SOME FIXED POINT THEOREMS FOR NONSELF GENERALIZED CONTRACTION

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Abstract. In this paper we give a new proof of a result by S. Reich and A.J. Zaslavski (S. Reich and A.J. Zaslavski, A fixed point theorem for Matkowski contractions, Fixed Point Theory, 8(2007), No. 2, 303–307). Some new fixed point theorems for nonself generalized contractions are also given.

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1. INTRODUCTION

There are many techniques in the fixed point theory of nonself operators (see [10], [4], [6], [9], [19], [20], [2], . . . ). An exotic result is given in [14] (see also, [13] and [15]). This result read as follows:

Theorem 1. Let \((X,d)\) be a complete metric space, \(Y \subseteq X\) a nonempty closed subset and \(f : Y \rightarrow X\) be a \(\varphi\)-contraction, where \(\varphi\) is a comparison function. We suppose that there exists a bounded sequence \((x_n)_{n \in \mathbb{N}^*}\) such that \(f^n(x_n)\) is defined for all \(n \in \mathbb{N}^*\). Then \(f\) has a unique fixed point \(x^*\) and \(f^n(x_n) \rightarrow x^*\).

The aim of this paper is to give a new proof of this theorem and to obtain other results of this type.

2. PRELIMINARIES

2.1. Notations

\(\mathbb{N} = \{0, 1, 2, \ldots\}\), \(\mathbb{N}^* = \{1, 2, 3, \ldots\}\).
\(\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}\), \(\mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}\)

Let \((X,d)\) be a metric space. We will use the following symbols:
\(\mathcal{P}(X) = \{Y \mid Y \subseteq X\}\)

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P(X) = \{Y \subset X \mid Y \text{ is nonempty}\}, P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded}\},
P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}, P_{b,cl}(X) := P_b(X) \cap P_{cl}(X).
If \( f : X \to X \) is an operator then \( F_f := \{x \in X \mid x = f(x)\} \) denotes the fixed point set of the operator \( f \). In the case when the operator \( f \) has an unique fixed point \( x^* \in X \) then we write \( F_f = \{x^*\} \).

The diameter functional \( \delta : P(X) \to \mathbb{R}_+ \cup \{+\infty\} \) is defined by
\[
\delta(A) := \sup\{d(a, b) \mid a, b \in A\}.
\]

2.2. Comparison functions

Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a function. We consider the following conditions relative to \( \varphi \):
\begin{itemize}
  \item (i) \( \varphi \) is increasing;
  \item (ii) \( \varphi(t) < t, \forall t > 0 \);
  \item (iii) \( \varphi(0) = 0 \);
  \item (iv) \( \varphi^n(t) \to 0 \) as \( n \to \infty \), \( \forall t \in \mathbb{R}_+ \);
  \item (v) \( t - \varphi(t) \to \infty \) as \( t \to \infty \);
  \item (vi) \( \sum_{n=0}^{\infty} \varphi^n(t) < +\infty, \forall t \in \mathbb{R}_+ \).
\end{itemize}

Definition 1 (I.A. Rus [17]). By definition the function \( \varphi \) is a comparison function if it satisfies the conditions (i) and (iv).

Definition 2. A comparison function is:
\begin{itemize}
  \item (a) strict comparison function if it satisfies the condition (v);
  \item (b) strong comparison function if it satisfies the condition (vi).
\end{itemize}

It is clear that if \( \varphi \) is a comparison function then \( \varphi(t) < t, \forall t > 0 \), and \( \varphi(0) = 0 \).

If \( \varphi \) is a strong comparison function then the functions \( \varphi \) and \( \sum_{n=0}^{\infty} \varphi^n \) are continuous in \( t = 0 \).

For example, if \( \varphi(t) := at, t \in \mathbb{R}_+, a \in [0; 1] \), then \( \varphi \) is a strict and strong comparison function and \( \varphi(t) := \frac{t}{1+t}, t \in \mathbb{R}_+ \), is a strict comparison function which is not a strong comparison function.

Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a strict comparison function. In this case we define the function \( \theta_\varphi : \mathbb{R}_+ \to \mathbb{R}_+ \), defined by,
\[
\theta_\varphi(t) = \sup\{s \in \mathbb{R}_+ \mid s - \varphi(s) \leq t\}.
\]

We remark that \( \theta_\varphi \) is increasing and \( \theta_\varphi(t) \to 0 \) as \( t \to 0 \). The function \( \theta_\varphi \) appears when we study the data dependence of the fixed points.

