



Miskolc Mathematical Notes  
Vol. 16 (2015), No 1, pp. 503-506

HU e-ISSN 1787-2413  
DOI: 10.18514/MMN.2015.1180

# Remarks on Yu Miao and Shoufang Xu's paper "Almost sure convergence of weighted sums"

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## REMARKS ON YU MIAO AND SHOUFANG XU'S PAPER “ALMOST SURE CONVERGENCE OF WEIGHTED SUMS”

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*Received 26 March, 2014*

*Abstract.* It is pointed out that a general theorem presented in Yu Miao and Shoufang Xu's paper has a serious gap. In this small note, a counterexample for this theorem is provided exhibiting a triangular array of real numbers and a sequence of independent identically distributed random variables for which its conclusion fails. An reformulation of Yu Miao and Shoufang Xu's general theorem is performed making this statement true.

*2010 Mathematics Subject Classification:* 60F15

*Keywords:* almost sure convergence, triangular arrays, weighted sums

### 1. INTRODUCTION

In paper [11], the authors announced a general theorem (see Theorem 6 of [11], page 175) which supports important improvements in several well-known results about the almost sure convergence of the weighted sum

$$\sum_{k=1}^n a_{n,k} X_k \quad (1.1)$$

where  $\{X_n, n \geq 1\}$  is a sequence of random variables and  $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$  is a triangular array of constants.

Over the past years, many authors have studied the almost sure convergence of (1.1) (see, for instance, [2–4, 6, 7, 10, 13–18]). In 1995, Li, Rao and Wang proved a result in [10] which seems to say that for almost all choices of weights, the law of the Iterated Logarithm fails, establishing what they call the law of Logarithm. Notwithstanding, a specific example with a concrete array of real numbers and sequence of independent identically distributed random variables for which the law of the Iterated Logarithm fails was not included in the quoted paper. Only one year later, Li and Rao presented the desired example (see [9]) which can be used to claim

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This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project PEst-OE/MAT/UI0297/2014 (Centro de Matemática e Aplicações).

that Corollary 3 of [11] is false. Indeed, a simple adaptation of Li and Rao's example can be made to prove that Theorem 6 of [11] is also false. Contrary to what Miao and Xu declared in page 176 of [11], we cannot rearrange the random sequence  $\{a_{n,k}X_k, 1 \leq k \leq n, n \geq 1\}$  when the order of  $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$  does not satisfy  $a_{n,1} \geq a_{n,2} \geq \dots \geq a_{n,n}$ .

## 2. COUNTEREXAMPLE

Consider a sequence  $\{X_n, n \geq 1\}$  of independent identically distributed random variables each with normal distribution with zero mean and unit variance. Let  $\mathbf{H}_n$  be a sequence of Hadamard matrices of order  $n$  (see [1] or [8]) and suppose the  $\infty \times \infty$  matrix  $(h_{n,k})_{n,k \geq 1}$  given by

$$\begin{pmatrix} \mathbf{H}_1 & \mathbf{O} & \mathbf{O} & \dots \\ \mathbf{O} & \mathbf{H}_2 & \mathbf{O} & \dots \\ \mathbf{O} & \mathbf{O} & \mathbf{H}_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let  $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$  be the triangular array of constants defined by

$$a_{2n,k} = \frac{h_{n,k}}{\log \log \log(n+15) \sqrt{n \log \log(n+2)}},$$

$$a_{2n-1,k} = \frac{1}{\log \log \log(n+15) \sqrt{n \log \log(n+2)}}.$$

Since  $|h_{n,k}| \leq 1$  for each  $n, k$  we have

$$\max_{1 \leq k \leq n} |a_{n,k}| \leq \frac{1}{\log \log \log(n+15) \sqrt{n \log \log(n+2)}}$$

and the sequence  $b_n$  in Theorem 6 of Yu Miao and Shoufang Xu's paper is

$$b_n = \log \log \log(n+15) \sqrt{n \log \log(n+2)}.$$

From Hartman-Wintner law of the iterated logarithm (see [12], page 248) we get

$$\frac{X_1 + \dots + X_n}{b_n} \xrightarrow{\text{a.s.}} 0.$$

On the other hand,

$$\limsup_{n \rightarrow \infty} \left| \sum_{k=1}^n a_{n,k} X_k \right| \geq \limsup_{n \rightarrow \infty} \frac{1}{b_n} \left| \sum_{k=1}^{2n} h_{n,k} X_k \right| \quad \text{a.s.}$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{4n \log(2n)}} \left| \sum_{k=1}^{2n} h_{n,k} X_k \right| \geq \frac{1}{\sqrt{2}} \quad \text{a.s.}$$

(see [9]) which implies

$$\limsup_{n \rightarrow \infty} \left| \sum_{k=1}^n a_{n,k} X_k \right| = \infty \quad \text{a.s.}$$

Therefore, subsequent corollaries in [11] which use Theorem 6 cannot be validated until a correct proof be given.

### 3. GENERAL THEOREM

We shall announce the following definition.

**Definition 1.** Let  $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$  be an array of constants. We say that  $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$  is *rowwise increasing* (resp. *rowwise decreasing*) if  $a_{n,k+1} \geq a_{n,k}$ , for all  $1 \leq k \leq n-1, n \geq 2$  (resp.  $a_{n,k+1} \leq a_{n,k}$ , for all  $1 \leq k \leq n-1, n \geq 2$ ). An array of constants  $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$  is said to be *rowwise monotone* if it is rowwise increasing or rowwise decreasing.

The result below will permit us to obtain the desired almost sure convergence of (1.1) in all corollaries of [11] assuming additionally the rowwise monotonicity of the triangular array of constants. The proof uses Abel transformation (see [5], page 114) and is similar to Cèsaro lemma (see [19], page 116) being, therefore, omitted.

**Theorem 1.** Let  $\{x_n\}$  be a sequence of constants and  $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$  an rowwise monotone array of constants satisfying

$$\max_{1 \leq k \leq n} |a_{n,k}| \leq b_n^{-1}$$

where  $0 < b_n \uparrow \infty$ . If

$$\frac{1}{b_n} \sum_{k=1}^n x_k \rightarrow 0$$

then

$$\sum_{k=1}^n a_{n,k} x_k \rightarrow 0.$$

*Remark 1.* Let us observe that the assumption “ $\{X, X_n, n \geq 1\}$  be a sequence of identically distributed random variables” in Theorem 6 of [11] is totally superfluous.

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