(TxT^{-1}q - qx)y + TxT^{-1}(TyT^{-1}q - qy) - (TyT^{-1}q - qy)x
- TyT^{-1}(TxT^{-1}q - qx) = [x, y]_n \quad \text{for all} \ x, y \in I.
\]
We can see easily that if $T^{-1}q \in C$, then
\[
\delta(x) = TxT^{-1}q - qx = T(xT^{-1}q - T^{-1}qx) = T[x, T^{-1}q] = 0, \quad \text{a contradiction.}
\]
Thus $T^{-1}q \notin C$. With this,
\[
\phi(x, y) = (TxT^{-1}q - qx)y + TxT^{-1}(TyT^{-1}q - qy) - (TyT^{-1}q - qy)x - TyT^{-1}(TxT^{-1}q - qx) - [x, y]_n. \tag{2.4}
\]

Since by [2] or [1, Theorem 6.4.4], $I$ and $Q$ satisfy the same generalized polynomial identities, with this we can see easily that $\phi(x, y) = 0$ is a nontrivial generalized polynomial identity of $Q$. Let $\mathcal{F}$ be the algebraic closure of $C$ if $C$ is infinite, otherwise let $\mathcal{F}$ be $C$. By Fact 3, $\phi(x, y)$ is also a generalized polynomial identity of $Q \otimes_C \mathcal{F}$. Moreover, in view of [7, Theorem 3.5], $Q \otimes_C \mathcal{F}$ is a prime ring with $\mathcal{F}$ as its extended centroid. Thus $Q \otimes_C \mathcal{F}$ is a prime ring satisfies a nontrivial generalized polynomial identity and its extended centroid $\mathcal{F}$ is either an algebraically closed field or a finite field. Since both $Q$ and $Q \otimes_C \mathcal{F}$ are prime and centrally closed [7, Theorem 3.5], we may replace $R$ by $Q$ or $Q \otimes_C \mathcal{F}$. Thus we may assume that $R$ is centrally closed and the field $\mathcal{F}$ which is either algebraically closed or finite and $R$ satisfies generalized polynomial identity (2.4). By Martindale’s theorem [1, Corollary 6.1.7], $R$ is a primitive ring having nonzero socle with the field $\mathcal{D}$ as its associated division ring. By Jacobson theorem [10, p.75], $R$ is isomorphic to a dense subring of the ring of linear transformations on a vector space $V$ over $\mathcal{D}$ (or $\text{End}(V_D)$) in brief), containing nonzero linear transformations of finite rank.

We assume that $\text{dim}(V_D) \geq 2$, otherwise we are done.

**Step 1.** We want to show that $w$ and $T^{-1}qw$ are linearly $\mathcal{D}$-dependent for all $w \in \mathcal{V}$. If $T^{-1}qw = 0$ then $\{w, T^{-1}qw\}$ is linearly $\mathcal{D}$-dependent. Suppose on contrary that $w_0$ and $T^{-1}qw_0$ are linearly $\mathcal{D}$-independent for some $w_0 \in \mathcal{D}$.

If $T^{-1}w_0 \notin \text{Span}_\mathcal{D}\{w_0, T^{-1}qw_0\}$ then $\{w_0, T^{-1}qw_0, T^{-1}w_0\}$ are linearly $\mathcal{D}$-independent. By the density of $R$ there exist $x, y \in R$ such that
\[
\begin{align*}
xw_0 &= 0, & xT^{-1}qw_0 &= T^{-1}w_0, & xT^{-1}w_0 &= 0, \\yw_0 &= w_0, & yT^{-1}qw_0 &= 0, & yT^{-1}w_0 &= T^{-1}w_0.
\end{align*}
\]

With all these, we obtained from (2.4),
\[
-w_0 = ((TxT^{-1}q - qx)y + TxT^{-1}(TyT^{-1}q - qy) - (TyT^{-1}q - qy)x \\
- TyT^{-1}(TxT^{-1}q - qx) - [x, y]_n)w_0, \quad \text{a contradiction.}
\]