For more considerations on comparison functions see [17], [1], [21] and [5].
2.3. Maximal displacement functional

Let \((X,d)\) be a metric space, \(Y \subseteq P_{cl}(X)\) and \(f : Y \to X\) be a continuous nonself operator. By the maximal displacement functional corresponding to \(f\) we understand the functional \(E_f : P(Y) \to \mathbb{R}_+ \cup \{+\infty\}\) defined by

\[
E_f(A) := \sup \{d(x,f(x)) \mid x \in A\}.
\]

We have that:

(i) \(A, B \in P(Y), A \subseteq B\) imply \(E_f(A) \leq E_f(B)\);
(ii) \(E_f(A) = E_f(\overline{A})\) for all \(A \in P(Y)\).

Definition 3. An operator \(f : Y \to X\) is \(\alpha\)-graphic contraction if \(0 \leq \alpha < 1\) and \(x \in Y, f(x) \in Y\) imply

\[
d\left(f^2(x), f(x)\right) \leq \alpha d(x, f(x)).
\]

Example 1. If \(f : Y \to X\) is \(\alpha\)-contraction then \(f\) is \(\alpha\)-graphic contraction.

Example 2. If \(f : Y \to X\) is \(\alpha\)-Kannan operator, i.e., \(0 \leq \alpha < \frac{1}{2}\), and

\[
d\left(f(x), f(y)\right) \leq \alpha \left[d(x, f(x)) + d(y, f(y))\right], \forall x, y \in Y,
\]

then \(f\) is \(\frac{\alpha}{1-\alpha}\)-graphic contraction.

Also, we have that:

Lemma 1. Let \((X,d)\) be a metric space, \(Y \subseteq P_{cl}(X)\) and \(f : Y \to X\) be a continuous \(\alpha\)-graphic contraction. Then:

(a) \(E_f(f(A)) \leq \alpha E_f(A)\), for all \(A \subseteq Y\) with \(f(A) \subseteq Y\);
(b) \(E_f(f(A) \cap Y) \leq \alpha E_f(A)\), for all \(A \subseteq Y\) with \(f(A) \cap Y \neq \emptyset\).

Proof. The proof follows from the definition of \(E_f\). Let, for example, to prove (b). We have

\[
E_f(f(A) \cap Y) = \sup \{d(x, f(x)) \mid x \in f(A) \cap Y\} =
\leq \alpha \sup \{d(u, f(u)) \mid u \in A\} =
\leq \alpha E_f(A)
\]

2.4. Matrices convergent to 0

Definition 4. A matrix \(S \in \mathbb{R}_{+}^{m \times m}\) is called a matrix convergent to zero iff \(S^k \to 0\) as \(k \to +\infty\).

Theorem 2 (see [12], [16], [23], [10]). Let \(S \in \mathbb{R}_{+}^{m \times m}\). The following statements are equivalent:

(i) \(S\) is a matrix convergent to zero;
(ii) \(S^k x \to 0\) as \(k \to +\infty\), \(\forall x \in \mathbb{R}^m\);

(iii) \(I_m - S\) is non-singular and
\[
(I_m - S)^{-1} = I_m + S + S^2 + \cdots
\]

(iv) \(I_m - S\) is non-singular and \((I_m - S)^{-1}\) has nonnegative elements;

(v) \(\lambda \in \mathbb{C}\), \(\det(S - \lambda I_m) = 0\) imply \(|\lambda| < 1\);

(vi) there exists at least one subordinate matrix norm such that \(kSk < 1\).

The matrices convergent to zero were used by A. I. Perov [11] (see also [10] pp. 432-434) to generalize the contraction principle in the case of generalized metric spaces with the metric taking values in the positive cone of \(\mathbb{R}^m\).

3. A NEW PROOF OF THEOREM 1

Now we present a new proof of Theorem 1. Let \(A \in P_{b,c,l}(Y)\) be such that \(x_n \in A\), for all \(n \in \mathbb{N}^*\). We consider the following standard construction in the fixed point theory for the nonself operators (see for example [8] and [7]).

