Miskolc Mathematical Notes
Vol. 16 (2015), No. 2, pp. 713–719

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1207

ON THE ZERO OF A RADIAL MINIMIZER OF p-GINZBURG-LANDAU TYPE

YUZE CAI, YUTIAN LEI, AND BEI WANG

Received 16 April, 2014

Abstract. Zeros of minimizers of p-Ginzburg-Landau functional are helpful to understand the location of vortices of superconductivity and superfluid. When $p = 2$ and the degree of the boundary data around the boundary is ± 1, there exists only one zero of the p-Ginzburg-Landau minimizers in the bounded domain. When $p > 2$, this becomes a more complicated problem. This paper is concerned with the location of the zeros of a radial minimizer of a p-Ginzburg-Landau type functional. The authors use the method of moving planes and the idea of the proof of Pohozaev's identity to verify that the origin is the unique zero of the radial minimizer in the domain.

2010 *Mathematics Subject Classification:* 35J92; 35Q56

Keywords: location of zeros, method of moving planes, Pohozaev's identity, p-Ginzburg-Landau type functional

1. INTRODUCTION

Let $B = \{x \in \mathbf{R}^2; |x| < 1\}$ and $p > 2$. Consider the p-Ginzburg-Landau type functional

$$E_\varepsilon(u, B) = \frac{1}{p}\int_B |\nabla u|^p + \frac{1}{2p\varepsilon^p}\int_B (1-|u|^p)^2.$$

When $p = 2$, the minimizer u_ε of $E_\varepsilon(u, B)$ in the class $H^1_g(B, \mathbf{R}^2)$ was well-studied by many papers (see [4], [5] and the references therein), where $g : \partial B \to \partial B$ is a smooth map. In particular, the paper [4] pointed out that there is only one zero of u_ε in B if $\deg(g, \partial B) = \pm 1$. When $p \neq 2$, the minimizer in the class $W^{1,p}_g(B, \mathbf{R}^2)$ was also researched in [2], [3], [10], [11] and [16]. The paper [10] proved that when $p > 2$, the zeros of the minimizer u_ε are located near the finite singularities of a p-harmonic map. The same result was generalized to higher dimensions (cf. [9]).

The second author was supported in part by the NSFC (No. 11171158, No. 11471164), and the NSF of Jiangsu (No. BK2012846). This research was also supported by the PAPD of Jiangsu Higher Education Institutions.

The third author was supported by Natural Science Foundation of Jiangsu Higher Education Institutions (No. 13KJB110003).

However, the relation between the number of singularities and the degree $\deg(g, \partial B)$ is still not clear.

In order to investigate this problem, we consider a special case. Investigate the minimization problem of $E_\varepsilon(u, B)$ in the class

$$W = \{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B, \mathbf{R}^2); f(1) = 1\},$$

where $r = |x|$. By the direct method in the calculus of variations, the minimizer u_ε exists, and it is called the *radial minimizer*. When $p = 2$, many papers investigated the properties of the radial minimizer. Papers [13] and [14] discussed the stability properties of $f_\varepsilon(r)$, the modulus of u_ε. The paper [8] also pointed out that

$$f_\varepsilon(r) > 0, \quad (f_\varepsilon(r))_r > 0, \quad in \quad (0, 1].$$

Here $(f_\varepsilon)_r$ is the first order derivative of f_ε. This means the origin is the unique zero of f_ε in B.

In this paper, we will also prove that the origin is the unique zero of f_ε in the case of $p > 2$. This improves the result in [12], which was only showed that the zeros of u_ε are located near the origin.

If we define

$$V = \{f(r); u(x) = f(r)\frac{x}{|x|} \in W\},$$

then, $V \subset \{f \in C[0, 1]; f(0) = 0\}$ (cf. Proposition 1.1 in [12]). Clearly, u_ε is a minimizer of $E_\varepsilon(u, B)$ in W if and only if f_ε is a minimizer of

$$E_\varepsilon(f) = \int_0^1 [\frac{1}{p}A^{p/2} + \frac{1}{2p\varepsilon^p}(1 - f^p)^2] r dr$$

in V, where $A = (f_r)^2 + (f/r)^2$. According to Proposition 2.2 in [12], the Euler-Lagrange equation which the minimizer f_ε satisfies is the following ODE

$$-(A^{\frac{p-2}{2}} f_r r)_r + A^{\frac{p-2}{2}} \frac{f}{r} = \frac{r}{\varepsilon^p} f^{p-1}(1 - f^p). \tag{1.1}$$

