Miskolc Mathematical Notes HU e-ISSN 1787-2413
Vol. 16 (2015), No. 2, pp. 637–646 DOI: 10.18514/MMN.2015.1192

ON THE METRIC DIMENSION OF BARYCENTRIC SUBDIVISION OF CAYLEY GRAPHS $Cay(Z_n \oplus Z_m)$

A. AHMAD, M. IMRAN, O. AL-MUSHAYT, AND S. A. U. H. BOKHARY

Received 03 April, 2014

Abstract. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. W is called a *resolving set* or a *locating set* if every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a *metric basis* for G and this cardinality is the *metric dimension* of G, denoted by $dim(G)$.
Metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists).
In this paper, we study the metric dimension of barycentric subdivision of Cayley graphs $Cay(Z_n \oplus Z_m)$. We prove that these subdivisions of Cayley graphs have constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of Cayley graphs $Cay(Z_n \oplus Z_m)$.

2010 *Mathematics Subject Classification:* 05C12

Keywords: metric dimension, basis, resolving set, barycentric subdivision, Cayley graph

1. INTRODUCTION AND PRELIMINARY RESULTS

Metric dimension is a parameter that has appeared in various applications of graph theory, as diverse as, pharmaceutical chemistry [3], robot navigation [13], combinatorial optimization [15] and sonar and coast guard Loran [16], to name a few. Metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists).
In a connected graph G, the *distance* $d(u,v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), d(v,w_3), \ldots, d(v,w_k))$. W is called a *resolving set* [3] or *locating set* [16] if every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is

The research for this article was supported by Jazan University Grant 26/4/33.

called a *basis* for G and this cardinality is the *metric dimension* or *location number* of G, denoted by $\beta(G)$ [2].
For a given ordered set of vertices $W = \{w_1, w_2, \ldots, w_k\}$ of a graph G, the ith component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \backslash W$.
A useful property in finding $\beta(G)$ is the following lemma:

Lemma 1 ([17])**.** *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \emptyset$.*

Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n \geq 1}$ depending on n as follows: the order $|V(G)| = \varphi(n)$ and $\lim\limits_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $\beta(G_n) \leq C$ for every $n \geq 1$, then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has unbounded metric dimension.
If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on n), $\mathcal{F}$ is called a family with constant metric dimension [12]. Some classes of *regular graphs* with constant metric dimension have been studied in [1, 10] recently while metric dimension of some classes of *convex polytopes* has been determined in [7] and [9].
Other families of graphs have unbounded metric dimension: if W_n denotes a *wheel* with n spokes and J_{2n} the graph deduced from the wheel W_{2n} by alternately deleting n spokes, then $\beta(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for every $n \geq 7$ [2] and $\beta(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ [18] for every $n \geq 4$. The generalized Petersen graphs $P(n,3)$ have bounded metric dimension [8].
The graphs having metric dimension 1 are characterized in the following theorem.

Theorem 1 ([3])**.** *The metric dimension of a graph G is 1 if and only if $G \cong P_n$, where P_n denotes a path of length $n-1$ or G is one-way infinite path.*

The next theorem gives a nice property of the graphs with metric dimension 2.

Theorem 2 ([14])**.** *Let G be a graph with metric dimension 2 and let $\{v_1, v_2\} \subseteq V(G)$ be a metric basis in G, then the degree of both v_1 and v_2 is at most 3.*

Geometrically, subdividing an edge is an operation that inserts a new vertex into the interior that results in splitting that edge into two edges. *Subdividing a graph G* means performing a sequence of edge-subdivision operations. The resulting graph is called a *subdivision of the graph G*. The operation of subdivision can be used to convert a general graph into a simple graph. The *barycentric subdivision* of a graph G is the subdivision in which one new vertex is inserted in the interior of each edge. The following propositions give some nice results related to barycentric subdivision of a graph (see [6]).

- The barycentric subdivision of any graph is a bipartite graph.
- The barycentric subdivision of any graph yields a loopless graph.

- The barycentric subdivision of any loopless graph yields a simple graph.

A graph G is *planar* if it can be drawn in the plane without edge crossings. Subdivision of graphs play a very important role in characterization of planar graphs. A graph G is planar if and only if every subdivision of G is planar. Two graphs are said to be homeomorphic if they are subdivisions of same graph G. The next theorem gives a nice characterization of planar graphs.