Let \(A_1 := f(A)\), \(A_2 := f(A_1 \cap A)\), \(\ldots\), \(A_{n+1} := f(A_n \cap A), n \in \mathbb{N}^*.\) We remark that:

(a) \(A_{n+1} \subset A_n, \forall n \in \mathbb{N}^*\);

(b) \(f^n(x_n) \in A_n, \forall n \in \mathbb{N}^*,\) so \(A_n \neq \emptyset, \forall n \in \mathbb{N}^*\).

Since \(f\) is a \(\varphi\)-contraction, i.e., \(\varphi: \mathbb{R}_+ \to \mathbb{R}_+\) is a comparison function such that
\[
d(f(x), f(y)) \leq \varphi(d(x, y)), \forall x, y \in Y,
\]
it follows that
\[
\delta(f(B)) \leq \varphi(\delta(B)), \forall B \in P_b(Y).
\]

From the properties of \(\varphi\) and \(\delta\) we have
\[
\delta(A_{n+1}) = \delta(f(A_n \cap A)) = \delta(f(A_n)) \leq \varphi(\delta(A_n)) \leq \cdots \leq \varphi^{n+1}(\delta(A)) \to 0
\]
as \(n \to +\infty\). From Cantor intersection lemma we have
\[
A_\infty := \bigcap_{n \in \mathbb{N}} A_n \neq \emptyset, \delta(A_\infty) = 0 \text{ and } f(A_\infty \cap A) \subset A_\infty.
\]

From \(A_\infty \neq \emptyset\) and \(\delta(A_\infty) = 0\), we have that \(A_\infty = \{x^*\}\). On the other hand \(f^n(x_n) \in A_n\) and \(f^{n-1}(x_n) \in A_{n-1} \cap Y\). This implies that \(\{f^n(x_n)\}_{n \in \mathbb{N}}\) and \(\{f^{n-1}(x_n)\}_{n \in \mathbb{N}}\) are fundamental sequences. Since \(A_n, n \in \mathbb{N}^*,\) are closed, it follows that
\[
f^{n-1}(x_n) \to x^* \text{ and } f^n(x_n) \to x^* \text{ as } n \to +\infty.
\]

Since \(f\) is continuous then \(f^n(x_n) \to f(x^*),\) so \(f(x^*) = x^*\).

With respect to the data dependence of the fixed point, in Theorem 1, we have the following result:
Theorem 3. Let \( f : Y \to X \) be as in Theorem 1, where \( \varphi \) is a strict comparison function. Then:

(a) \( d \left( f^n (x_n), x^* \right) \leq \varphi \left( d \left( x_n, x^* \right) \right), \forall n \in \mathbb{N}^* \);

(b) \( d \left( x, x^* \right) \leq \theta_{\varphi} \left( d \left( x, f \left( x \right) \right) \right), \forall x \in Y \);

(c) Let \( g : Y \to X \) be such that:
   (1) there exists \( \eta > 0 \) such that \( d \left( f \left( x \right), g \left( x \right) \right) \leq \eta, \forall x \in Y \);
   (2) \( F_g \neq \emptyset \).

Then
\[
d \left( x^*, y^* \right) \leq \theta_{\varphi} \left( \eta \right), \forall y^* \in F_g.
\]

Proof. Let us prove (b) and (c).

(b). The conclusion (b) follows from the following estimation
\[
d \left( x, x^* \right) \leq d \left( x, f \left( x \right) \right) + d \left( f \left( x \right), x^* \right) \leq d \left( x, f \left( x \right) \right) + \varphi \left( d \left( x, x^* \right) \right), \forall x \in Y.
\]

So,
\[
d \left( x, x^* \right) - \varphi \left( d \left( x, x^* \right) \right) \leq d \left( x, f \left( x \right) \right), \forall x \in Y.
\]

(c). Let \( y^* \in F_g \) then from (b) it follows
\[
d \left( x^*, y^* \right) \leq \theta_{\varphi} \left( d \left( y^*, f \left( y^* \right) \right) \right) = \theta_{\varphi} \left( d \left( g \left( y^* \right), f \left( y^* \right) \right) \right) \leq \theta_{\varphi} \left( \eta \right).
\]

\[\Box\]

For more considerations on data dependence of the fixed points for nonself \( \varphi \)-contractions see [3], [18] and [22].