Similar to the argument of regularity results in [3], we see that the minimizer f_ε is a classical solution of the equation above. By the proof of Proposition 2.1 in [3], we can always assume

$$0 \le f_\varepsilon(r) \le 1, \quad r \in [0, 1].$$

Remark 1. Theorem 3.5 in [12] shows that for any given $\eta \in (0, 1)$, there exists $h = h(\eta) > 0$ and $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that as $\varepsilon \in (0, \varepsilon_0)$,

$$f_\varepsilon(r) > 1 - \eta, \quad in \quad [h\varepsilon, 1].$$

This means that all zeros of f_ε are located in $[0, h\varepsilon)$ for each $\varepsilon \in (0, \varepsilon_0)$. Therefore, we can search the zeros only in this interval. On the other hand, Theorem IV.1 in [5] shows the relation between h and η:

$$\lambda \le h \le \lambda 9^{Card\ J},$$

where $Card\ J$ is a positive integer which is independent of ε (cf. Proposition 3.4 in [12]), and λ is defined in the proof of Proposition 3.3 in [12] as

$$\lambda = (\frac{\eta}{2C})^{\frac{p}{p-2}}.$$

These results mean that h can be suitably small as long as η is chosen small properly.

The main result of this paper, which will be proved in §2, is read as follows

Theorem 1. *Let f_ε be the modulus of a radial minimizer of $E_\varepsilon(u, B)$. Then there exists $\varepsilon_0 \in (0, 1)$, such that as $\varepsilon \in (0, \varepsilon_0)$,*

$$f_\varepsilon > 0, \quad in \quad (0, h\varepsilon], \tag{1.2}$$

$$(f_\varepsilon)_r > 0, \quad in \quad [0, h\varepsilon]. \tag{1.3}$$

Remark 2. Clearly, Remark 1, (1.2) and (1.3) show that the origin is the unique zero of the radial minimizer u_ε in B.

An analogous result for minimizer of $E(u)$ was proved in [3], where $E(u) = \int_B [p^{-1}|\nabla u|^p + (4\varepsilon^p)^{-1}(1 - |u|^2)^2]dx$ is equipped with a different penalization. Here we can also locate the zero of minimizers of $E_\varepsilon(u, B)$ by another technique-the method of moving planes. Such a method was first proposed by Alexandrov [1] and developed by Serrin [15], Gidas-Ni-Nirenberg [7], and Chen-Li [6]. Now, this method has been a powerful tool to handle the monotonicity of solutions of differential equations.

2. Proof of Theorem 1

Let $f = f_\varepsilon$ be a minimizer of $E_\varepsilon(f)$ in V. In view of $f(0) = 0$ and $f(h\varepsilon) > 1 - \eta > 0$, we can set

$$\sigma = \sup\{R \in [0, h\varepsilon); f(r) \equiv 0, \quad in \quad [0, R]\}.$$

Proposition 1. *We have*

$$f > 0, \quad in \quad (\sigma, 1];$$

$$f_r > 0, \quad in \quad [\sigma, h\varepsilon].$$

Proof. The idea of the method of moving planes is used here. Thus, consider the equation (1.1) multiplied with r^{-1},

$$-A^{(p-2)/2} f_{rr} - (A^{(p-2)/2})_r f_r - A^{(p-2)/2}\frac{f_r}{r} + A^{(p-2)/2}\frac{f}{r^2} = \frac{f^{p-1}}{\varepsilon^p}(1 - f^p). \tag{2.1}$$

Step 1. We claim that $f_r(h\varepsilon - 0) > 0$.

In fact, if the claim is not true, then we can consider two cases:

$$Case \quad I: \quad f(r) \equiv f(h\varepsilon), \quad in \quad (h\varepsilon - \delta, h\varepsilon);$$

$$Case \quad II: \quad f(r) > f(h\varepsilon), \quad in \quad (h\varepsilon - \delta, h\varepsilon)$$

when δ is sufficiently small. In Case I, (2.1) becomes

$$\frac{f^{p-1}}{r^p} = \frac{f^{p-1}}{\varepsilon^p}(1-f^p), \quad in \quad (h\varepsilon-\delta, h\varepsilon).$$

By virtue of $r < h\varepsilon$, the result above implies

$$\frac{f^{p-1}}{\varepsilon^p}(h^{-p}-(1-f^p)) < 0.$$

Noting $r > h\varepsilon-\delta > \sigma$, we see that $f > 0$. Therefore,

$$h^{-p}-(1-f^p) < 0.$$

It is impossible if we let η in h be sufficiently small (cf. Remark 1 in \$1).

In Case II, we can find a maximizer $r_0 \in (\sigma, h\varepsilon)$ of f. Then,

$$f(r_0) > 0, \quad f_r(r_0) = 0, \quad and \quad f_{rr}(r_0) < 0.$$

Hence, (2.1) with $r = r_0$ leads to

$$\frac{f^{p-1}(r_0)}{\varepsilon^p}[h^{-p}-(1-f^p(r_0))] < 0.$$

This is also impossible when we choose η sufficiently small such that $h < 1$.