Theorem 3 ([6]). *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

Note that the problem of determining whether $\beta(G) < k$ is an NP-complete problem [5].
In this paper, we study the metric dimension of barycentric subdivision of Cayley graphs $Cay(Z_n \oplus Z_m)$. We prove that these subdivisions of Cayley graphs have constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of these subdivision of Cayley graphs $Cay(Z_n \oplus Z_m)$.

2. THE METRIC DIMENSION OF BARYCENTRIC SUBDIVISION OF CAYLEY GRAPHS $Cay(Z_n \oplus Z_m)$

Let G be a semigroup, and let S be a nonempty subset of G. The Cayley graph $Cay(G,S)$ of G relative to S is defined as the graph with vertex set G and edge set $E(S)$ consisting of those ordered pairs (x,y) such that $sx = y$ for some $s \in S$. Cayley graphs of groups are significant both in group theory and in constructions of interesting graphs with nice properties. The Cayley graph $Cay(G,S)$ of a group G is symmetric or undirected if and only if $S = S^{-1}$.
The Caylay graphs $Cay(Z_n \oplus Z_m), n \geq 3, m \geq 2$, is a graph which can be obtained as the cartesian product $P_m \Box C_n$ of a path on m vertices with a cycle on n vertices. The vertex set and edge set of $Cay(Z_n \oplus Z_m)$ defined as: $V(Cay(Z_n \oplus Z_m)) = \{(x_i, y_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(Cay(Z_n \oplus Z_m)) = \{(x_i, y_j)(x_{i+1}, y_j) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(x_i, y_j)(x_i, y_{j+1}) : 1 \leq i \leq n, 1 \leq j \leq m-1\}$. We have $|V(Cay(Z_n \oplus Z_m)| = mn$, $|E(Cay(Z_n \oplus Z_m)| = (2m-1)n$, where $|V(Cay(Z_n \oplus Z_m)|, |E(Cay(Z_n \oplus Z_2)|$ denote the number of vertices, edges of the Cayley graphs $Cay(Z_n \oplus Z_m)$, respectively.

The metric dimension of Cayley graphs $Cay(Z_n \oplus Z_2)$ has been determined in [4] while the metric dimension of Cayley graphs $Cay(\mathbb{Z}_n : S)$ for all $n \geq 7$ and $S = \{\pm 1, \pm 3\}$ hase been determined in [11].
The barycentric subdivision graph $S(Cay(Z_n \oplus Z_m))$ can be obtained by adding a new vertex u_i^j between (x_i, y_j) and (x_{i+1}, y_j) and adding a new vertex v_i^j between (x_i, y_j) and (x_i, y_{j+1}). Clearly, $S(Cay(Z_n \oplus Z_m))$ has $3nm - n$ vertices and $4nm - 2n$ edges.
The metric dimension of $P_m \Box C_n$ has been determined in [4] and Cayley graphs $Cay(Z_n \oplus Z_2)$ is actually the cartesian product of $P_2 \Box C_n$. In the next theorem, we

prove that the metric dimension of the barycentric subdivision $S(Cay(Z_n \oplus Z_m))$ is constant and only three vertices appropriately chosen suffice to resolve all the vertices of the $S(Cay(Z_n \oplus Z_m))$. Note that the choice of appropriate basis vertices (also refereed to as landmarks in [14]) is the core of the problem. For our purpose, we call the sets of points as $S_1 = \{(x_i, y_j) : 1 \le i \le n, 1 \le j \le m\}$, $S_2 = \{u_i^j : 1 \le i \le n, 1 \le j \le m\}$ and $S_3 = \{v_i^j : 1 \le i \le n, 1 \le j \le m-1\}$.