4. A fixed point theorem for nonself Kannan operators

We have:

Theorem 4. Let \( (X, d) \) be a complete metric space, \( Y \subseteq X \) a nonempty bounded closed subset and \( f : Y \to X \) a continuous operator. We suppose that:

(i) \( f \) is \( \alpha \)-Kannan operator;

(ii) there exists a sequence \( (x_n)_{n \in \mathbb{N}^*} \) in \( Y \) such that \( f^n \left( x_n \right) \) is defined for all \( n \in \mathbb{N}^* \);

(iii) \( E_f \left( Y \right) < +\infty \).

Then:

(a) \( F_f = \{ x^* \} \);

(b) \( f^{n-1} \left( x_n \right) \to x^* \) and \( f^n \left( x_n \right) \to x^* \) as \( n \to +\infty \);

(c) \( d \left( x, x^* \right) \leq \left( 1 + \alpha \right) d \left( x, f \left( x \right) \right), \forall x \in Y \);

(d) \( d \left( f^{n-1} \left( x_n \right), x^* \right) \leq \alpha^{n-1} \left( 1 - \alpha \right)^{1-n} \left( 1 + \alpha \right) d \left( x_n, f \left( x_n \right) \right), \forall n \in \mathbb{N}^* \);

(e) Let \( g : Y \to X \) be such that:
   (1) there exists \( \eta > 0 \) such that \( d \left( f \left( x \right), g \left( x \right) \right) \leq \eta, \forall x \in Y \);
   (2) \( F_g \neq \emptyset \).
Then
\[ d(x^*, y^*) \leq (1 + \alpha) \eta, \forall y^* \in F_g. \]

Proof. (a) Let \( Y_1 := f(Y), Y_2 := f(Y_1 \cap Y), \ldots, Y_{n+1} := f(Y_n \cap Y) \), \( n \in \mathbb{N}^* \). We remark that \( Y_{n+1} \subseteq Y_n \) and \( f^n(x_n) \in Y_n \), so \( Y_n \neq \emptyset, n \in \mathbb{N}^* \). Since \( f \) is \( \alpha \)-Kannan operator, from Example 2 and Lemma 1, we have that:
\[
\delta(Y_{n+1}) = \delta \left( \frac{f(Y_n \cap Y)}{f(Y_n \cap Y)} \right) = \delta \left( f(Y_n \cap Y) \right) \leq 2\alpha \cdot E_f(Y_n \cap Y) =
\[
= 2\alpha \cdot E_f \left( f(Y_{n-1} \cap Y) \cap Y \right) = 2\alpha \cdot E_f \left( f(Y_{n-1} \cap Y) \cap Y \right) \leq
\[
\leq \frac{2\alpha^n + 1}{1 - \alpha} E_f(Y_{n-1} \cap Y) \leq \cdots \leq \frac{2\alpha^n + 1}{(1 - \alpha)n} E_f(Y) \to 0 \text{ as } n \to +\infty.
\]
Now the proof is similar with the proof of Theorem 1.

(c). Let \( x \in Y \). From the definition of the Kannan operator we have:
\[
d(x, x^*) \leq d(x, f(x)) + d(f(x), x^*) \leq d(x, f(x)) + \alpha d(x, f(x)), \forall x \in Y.
\]
(d) and (e) follow from (c).

5. Other Nonself Generalized Contractions

5.1. Ćirić-Reich-Rus operators

Let \((X, d)\) be a metric space, \( Y \in P_{cl}(X) \) and \( f : Y \to X \) be a nonself operator. An operator \( f : Y \to X \) is a Ćirić-Reich-Rus operator (see [4], [20], [22], ...) if there exist \( a, b \in \mathbb{R}^+ \) with \( a + 2b < 1 \) such that
\[
d(f(x), f(y)) \leq ad(x, y) + b[d(x, f(x)) + d(y, f(y))], \forall x, y \in Y.
\]

**Lemma 2.** Let \((X, d)\) be a metric space, \( Y \in P_{cl}(X) \) and \( f : Y \to X \) a nonself Ćirić-Reich-Rus operator then \( f \) is a nonself \( \alpha \)-graphic contraction with \( \alpha = \frac{a + b}{1 - b} \).

