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A unified proof of several inequalities and some new inequalities involving Neuman-Sándor mean

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A UNIFIED PROOF OF SEVERAL INEQUALITIES AND SOME NEW INEQUALITIES INVOLVING NEUMAN-SÁNDOR MEAN

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Abstract. In the paper, by finding linear relations of differences between some means, the authors supply a unified proof of several double inequalities for bounding Neuman-Sándor means in terms of the arithmetic, harmonic, and contra-harmonic means and discover some new sharp inequalities involving Neuman-Sándor, contra-harmonic, root-square, and other means of two positive real numbers.

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1. INTRODUCTION

It is well known that the quantities

$$A(a,b)=\frac{a+b}{2}, \qquad G(a,b)=\sqrt{ab}\,,$$
$$H(a,b)=\frac{2ab}{a+b}, \qquad \overline{C}(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)},$$
$$C(a,b)=\frac{a^2+b^2}{a+b}, \qquad P(a,b)=\frac{a-b}{4\arctan\sqrt{a/b}-\pi},$$
$$Q(a,b)=\sqrt{\frac{a^2+b^2}{2}}\,, \qquad T(a,b)=\frac{a-b}{2\arctan\frac{a-b}{a+b}}$$

are respectively called in the literature the arithmetic, geometric, harmonic, centroidal, contra-harmonic, first Seiffert, root-square, and second Seiffert means of two positive real numbers a and b with $a\neq b$.

For $a,b>0$ with $a\neq b$, Neuman-Sándor mean $M(a,b)$ is defined in [11] by

$$M(a,b)=\frac{a-b}{2\operatorname{arcsinh}\frac{a-b}{a+b}},$$

where $\operatorname{arcsinh} x = \ln(x + \sqrt{x^2+1})$ is the inverse hyperbolic sine function. At the same time, a chain of inequalities

$$G(a,b) < L_{-1}(a,b) < P(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b)$$

were given in [11], where

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq -1,0 \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0 \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1 \end{cases}$$

is the p-th generalized logarithmic mean of a and b with $a \neq b$.

In [11, 12], it was established that

$$A(a,b) < M(a,b) < T(a,b), \quad P(a,b) < M(a,b) < T^2(a,b),$$
$$A(a,b)T(a,b) < M^2(a,b) < \frac{A^2(a,b)+T^2(a,b)}{2}$$

for $a,b > 0$ with $a \neq b$.

For $0 < a,b < \frac{1}{2}$ with $a \neq b$, Ky Fan type inequalities

$$\frac{G(a,b)}{G(1-a,1-b)} < \frac{L_{-1}(a,b)}{L_{-1}(1-a,1-b)} < \frac{P(a,b)}{P(1-a,1-b)}$$
$$< \frac{A(a,b)}{A(1-a,1-b)} < \frac{M(a,b)}{M(1-a,1-b)} < \frac{T(a,b)}{T(1-a,1-b)}$$

were presented in [11, Proposition 2.2].

In [8], it was showed that the double inequality

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ and for $p_0 = 1.843\ldots$, where p_0 is the unique solution of the equation $(p+1)^{1/p} = 2\ln(1+\sqrt{2})$.

In [10], Neuman proved that the double inequalities

$$\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b)$$

and

$$\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu C(a,b) + (1-\mu)A(a,b) \tag{1.1}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if

$$\alpha \leq \frac{1-\ln(1+\sqrt{2})}{(\sqrt{2}-1)\ln(1+\sqrt{2})} = 0.3249\ldots, \quad \beta \geq \frac{1}{3}$$

and

$$\lambda \le \frac{1-\ln(1+\sqrt{2})}{\ln(1+\sqrt{2})} = 0.1345\dots, \quad \mu \ge \frac{1}{6}.$$

In [20, Theorems 1.1 to 1.3], it was found that the double inequalities

$$\alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 H(a,b) + (1-\beta_1)Q(a,b),$$

$$\alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) < \beta_2 G(a,b) + (1-\beta_2)Q(a,b),$$

and

$$\alpha_3 H(a,b) + (1-\alpha_3)C(a,b) < M(a,b) < \beta_3 H(a,b) + (1-\beta_3)C(a,b) \tag{1.2}$$

hold for all $a,b>0$ with $a \ne b$ if and only if

$$\alpha_1 \ge \frac{2}{9} = 0.2222\dots, \quad \beta_1 \le 1 - \frac{1}{\sqrt{2}\ln(1+\sqrt{2})} = 0.1977\dots,$$

$$\alpha_2 \ge \frac{1}{3} = 0.3333\dots, \quad \beta_2 \le 1 - \frac{1}{\sqrt{2}\ln(1+\sqrt{2})} = 0.1977\dots,$$

$$\alpha_3 \ge 1 - \frac{1}{2\ln(1+\sqrt{2})} = 0.4327\dots, \quad \beta_3 \le \frac{5}{12} = 0.4166\dots$$

In [19, Theorem 3.1], it was established that the double inequality

$$\alpha I(a,b) + (1-\alpha)Q(a,b) < M(a,b) < \beta I(a,b) + (1-\beta)Q(a,b)$$

holds for all $a,b>0$ with $a \ne b$ if and only if

$$\alpha \ge \frac{1}{2} \quad \text{and} \quad \beta \le \frac{e\left[\sqrt{2}\ln(1+\sqrt{2})-1\right]}{(\sqrt{2}e-2)\ln(1+\sqrt{2})} = 0.4121\dots$$

For more information on this topic, please refer to [1–3, 5, 7–10, 12–14, 16–18, 20] and plenty of references cited therein.