Theorem 4. *Let $S(Cay(Z_n \oplus Z_m))$ be the barycentric subdivision of Cayley graphs $Cay(Z_n \oplus Z_m)$; then $\beta(S(Cay(Z_n \oplus Z_m))) = 3$ for every $n \ge 6$.*

Proof. We will prove the above equality by double inequalities.
Case 1. When n is even.
Let $W = \{(x_1, y_1), (x_2, y_1), (x_{\frac{n}{2}+1}, y_1)\} \subset V(S(Cay(Z_n \oplus Z_m)))$, we show that W is a resolving set for $S(Cay(Z_n \oplus Z_m))$ in this case. For this we give representations of any vertex of $V(S(Cay(Z_n \oplus Z_m)))W$ with respect to W.
Representations for the vertices of $S(Cay(Z_n \oplus Z_m))$ are

$$r((x_i, y_j)|W) = \begin{cases} (2(j-1), 2j, n+2(j-1)), i = 1 \ \& \ 1 \le j \le m \\ (2j, 2(j-1), n+2(j-2)), i = 2 \ \& \ 1 \le j \le m \\ (2(i+j-2), 2(i+j-3), 2(\frac{n}{2}+j-i)), 3 \le i \le \frac{n}{2} \& 1 \le j \le m \\ (n+2(j-1), n+2(j-2), 2(j-1)), i = \frac{n}{2}+1 \ \& \ 1 \le j \le m \\ (2(n+j-i), 2(n+j+1-i), 2(j+i-2)-n), \frac{n}{2}+2 \le i \le n \\ \qquad \& \ 1 \le j \le m \end{cases}$$

and

$$r(u_i^j|W) = \begin{cases} (2j-1, 2j-1, n+2j-3), i = 1 \ \& \ 1 \le j \le m \\ (2(i+j)-3, 2(i+j)-5, n-1-2(i-j)), 2 \le i \le \frac{n}{2} \\ \qquad \& \ 1 \le j \le m \\ (n+2j-3, n+2j-3, 2j-1), i = \frac{n}{2}+1 \ \& \ 1 \le j \le m \\ (2(n+j-i)-1, 2(n+j-i)+1, 2(j+i)-n-3), \frac{n}{2}+2 \le i \le n \\ \qquad \& \ 1 \le j \le m \end{cases}$$

and

$$r(v_i^j|W) = \begin{cases} (2j-1, 2j+1, n+2j-1), i = 1 \ \& \ 1 \le j \le m-1 \\ (2(i+j)-3, 2(i+j)-5, n+1-2(i-j)), 2 \le i \le \frac{n}{2}+1 \\ \qquad \& \ 1 \le j \le m-1 \\ (2(n+j-i)+1, 2(n+j-i)+3, 2(j+i)-n-3), \frac{n}{2}+2 \le i \le n \\ \qquad \& \ 1 \le j \le m-1 \end{cases}$$

We note that there are no two vertices having the same representations implying that $\beta(S(Cay(Z_n \oplus Z_m))) \le 3$.
On the other hand, we show that $\beta(S(D_n)) \ge 3$. Suppose on contrary that $\beta(S(Cay(Z_n \oplus Z_m))) = 2$, then there are the following possibilities to be discussed. The behavior of the outmost cycle and innermost cycle are same.
(1) Both vertices are in the same set. Here are the following subcases.
• Both vertices belong to the set S_1. When Without loss of generality, we can suppose that one resolving vertex is (x_1, y_1). Suppose that the second resolving vertex

is (x_p, y_j) $(2 \le p \le \frac{n}{2}+1, 1 \le j \le m)$. Then for $2 \le p \le \frac{n}{2}, 1 \le j \le m-1$, we have

$$r(u_n^j|\{(x_1,y_j),(x_p,y_j)\}) = r(v_1^j|\{(x_1,y_j),(x_p,y_j)\}) = (1,2p-1).$$

For $j = m$, we have

$$r(u_n^m|\{(x_1,y_m),(x_p,y_m)\}) = r(v_1^{m-1}|\{(x_1,y_m),(x_p,y_m)\}) = (1,2p-1)$$

and for $p = \frac{n}{2}+1, 1 \le j \le m$ we have

$$r(u_1^j|\{(x_1,y_j),(x_p,y_j)\}) = r(u_n^j|\{(x_1,y_j),(x_p,y_j)\}) = (1,n-1),$$

a contradiction.