Proof. Let \( x \in Y \) such that \( f(x) \in Y \) then
\[
d(f^2(x), f(x)) \leq ad(f(x), x) + b[d(f(x), f^2(x)) + d(x, f(x))]
\]
so
\[
d(f^2(x), f(x)) \leq \frac{a + b}{1 - b} d(x, f(x)).
\]

**Lemma 3.** Let \((X, d)\) be a metric space, \( Y \in P_{cl}(X) \) and \( f : Y \to X \) a nonself Ćirić-Reich-Rus operator then:

(a) \( \delta(f(A) \cap Y) \leq a\delta(A) + 2bE_f(A) \), for all \( A \subseteq Y \);
(b) \( E_f(f(A) \cap Y) \leq \alpha E_f(A) \), for all \( A \subseteq Y \), where \( \alpha = \frac{a + b}{1 - b} \).
Proof. (a). Let $A \subset Y$ then
\[
\delta (f (A) \cap Y) = \sup \{ d(x, y) \mid x, y \in f (A) \cap Y \} = \\
= \sup \{ d(f(u), f(v)) \mid u, v \in A, f(u), f(v) \in Y \} \leq \\
\leq a \sup \{ d(u, v) \mid u, v \in A \} + 2b \sup \{ d(u, f(u)) \mid u \in A \} = \\
= a \delta (A) + 2b E_f (A)
\]
(b). The proof follows from Lemma 2 and Lemma 1. □

Also, for the next result we need the following lemma

Lemma 4 (Cauchy Lemma, [21]). Let $a_n, b_n \in \mathbb{R}, n \in \mathbb{N}$. We suppose that:

(i) $\sum_{k=0}^{\infty} a_k < +\infty$;
(ii) $b_n \to 0$ as $n \to \infty$.

Then
\[
\sum_{k=0}^{n} a_{n-k} b_k \to 0 \text{ as } n \to \infty.
\]

Theorem 5. Let $(X, d)$ be a complete metric space, $Y \subset X$ a nonempty bounded closed subset and $f : Y \to X$ a continuous operator. We suppose that:

(i) $f$ is Ćirić-Reich-Rus operator;
(ii) there exists a sequence $(x_n)_{n \in \mathbb{N}^*}$ in $Y$ such that $f^n (x_n)$ is defined for all $n \in \mathbb{N}^*$;
(iii) $E_f (Y) < +\infty$.

Then:

(a) $F_f = \{ x^* \}$;
(b) $f^n(x_n) \to x^*$ and $f^n(x_n) \to x^*$ as $n \to +\infty$;
(c) $d(x, x^*) \leq (1 + b) (1 - a)^{-1} d(x, f(x)), \forall x \in Y$;
(d) $d\left(f^{n-1}(x_n), x^* \right) \leq (1 + b) (1 - a)^{-1} a^n d\left(x_n, f(x_n) \right), \forall n \in \mathbb{N}^*$, where $\alpha = \frac{a + b}{1 - b}$.
(e) Let $g : Y \to X$ be such that:
(1) there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta, \forall x \in Y$;
(2) $F_g \neq \emptyset$.

Then
\[
d\left(x^*, y^* \right) \leq (1 + b) (1 - a)^{-1} \eta, \forall y^* \in F_g.
\]

Proof. (a) + (b). Let $Y_1 := f(Y), Y_2 := f(Y_1 \cap Y), \ldots, Y_{n+1} := f(Y_n \cap Y), n \in \mathbb{N}^*$. We remark that $Y_{n+1} \subset Y_n$ and $f^n (x_n) \in Y_n$, so $Y_n \neq \emptyset, n \in \mathbb{N}^*$. Since $f$ is Ćirić-Reich-Rus operator, from Lemma 3 (a), we have that:
\[
\delta (Y_{n+1}) = \delta \left(f \left(Y_n \cap Y \right) \right) = \delta \left(f (Y_n \cap Y) \right) \leq
\]
On the other hand, from Lemma 3 (b) we get
\[
E_f (Y_k \cap Y) = E_f \left( f(Y_{n-1} \cap Y) \right) = E_f \left( f (Y_{n-1} \cap Y) \cap Y \right) \leq \alpha E_f (Y_{k-1} \cap Y) \leq \cdots < \alpha^k E_f (Y), \; k \in \mathbb{N}^*,
\]
where \( \alpha = \frac{a+b}{1-b} \). Applying Lemma 4 for \( a_n = a^n \) and \( b_n = 2b \cdot E_f (Y_n \cap Y) \) and we get that
\[
\delta (Y_n) \to 0 \text{ as } n \to +\infty
\]
and the proof is similar with the proof of Theorem 1.

c. Let \( x \in Y \). From the definition of the Ćirić-Reich-Rus operator we have:
\[
d (x, x^*) \leq d (x, f (x)) + d (f (x), x^*) \leq d (x, f (x)) + ad (x, x^*) + bd (x, f (x)), \; \forall x \in Y;
\]
so
\[
d (x, x^*) \leq \frac{1+b}{1-a} d (x, f (x)), \; \forall x \in Y.
\]
(d) and (e) follow from (c).