If $T^{-1}w_0 \in \text{Span}_\mathcal{D}\{w_0, T^{-1}qw_0\}$ then $T^{-1}w_0 = w_0\beta + T^{-1}qw_0\gamma$ for some $\beta, \gamma \in \mathcal{D}$ and $\beta \neq 0$. Since $w_0$ and $T^{-1}qw_0$ are linearly $\mathcal{D}$-independent, by the density of $R$ there exist $x, y \in R$ such that
\[
\begin{align*}
xw_0 &= 0, & xT^{-1}qw_0 &= w_0\beta + T^{-1}qw_0\gamma, \\
yw_0 &= w_0, & yT^{-1}qw_0 &= 0.
\end{align*}
\]
The application of (2.4) implies that
\[ 0 = (T x T^{-1} q - q x) y + T x T^{-1} (T y T^{-1} q - q y) - (T y T^{-1} q - q y) x \]
and we arrive at a contradiction. So we conclude that \( \{ w_0, T^{-1} w_0 \} \) are linearly \( D \)-dependent, for all \( w_0 \in V \) as claimed.

**Step 2.** By using the arguments presented above, we prove that \( T^{-1} q w_0 = w_0 \mu(w) \) for all \( w \in V \), where \( \mu(w) \in D \) depends on \( w \in V \). In fact, it is easy to check that \( \mu(w) \) is independent of choice \( w \in V \). Indeed, for any \( w, z \in V \), in view of above situation, there exist \( \mu(w), \mu(z), \mu(w + z) \in D \) such that
\[ T^{-1} q w = w \mu(w), \quad T^{-1} q z = z \mu(z), \quad T^{-1} q (w + z) = (w + z) \mu(w + z) \]
and therefore,
\[ w \mu(w) + z \mu(z) = T^{-1} q (w + z) = (w + z) \mu(w + z). \]
Hence,
\[ w (\mu(w) - \mu(w + z)) + z (\mu(z) - \mu(w + z)) = 0. \]
Since \( w \) and \( z \) are \( D \)-independent, then \( \mu(w) = \mu(z) = \mu(w + z) \). Otherwise, \( w \) and \( z \) are \( D \)-dependent, say \( w = \lambda z \) for some \( \lambda \in D \). Thus,
\[ w \mu(w) = T^{-1} q w = T^{-1} q z = \lambda T^{-1} q z = \lambda z \mu(z) = w \mu(z) \]
i.e., \( V (\mu(w) - \mu(z)) = 0 \). Since \( V \) is faithful, we get \( \mu(w) = \mu(z) \). Hence, we conclude that there exists \( \chi \in D \) such that \( T^{-1} q w = w \chi \) for all \( w \in V \).

At last, we want to show that \( \chi \in Z(D) \) (the center of \( D \)). Indeed, for any \( \eta \in D \), we have
\[ T^{-1} q (w \eta) = (w \eta) \chi = w (\eta \chi). \]
and on the other hand,
\[ T^{-1} q (w \eta) = (T^{-1} q w) \eta = (w \chi) \eta = w (\chi \eta). \]
Therefore, \( V (\eta \chi - \chi \eta) = 0 \) and thus, \( \eta \chi = \chi \eta \), which implies that \( \chi \in Z(D) \). Hence, \( T^{-1} q \in C \), a contradiction. With this completes the proof of the theorem. \( \square \)

The following example demonstrates that the hypothesis of primeness of \( R \) is essential in Theorem 1.

**Example 1.** Let \( S \) be the set of all integers. Consider
\[ R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S \right\}. \] Define maps \( \varphi : R \to R \) by \( \varphi \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & 0 \end{pmatrix} \) and \( \delta : R \to R \) by \( \delta \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -2b \\ 0 & 0 \end{pmatrix} \). The fact that \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0 \) and \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 \) implies that \( R \) is not
prime. It is easy to check that $I$ is a nonzero ideal of $R$ and $(\delta, \varphi)$ is a skew derivation of $R$ such that $\delta([x, y]) = [x, y]_n$ for all $x, y \in I$. However, $R$ is not commutative.

**Remark 1.** In view of the above result, it is an obvious question, what about the commutativity of $R$, if $\delta([x, y])^m = [x, y]_n$ for all $x, y \in I$(or a Lie ideal $L$). Unfortunately, we are unable to solve it and leave as an open question whether or not this result can be prove.

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