• Both vertices belong to the set S_2. Without loss of generality, we can suppose that one resolving vertex is u_1^j. There are two possibilities. When second resolving vertex on the same level. Suppose that the second resolving vertex is u_p^j $(2 \le p \le \frac{n}{2}+1)$. Then for $2 \le p \le \frac{n}{2}$, we have

$$r(u_n^j|\{u_1^j,u_p^j\}) = r(v_1^j|\{u_1^j,u_p^j\}) = (2,2p),$$

for $j = m$, we have

$$r(u_n^m|\{u_1^m,u_p^m\}) = r(v_1^{m-1}|\{u_1^m,u_p^m\}) = (2,2p)$$

and for $p = \frac{n}{2}+1$, we have

$$r((x_1,y_j)|\{u_1^j,u_{\frac{n}{2}+1}^j\}) = r((x_2,y_j)|\{u_1^j,u_{\frac{n}{2}+1}^j\}) = (1,n-1),$$

a contradiction. When second resolving vertex on the different level. Suppose that the second resolving vertex is u_p^s $(1 \le p \le \frac{n}{2}+1, s > j)$. Then for $p = 1$, we have

$$r((x_1,y_j)|\{u_1^j,u_1^s\}) = r((x_2,y_j)|\{u_1^j,u_1^s\})) = (1,2(s-j)+1),$$

for $p = 2$, we have

$$r(v_3^{s-1}|\{u_1^j,u_2^s\}) = r(u_1^s|\{u_1^j,u_2^s\}) = (2(s+1-j),2)$$

and when $3 \le p \le \frac{n}{2}+1$, we have

$$r(v_3^j|\{u_1^j,u_p^s\}) = r(u_2^{j+1}|\{u_1^j,u_p^s\}) = (4,2(p+s-j-3)),$$

a contradiction.

• Both vertices belong to the set S_3. Without loss of generality, we can suppose that one resolving vertex is v_1^j, $1 \le j \le m-1$. Suppose that the second resolving vertex is v_p^s $(2 \le p \le \frac{n}{2}+1, 1 \le j,s \le m-1, j = s)$. Then for $2 \le p \le \frac{n}{2}+1$, we have

$$r((x_1,y_j)|\{v_1^j,v_p^j\}) = r((x_1,y_{j+1})|\{v_1^j,v_p^j\}) = (1,2p-1),$$

a contradiction. If one resolving vertex is v_p^j and second resolving vertex is v_p^s for $1 \le p \le \frac{n}{2}+1, 1 \le j,s \le m-1, s > j$, then we have

$$r(u_p^j|\{v_p^j,v_p^s\}) = r(u_{p-1}^j|\{v_p^j,v_p^s\}) = (2j,2s),$$

a contradiction. Now if we can suppose that one resolving vertex is $v_1^j, 1 \le j \le m-1$. Suppose that the second resolving vertex $v_p^s, (2 \le p \le \frac{n}{2}+1, 1 \le s \le m-1, 1 \le j \le m-2, s > j)$. For $2 \le p \le \frac{n}{2}$, we have

$$r(u_1^j|\{v_1^j, v_p^s\}) = r(u_n^{j+1}|\{v_1^j, v_p^s\}) = (2, 2(s+p-j-1)),$$

and for $p = \frac{n}{2}+1$, we have

$$r(u_1^j|\{v_1^j, v_p^s\}) = r(u_n^j|\{v_1^j, v_p^s\}) = (2, n+2s-2j),$$

a contradiction.

(2) Both vertices are not in the same set. Here are the following subcases.

• One vertex in the set S_1 and the other one in the set S_2 but in the same level. Without loss of generality, we can suppose that one resolving vertex is $(x_1, y_j), 1 \le j \le m$. Suppose that the second resolving vertex is u_p^j $(1 \le p \le \frac{n}{2}+1, 1 \le j \le m)$. Then for $1 \le p \le \frac{n}{2}, 1 \le j \le m-1$, we have

$$r(u_n^j|\{(x_1, y_j), u_p^j\}) = r(v_1^j|\{(x_1, y_j), u_p^j\}) = (1, 2p)$$

and for $1 \le p \le \frac{n}{2}, j = m$, we have

$$r(u_n^m|\{(x_1, y_m), u_p^m\}) = r(v_1^{m-1}|\{(x_1, y_m), u_p^m\}) = (1, 2p),$$

a contradiction. For $p = \frac{n}{2}+1, 1 \le j \le m-1$ we have

$$r(u_1^j|\{(x_1, y_j), u_{\frac{n}{2}+1}^j\}) = r(v_1^j|\{(x_1, y_j), u_{\frac{n}{2}+1}^j\}) = (1, n)$$