The first goal of this paper is, by finding linear relations of differences between some means, to supply a unified proof of inequalities (1.1) and (1.2).

The second purpose of this paper is to establish some new sharp inequalities involving Neuman-Sándor, centroidal, contra-harmonic, and root-square means of two positive real numbers a and b with $a \ne b$.

2. Lemmas

In order to attain our aims, the following lemmas are needed.

Lemma 1 ([15, Lemma 1.1]). *Suppose that the power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *have the radius of convergence* $r > 0$ *and* $b_n > 0$ *for all* $n \in \mathbb{N} = \{0,1,2,\dots\}$. *Let* $h(x) = \frac{f(x)}{g(x)}$. *Then the following statements are true.*

(1) *If the sequence* $\{\frac{a_n}{b_n}\}_{n=0}^{\infty}$ *is (strictly) increasing (decreasing), then* $h(x)$ *is also (strictly) increasing (decreasing) on* $(0,r)$.

(2) *If the sequence* $\{\frac{a_n}{b_n}\}$ *is (strictly) increasing (decreasing) for* $0<n\le n_0$ *and (strictly) decreasing (increasing) for* $n>n_0$*, then there exists* $x_0\in(0,r)$ *such that* $h(x)$ *is (strictly) increasing (decreasing) on* $(0,x_0)$ *and (strictly) decreasing (increasing) on* (x_0,r).

Lemma 2. *Let*

$$h_1(x)=\frac{\sinh x-x}{2x\sinh^2 x}. \tag{2.1}$$

Then $h_1(x)$ *is strictly decreasing on* $(0,\infty)$ *and has the limit* $\lim\limits_{x\to0^+}h_1(x)=\frac{1}{12}$.

Proof. Let $f_1(x)=\sinh x-x$ and $f_2(x)=2x\sinh^2 x=x\cosh 2x-x$. Using the power series

$$\sinh x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\quad\text{and}\quad\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}, \tag{2.2}$$

we can express the functions $f_1(x)$ and $f_2(x)$ as

$$f_1(x)=\sum_{n=0}^{\infty}\frac{x^{2n+3}}{(2n+3)!}\quad\text{and}\quad f_2(x)=\sum_{n=0}^{\infty}\frac{2^{2n+2}x^{2n+3}}{(2n+2)!}. \tag{2.3}$$

Hence, we have

$$h_1(x)=\frac{\sum_{n=0}^{\infty}a_nx^{2n}}{\sum_{n=0}^{\infty}b_nx^{2n}}, \tag{2.4}$$

where $a_n=\frac{1}{(2n+3)!}$ and $b_n=\frac{2^{2n+2}}{(2n+2)!}$. Let $c_n=\frac{a_n}{b_n}$. Then $c_n=\frac{1}{(2n+3)2^{2n+2}}$ and

$$c_{n+1}-c_n=\frac{-(6n+17)}{(2n+3)(2n+5)2^{2n+4}}<0.$$

As a result, by Lemma 1, it follows that the function $h_1(x)$ is strictly decreasing on $(0,\infty)$.

From (2.4), it is easy to see that $\lim_{x\to0^+}h_1(x)=\frac{a_0}{b_0}=\frac{1}{12}$. The proof of Lemma 2 is complete. □

Lemma 3. *Let*

$$h_2(x)=\frac{1-\frac{\sinh x}{x}+\frac{\sinh^2 x}{3}}{\cosh x-\frac{\sinh x}{x}}. \tag{2.5}$$

Then $h_2(x)$ *is strictly increasing on* $(0,\infty)$ *and has the limit* $\lim\limits_{x\to0^+}h_2(x)=\frac{1}{2}$.