5.2. Perov operators

Let \((X, d)\) be a generalized metric space with \( d : X \times X \to \mathbb{R}^m_+ \), \( Y \subset X \) a nonempty bounded closed subset and \( f : Y \to X \) a nonself operator. By definition (see [17], [20]) \( f : Y \to X \) is a nonself Perov operator if there exists a matrix convergent to zero \( S \in \mathbb{R}^{m \times m}_+ \) such that
\[
d (f (x), f (y)) \leq S \cdot d (x, y), \; x, y \in Y.
\]
We have the following fixed point results in the case of nonself Perov operators:

**Theorem 6.** Let \((X, d)\) be a complete generalized metric space with \( d : X \times X \to \mathbb{R}^m_+ \), \( Y \subset X \) a nonempty bounded closed subset and \( f : Y \to X \) an operator. We suppose that:

(i) \( f \) is a Perov operator;

(ii) there exists a sequence \((x_n)_{n \in \mathbb{N}^*}\) in \( Y \) such that \( f^n (x_n) \) is defined for all \( n \in \mathbb{N}^* \).

Then:

(a) \( F_f = \{x^*\} \);

(b) \( f^{n-1} (x_n) \to x^* \) and \( f^n (x_n) \to x^* \) as \( n \to +\infty \).

(c) \( d (x, x^*) \leq (I_m - S)^{-1} d (x, f (x)), \; \forall x \in Y; \)
(d) \( d\left(f^n(x_n), x^*\right) \leq S^n d\left(x_n, x^*\right), \forall n \in \mathbb{N}^*; \)

(e) Let \( g : Y \to X \) be such that:

(1) there exists \( \eta \in \left(\mathbb{R}_+^n\right)^m \) such that \( d\left(f(x), g(x)\right) \leq \eta, \forall x \in Y; \)

(2) \( F_g \neq \emptyset. \)

Then

\[ d\left(x^*, y^*\right) \leq (I_m - S)^{-1} \eta, \forall y^* \in F_g. \]

**Proof.** (a) + (b). Let \( Y_1 := f(Y), Y_2 := f(Y_1 \cap Y), \ldots, Y_{n+1} := f(Y_n \cap Y), \)

\( n \in \mathbb{N}^*. \) We remark that \( Y_{n+1} \subset Y_n \) and \( f^n(x_n) \in Y_n, \) so \( Y_n \neq \emptyset, n \in \mathbb{N}^*. \) Since \( f \)

is a Perov we have:

\[ \delta(Y_{n+1}) = \delta\left(f(Y_n \cap Y)\right) = \delta(f(Y_n \cap Y)) \leq S \cdot \delta(Y_n \cap Y) \leq \]

\[ \leq S \cdot \delta(Y_n) \leq \cdots \leq S^{n+1} \cdot \delta(Y) \to 0 \text{ as } n \to +\infty. \]

Now the proof is similar with the proof of Theorem 1.

(c). Let \( x \in Y \) then we have:

\[ d\left(x, x^*\right) \leq d\left(x, f(x)\right) + d\left(f(x), x^*\right) \leq d\left(x, f(x)\right) + S d\left(x, x^*\right), \forall x \in Y, \]

so

\[ d\left(x, x^*\right) \leq (I_m - S)^{-1} d\left(x, f(x)\right), \forall x \in Y. \]

(d) follows from the definition of the Perov operator and (e) is obtained from (c) for \( x := y^* \in F_g. \)

6. AN OPEN PROBLEM

The above considerations give rise to the following problem:

**Problem 1.** Let \((X, d)\) be a complete metric space, \( Y \) a nonempty bounded and closed subset of \( X \) and \( f : Y \to X \) a nonself operator. We suppose that there exists a sequence \((x_n)_{n \in \mathbb{N}^*}\) such that \( f^n(x_n) \) is defined for all \( n \in \mathbb{N}^* \). In which additional conditions on \( f \) we have that:

(a) \( F_f \neq \emptyset \)?

(b) \( F_f = \{x^*\} \)?

**References**


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