Step 2. Moving planes from σ to $h\varepsilon$. For $\mu \in [\sigma, h\varepsilon]$, let $r_\mu = 2\mu - r$ be the reflection point of r about the point μ. Define

$$F_\mu(r) = f(r_\mu) - f(r), \quad as \quad r \in (\mu-\delta, \mu).$$

We claim that the set

$$S := \quad \{\mu \in [\sigma, h\varepsilon]; \quad \text{there is} \quad r' \in (\mu-\delta, \mu)$$
$$\text{such that} \quad F_\mu(r') = 0\} = \varnothing.$$

Otherwise, we can suppose

$$\mu_* = \min_{\mu \in S} \mu.$$

Then, we claim $\mu_* > \sigma$. In fact, if $\mu_* = \sigma$, then there exists $r' \in (\sigma-\delta, \sigma)$ such that

$$f(r'_\sigma) - f(r') = 0.$$

On the other hand, in view of the definition of σ, we get

$$f(r') = 0, \quad f(r'_\sigma) > 0.$$

Therefore, we see the contradiction and hence the claim means

$$\mu_* \in (\sigma, h\varepsilon].$$

This consequence implies either $f \equiv Constant$ in some open interval $I \in (r', r'_\mu)$, or a maximizer of f in (r', r'_μ). Similar to the argument in Cases I and II of Step 1, we can also deduce the contradiction. Thus, $S = \varnothing$.

The result $S=\varnothing$ shows that $f(r')<f(r'_\mu)$, $\forall r'\in(\mu-\delta,\mu)$. After μ moves from σ to $h\varepsilon$, it follows that $f(r_1)<f(r_2)$ as long as $\sigma<r_1<r_2\le h\varepsilon$. Therefore,

$$f_r(r)>0,\quad \forall r\in(\sigma,h\varepsilon]. \tag{2.2}$$

By virtue of the definition of σ, we have

$$f(\sigma)=0;\quad f(r)>0\quad \forall r\in(\sigma,h\varepsilon]. \tag{2.3}$$

This implies $f_r(\sigma+0)>0$. Combining with (2.2) yields

$$f_r(r)>0,\quad \forall r\in[\sigma,h\varepsilon].$$

Combining (2.3) with Remark 1 in §1, we can see $f(r)>0$ in $(\sigma,1]$.

Thus, we complete the proof of Proposition 1. □

Proposition 2. *Let $\sigma\in[0,h\varepsilon)$ satisfy*

$$f(r)\equiv 0,\quad in\quad [0,\sigma];$$

$$f_r(r)>0,\quad in\quad [\sigma,h\varepsilon].$$

Then $\sigma=0$.

Proof. Suppose $\sigma>0$. Since the Pohozaev identity can show the properties of two end points of an interval by investigating the integrals on this interval, we use the idea of proof of Pohozaev's identity here to deduce a contradiction.

Multiplying (1.1) by f_r and integrating on $(0,\sigma)$, we have

$$\begin{aligned}&-\int_0^\sigma (A^{\frac{p-2}{2}}f_r r)_r f_r dr+\int_0^\sigma A^{\frac{p-2}{2}}\frac{f}{r^2}f_r r dr\\&=\frac{1}{\varepsilon^p}\int_0^\sigma f^{p-1}(1-f^p)f_r r dr.\end{aligned} \tag{2.4}$$

By $f(r)=0$ in $[0,\sigma]$, (2.4) becomes

$$\int_0^\sigma (A^{\frac{p-2}{2}}f_r r)_r f_r dr=0. \tag{2.5}$$

Calculating the left hand side of (2.5), we obtain that

$$\begin{aligned}&-\int_0^\sigma (A^{\frac{p-2}{2}}f_r r)_r f_r dr\\&=-\int_0^\sigma A^{\frac{p-2}{2}}f_r f_{rr} r dr-\int_0^\sigma (A^{\frac{p-2}{2}})_r (f_r)^2 r dr-\int_0^\sigma A^{\frac{p-2}{2}}(f_r)^2 dr\\&=-\int_0^\sigma A^{\frac{p-2}{2}}f_r f_{rr} r dr-A^{\frac{p-2}{2}}(f_r)^2 r|_{r=\sigma}\\&\quad+\int_0^\sigma A^{\frac{p-2}{2}}((f_r)^2 r)_r dr-\int_0^\sigma A^{\frac{p-2}{2}}(f_r)^2 dr\\&=\int_0^\sigma A^{\frac{p-2}{2}}f_r f_{rr} r dr-A^{\frac{p-2}{2}}(f_r)^2 r|_{r=\sigma}.\end{aligned}$$

This result, together with (2.5), implies

$$\int_0^\sigma A^{\frac{p-2}{2}} f_r f_{rr} r dr = A^{\frac{p-2}{2}} (f_r)^2 r|_{r=\sigma}. \tag{2.6}$$

Noting $f(r) = 0$ in $(0, \sigma)$, we can deduce that the left hand side of (2.6) is zero. In view of $f_r(\sigma) > 0$, we obtain that the right hand side of (2.6) is nonzero, which leads to a contradiction. □

Combining Propositions 1 and 2, we can complete the proof of Theorem 1.