and for $p = \frac{n}{2}+1, j = m$ we have

$$r(u_1^m|\{(x_1, y_m), u_{\frac{n}{2}+1}^m\}) = r(v_1^{m-1}|\{(x_1, y_m), u_{\frac{n}{2}+1}^m\}) = (1, n),$$

a contradiction. Now one vertex in the set S_1 and the other one in the set S_2 but in the different level $(1 \le p \le \frac{n}{2}+1, 1 \le j \le m-1, 1 \le s \le m, s > j)$. Without loss of generality, we can suppose that one resolving vertex is $(x_1, y_j), 1 \le j \le m$. Suppose that the second resolving vertex is u_p^j $(1 \le p \le \frac{n}{2}+1, 1 \le j \le m)$. Then for $p = 1$, we have

$$r(u_1^j|\{(x_1, y_j), u_1^s\}) = r(u_n^j|\{(x_1, y_j), u_1^s\}) = (1, 2(s-j+1)),$$

and for $2 \le p \le \frac{n}{2}$, we have

$$r(u_1^j|\{(x_1, y_j), u_p^s\}) = r(v_1^j|\{(x_1, y_j), u_p^s\}) = (1, 2(s+p-j-1)).$$

For $p = \frac{n}{2}+1$, we have

$$r(u_n^j|\{(x_1, y_j), u_p^s\}) = r(v_1^j|\{(x_1, y_j), u_p^s\}) = (1, 2(s-j-1)+n),$$

a contradiction.

• One vertex in the set S_1 and the other one in the set S_3 but in the same level. Without loss of generality, we can suppose that one resolving vertex is $(x_1, y_j), 1 \le j \le m$.

Suppose that the second resolving vertex is v_p^s $(1 \leq p \leq \frac{n}{2}+1, 1 \leq s \leq m-1)$. Then for $p=1, 1 \leq j, s \leq m-1, s \geq j$, we have

$$r(u_1^j|\{(x_1,y_j),v_p^s\}) = r(u_n^j|\{(x_1,y_j),v_p^s\}) = (1, 2(s-j+1)),$$

a contradiction. For $2 \leq p \leq \frac{n}{2}, 1 \leq j, s \leq m-1, s = j$, we have

$$r(u_n^j|\{(x_1,y_j),v_p^j\}) = r(v_1^j|\{(x_1,y_j),v_p^j\}) = (1, 2p),$$

for $p = \frac{n}{2}+1$, we have

$$r(u_1^j|\{(x_1,y_j),v_p^j\}) = r(u_n^j|\{(x_1,y_j),v_p^j\}) = (1, n),$$

a contradiction. For $2 \leq p \leq \frac{n}{2}, 1 \leq j, s \leq m-1, s > j$, we have

$$r(u_1^j|\{(x_1,y_j),v_p^s\}) = r(v_1^j|\{(x_1,y_j),v_p^s\}) = (1, 2(s+p-j-1)),$$

for $p = \frac{n}{2}+1$, we have

$$r(u_1^j|\{(x_1,y_j),v_p^s\}) = r(u_n^j|\{(x_1,y_j),v_p^s\}) = (1, 2(s+p-j-1)),$$

a contradiction.

• One vertex in the set S_2 and the other one in the set S_3 but in the same level. Without loss of generality, we can suppose that one resolving vertex is $u_1^j, 1 \leq j \leq m$. Suppose that the second resolving vertex is v_1^s $(1 \leq p \leq \frac{n}{2}+1, 1 \leq s \leq m-1)$. Then for $p=1, 1 \leq j, s \leq m-1, s \geq j$, we have

$$r(u_1^{s+1}|\{u_1^j, v_1^s\}) = r(u_n^{s+1}|\{u_1^j, v_1^s\}) = (2(s-j+2), 2),$$

a contradiction. For $2 \leq p \leq \frac{n}{2}, 1 \leq j \leq m, 1 \leq s \leq m-1, s = j$, we have

$$r(u_n^j|\{u_1^j, v_p^j\}) = r(v_1^j|\{u_1^j, v_p^j\}) = (2, 2p),$$

for $p = \frac{n}{2}+1$, we have

$$r(u_n^j|\{u_1^j, v_p^j\}) = r(v_2^j|\{u_1^j, v_p^j\}) = (2, n),$$

a contradiction. For $p = 2, 1 \leq j, s \leq m-1, s > j$, we have

$$r(u_2^j|\{u_1^j, v_p^s\}) = r(v_n^j|\{u_1^j, v_p^s\}) = (2, 2(s-j+1)),$$

for $3 \leq p \leq \frac{n}{2}+1$, we have

$$r(u_2^j|\{u_1^j, v_p^s\}) = r(v_2^j|\{u_1^j, v_p^s\}) = (2, 2(s+p-j-2)),$$

a contradiction.