ACKNOWLEDGEMENT

The authors express their gratitude to the referees for giving helpful comments. Their suggestions have improved this paper.

REFERENCES

[1] A. Alexandrov, "Uniqueness theorems for surfaces in the large," *Ameri. Math. Soc. Transl. Ser.2.*, vol. 21, pp. 412–416, 1962.

[2] Y. Almog, L. Berlyand, D. Golovaty, and I. Shafrir, "Global minimizers for a p-ginzburg-landau-type energy in $\mathbf{r}^2$," *J. Funct. Anal.*, vol. 256, pp. 2268–2290, 2009, doi: 10.1016/j.jfa.2008.09.020.

[3] Y. Almog, L. Berlyand, D. Golovaty, and I. Shafrir, "Radially symmetric minimizers for a p-ginzburg landau type energy in $\mathbf{r}^2$," *Calc. Var. Partial Differential Equations*, vol. 42, pp. 517–546, 2011, doi: 10.1007/s00526-011-0396-9.

[4] P. Bauman, N. Carlson, and D. Phillips, "On the zeros of solutions to ginzburg-landau type systems," *SIAM J. Math. Anal.*, vol. 24, pp. 1283–1293, 1993, doi: 10.1137/0524073.

[5] F. Bethuel, H. Brezis, and F. Helein, *Ginzburg-Landau Vortices*. Berlin: Birkhauser, 1994.

[6] W. Chen and C. Li, "Classification of solutions of some nonlinear elliptic equations," *Duke Math. J.*, vol. 63, pp. 615–622, 1991, doi: 10.1215/S0012-7094-91-06325-8.

[7] B. Gidas, W.-M. Ni, and L. Nirenberg, "Symmetry and related properties via the maximum principle," *Comm. Math. Phys.*, vol. 68, pp. 209–243, 1979, doi: 10.1007/BF01221125.

[8] R. M. Herve and M. Herve, "Etude qualitative des solutions reelles d'une equation differentielle liee a l'equation de ginburg-landau," *Ann. Inst. H. Poincare Anal. NonLineaire*, vol. 11, pp. 427–440, 1994.

[9] Y. Lei, "Asymptotic estimation for a p-ginzburg-landau type minimizer in higher dimensions," *Pacific J. Math.*, vol. 226, pp. 103–135, 2006, doi: 10.2140/pjm.2006.226.103.

[10] Y. Lei, "Asymptotic estimations for a p-ginzburg-landau type minimizer," *Math. Nachr.*, vol. 280, pp. 1559–1576, 2007, doi: 10.1002/mana.200410565.

[11] Y. Lei, "Singularity analysis of a p-ginzburg-landau type minimizer," *Bull. Sci. Math.*, vol. 134, pp. 97–115, 2010, doi: 10.1016/j.bulsci.2008.02.004.

[12] Y. Lei, Z. Wu, and H. Yuan, "Radial minimizers of a ginzburg-landau type functional," *Electron. J. Diff. Equs.*, vol. 1999, no. 30, pp. 1–21, 1999.

[13] E. Lieb and M. Loss, "Symmetry of the ginzburg-landau minimizer in a disc," *Math. Res. Lett.*, vol. 1, pp. 701–715, 1995, doi: 10.4310/MRL.1994.v1.n6.a7.

[14] P. Mironescu, "On the stability of radial solution of the ginzburg-landau equation," *J. Funct. Anal.*, vol. 130, pp. 334–344, 1995, doi: 10.1006/jfan.1995.1073.

[15] J. Serrin, "A symmetry problem in potential theory," *Arch. Rational Mech. Anal.*, vol. 43, pp. 304–318, 1971, doi: 10.1007/BF00250468.

[16] C. Wang, "Limits of solutions to the generalized ginzburg-landau functional," *Comm. Partial Differetial Equations*, vol. 27, pp. 877–906, 2002, doi: 10.1081/PDE-120004888.

Authors' addresses

Yuze Cai
Shazhou Professional Institute of Technology, Department of Basic Science, Zhangjiagang, Jiangsu, 215600, China

Yutian Lei
Nanjing Normal University, Institute of Mathematics, School of Mathematical Sciences, Nanjing, Jiangsu, 210023, China
E-mail address: leiyutian@njnu.ede.cn

Bei Wang
Jiangsu Second Normal University, School of Mathematics and Information Technology, Nanjing, Jiangsu, 210013, China