Case 2. When n is even.

Let $W = \{(x_1,y_1),(x_2,y_1),u_{\frac{n}{2}+1}\} \subset V(S(Cay(Z_n \oplus Z_m)))$, we show that W is a resolving set for $S(Cay(Z_n \oplus Z_m))$ in this case. For this we give representations of any vertex of $V(S(Cay(Z_n \oplus Z_m)))W$ with respect to W.

Representations for the vertices of $S(Cay(Z_n \oplus Z_m))$ are

$$r((x_i,y_j)|W)=\begin{cases}(2(j-1),2j,n+2j-2), i=1 \ \& \ 1\le j\le m\\ (2j,2(j-1),n+2j-4), i=2 \ \& \ 1\le j\le m\\ (2(i+j-2),2(i+j-3),n-2(i-j)), 3\le i\le \frac{n+1}{2}\\ \qquad \& \ 1\le j\le m\\ (n+2j-3,n+2j-3,2j-1), i=\frac{n+3}{2} \ \& \ 1\le j\le m\\ (2(n+j-i),2(n+j+1-i),2(j+i)-n-3), \frac{n+5}{2}\le i\le n\\ \qquad \& \ 1\le j\le m\end{cases}$$

$$r(u_i^j|W)=\begin{cases}(2j-1,2j-1,n+2j-3), i=1 \ \& \ 1\le j\le m\\ (2(i+j)-3,2(i+j)-5,n-1+2(j-i)), 2\le i\le \frac{n-1}{2}\\ \qquad \& \ 1\le j\le m\\ (n+2j-2,n+2j-4,2j-2), i=\frac{n+1}{2} \ \& \ 1\le j\le m\\ (2(n+j-i)-1,2(n+j-i)+1,2(j+i)-n-3), \frac{n+3}{2}\le i\le n\\ \qquad \& \ 1\le j\le m\end{cases}$$

and

$$r(v_i^j|W)=\begin{cases}(2j-1,2j+1,n+2j-1), i=1 \ \& \ 1\le j\le m-1\\ (2(i+j)-3,2(i+j)-5,n-2i+2j+1)), 2\le i\le \frac{n+1}{2}\\ \qquad \& \ 1\le j\le m-1\\ (n+2(j-1),n+2(j-1),2j), i=\frac{n+3}{2} \ \& \ 1\le j\le m-1\\ (2(n+j-i)+1,2(n+j-i+1)+1,2j+2i-n-3),\\ \qquad \frac{n+5}{2}\le i\le n \& \ 1\le j\le m-1\end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $\beta(S(Cay(Z_n\oplus Z_m)))\le 3$.
On the other hand, suppose that $\beta(S(Cay(Z_n\oplus Z_m)))=2$, then there are the same possibilities as in case (1) and contradictions can be deduced analogously. This implies that $\beta(S(Cay(Z_n\oplus Z_m)))=3$ in this case, which completes the proof. □

3. CONCLUSION

The problem of determining whether $\beta(G)<k$ is an NP-complete problem. In this paper, we have studied the metric dimension of barycentric subdivision of Cayley graphs $Cay(Z_n\oplus Z_m)$. We proved that these subdivisions of Cayley graphs have constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of subdivisions of Cayley graphs $Cay(Z_n\oplus Z_m)$. It is natural to ask for characterization of graph classes with constant metric dimension.

ACKNOWLEDGEMENT

The authors are very grateful to the referee for his careful reading with corrections and useful comments, which improved this work very much.

REFERENCES

[1] M. Ali, G. Ali, M. Imran, A. Baig, and M. K. Shafiq, "On the metric dimension of mobius ladders," *Ars Combinatoria*, vol. 105, pp. 403–410, 2012.

[2] P. Buczkowski, G. Chartrand, C. Poisson, and P. Zhang, "On k-dimensional graphs and their bases," *Periodica Mathematica Hungarica*, vol. 46, no. 1, pp. 9–15, 2003, doi: 10.1023/A:1025745406160.

[3] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, "Resolvability in graphs and the metric dimension of a graph," *Discrete Applied Mathematics*, vol. 105, no. 1-3, pp. 99 – 113, 2000, doi: 10.1016/S0166-218X(00)00198-0.

[4] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, and D. R. Wood, "On the metric dimension of cartesian products of graphs," *SIAM Journal on Discrete Mathematics*, vol. 21, no. 2, pp. 423–441, 2007, doi: 10.1137/050641867.

[5] M. R. Garey and D. S. Johnson, *Computers and Intractability; A Guide to the Theory of NP-Completeness*. New York, NY, USA: W. H. Freeman & Co., 1990.

[6] J. L. Gross and J. Yellen, *Graph Theory and Its Applications, Second Edition*. New York, USA: Chapman and Hall/CRC, 2005.

[7] M. IMRAN, A. Q. BAIG, and A. AHMAD, "Families of plane graphs with constant metric dimension," *Utilitas Mathematica*, vol. 88, pp. 43 – 57, 2012.

[8] M. Imran, A. Baig, M. K. Shafiq, and I. Tomescu, "On metric dimension of generalized petersen graphs $p_(n,3)$," *Ars Combinntoria*, vol. 117, pp. 13–130, 2014.

[9] M. Imran, S. A. U. H. Bokhary, and A. Baig, "On families of convex polytopes with constant metric dimension," *Computers & Mathematics with Applications*, vol. 60, no. 9, pp. 2629 – 2638, 2010, doi: 10.1016/j.camwa.2010.08.090.

[10] M. IMRAN, S. A. ul Haq BOKHARY, A. AHMAD, and A. SEMANIČOVÁ-FEŇOVČÍKOVÁ, "On classes of regular graphs with constant metric dimension," *Acta Mathematica Scientia*, vol. 33, no. 1, pp. 187 – 206, 2013, doi: 10.1016/S0252-9602(12)60204-5.

[11] I. JAVAID, M. N. AZHAR, and M. SALMAN, "Metric dimension and determining number of cayley graphs," *World Applied Sciences Journal*, vol. 18, no. 12, pp. 1800 – 1812, 2012.

[12] I. JAVAID, M. T. Rahim, and K. ALI, "Families of regular graphs with constant metric dimension," *Utilitas Mathematica*, vol. 75, pp. 21 – 33, 2008.

[13] S. Khuller, B. Raghavachari, and A. Rosenfeld, *Localization in Graphs*, ser. Computer science technical report series. University of Maryland, 1994.

[14] S. Khuller, B. Raghavachari, and A. Rosenfeld, "Landmarks in graphs," *Discrete Applied Mathematics*, vol. 70, no. 3, pp. 217 – 229, 1996, doi: 10.1016/0166-218X(95)00106-2.

[15] A. Sebő and E. Tannier, "On metric generators of graphs," *Mathematics of Operations Research*, vol. 29, no. 2, pp. pp. 383–393, 2004, doi: 10.1287/moor.1030.0070.

[16] P. J. Slater, "Leaves of trees," *Congress Number*, vol. 14, no. 37, pp. 549–559, 1975.

[17] I. Tomescu and M. Imran, "On metric and partition dimensions of some infinite regular graphs," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 52, no. 4, pp. 461–472, 2009.

[18] I. Tomescu and I. Javaid, "On the metric dimension of the jahangir graph," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 50, no. 4, pp. 371–376, 2007.

Authors' addresses

A. Ahmad

Faculty of Computer Science & Information Systems, Jazan University, Jazan,, Kingdom of Saudi Arabia

E-mail address: ahmadsms@gmail.com

M. Imran

Department of Mathematics, School of Natural Sciences (SNS), National University of Sciences and Technology (NUST), Islamabad, Pakistan

E-mail address: imrandhab@gmail.com

O. Al-Mushayt

Faculty of Computer Science & Information Systems, Jazan University, Jazan, Kingdom of Saudi Arabia

Current address: MIS Department, Business College, King Khalid University, Kingdom of Saudi Arabia

E-mail address: oalmushayt@yahoo.com

S. A. U. H. Bokhary

Center for Advanced Studies in Pure and Applied Mathematics, Bahaudin Zakariya University, Multan, Pakistan

E-mail address: sihtsham@gmail.com