Proof. Let

$$f_3(x) = 1 - \frac{\sinh x}{x} + \frac{\sinh^2 x}{3} = 1 - \frac{\sinh x}{x} + \frac{\cosh 2x - 1}{6}$$

and

$$f_4(x) = \cosh x - \frac{\sinh x}{x}.$$

Making use of the power series in (2.2) shows that

$$f_3(x) = \sum_{n=0}^{\infty} \frac{(2n+3)2^{2n+2} - 6}{6(2n+3)!} x^{2n+2} \quad \text{and} \quad f_4(x) = \sum_{n=0}^{\infty} \frac{2n+2}{(2n+3)!} x^{2n+2}.$$

Therefore, we have

$$h_2(x) = \frac{\sum_{n=0}^{\infty} a_n x^{2n+2}}{\sum_{n=0}^{\infty} b_n x^{2n+2}}, \tag{2.6}$$

where $a_n = \frac{(2n+3)2^{2n+2}-6}{6(2n+3)!}$ and $b_n = \frac{2n+2}{(2n+3)!}$. Let $c_n = \frac{a_n}{b_n}$. Then

$$c_n = \frac{(2n+3)2^{2n+1} - 3}{6(n+1)}$$

and

$$c_{n+1} - c_n = \frac{3 + 7 \cdot 2^{2n+2} + 21n \cdot 2^{2n+1} + 3n^2 \cdot 2^{2n+2}}{6(n+1)(n+2)} > 0.$$

Accordingly, by Lemma 1, it follows that the function $h_2(x)$ is strictly increasing on $(0, \infty)$.

It is clear that $\lim_{x \to 0^+} h_2(x) = \frac{a_0}{b_0} = \frac{1}{2}$. The proof of Lemma 3 is complete. □

Lemma 4. *Let*

$$h_3(x) = \frac{\cosh x - \frac{\sinh x}{x}}{1 + \sinh^2 x - \frac{\sinh x}{x}}. \tag{2.7}$$

Then $h_3(x)$ is strictly decreasing on $(0, \infty)$ and has the limit $\lim_{x \to 0^+} h_3(x) = \frac{2}{5}$.

Proof. Let

$$f_5(x) = \cosh x - \frac{\sinh x}{x}$$

and

$$f_6(x) = 1 + \sinh^2 x - \frac{\sinh x}{x} = 1 - \frac{\sinh x}{x} + \frac{\cosh 2x - 1}{2}.$$

Utilizing the power series in (2.2) gives

$$f_5(x) = \sum_{n=0}^{\infty} \frac{2n+2}{(2n+3)!} x^{2n+2} \quad \text{and} \quad f_6(x) = \sum_{n=0}^{\infty} \frac{(2n+3)2^{2n+1} - 1}{(2n+3)!} x^{2n+2}.$$

This implies that

$$h_3(x) = \frac{\sum_{n=0}^{\infty} a_n x^{2n+2}}{\sum_{n=0}^{\infty} b_n x^{2n+2}}, \tag{2.8}$$

where $a_n = \frac{2n+2}{(2n+3)!}$ and $b_n = \frac{(2n+3)2^{2n+1}-1}{(2n+3)!}$. Let $c_n = \frac{a_n}{b_n}$. Then

$$c_n = \frac{2n+2}{(2n+3)2^{2n+1}-1}$$

and

$$c_{n+1} - c_n = -\frac{2(1+7\cdot 2^{2n+2}+21n\cdot 2^{2n+1}+3n^2\cdot 2^{2n+2})}{(3\cdot 2^{2n+1}+n\cdot 2^{2n+2}-1)(5\cdot 2^{2n+3}+n\cdot 2^{2n+4}-1)} < 0.$$

In light of Lemma 1, we obtain that the function $h_3(x)$ is strictly decreasing on $(0,\infty)$.

It is obvious that $\lim_{x\to 0^+} h_3(x) = \frac{a_0}{b_0} = \frac{2}{5}$. The proof of Lemma 4 is complete. □

3. A UNIFIED PROOF OF INEQUALITIES (1.1) AND (1.2)

Now we are in a position to supply a unified proof of inequalities (1.1) and (1.2) and, as corollaries, to establish some new inequalities involving Neuman-Sándor, contra-harmonic, centroidal, and root-square means of two positive real numbers a and b with $a \ne b$.

It is not difficult to see that the inequalities (1.1) and (1.2) can be rearranged respectively as

$$\lambda - 1 < \frac{M(a,b)-C(a,b)}{C(a,b)-A(a,b)} < \mu - 1 \tag{3.1}$$

and

$$-\alpha_3 < \frac{M(a,b)-C(a,b)}{C(a,b)-H(a,b)} < -\beta_3. \tag{3.2}$$

The denominators in (3.1) and (3.2) meet

$$2[C(a,b)-A(a,b)] = C(a,b)-H(a,b) = \frac{(a-b)^2}{a+b} \tag{3.3}$$

which were presented in [4, Eq. (4.4)]. This implies that the inequalities (1.1) and (1.2) are identical up to a scalar. Therefore, it is sufficient to prove one of the two inequalities (1.1) and (1.2).

By a direct calculation, we also find

$$\begin{aligned} 6[\overline{C}(a,b)-A(a,b)] &= 3[C(a,b)-\overline{C}(a,b)] = 2[A(a,b)-H(a,b)] \\ &= \frac{3}{2}[\overline{C}(a,b)-H(a,b)] = \frac{(a-b)^2}{a+b} \triangleq CH(a,b). \end{aligned} \tag{3.4}$$

So, it is natural to raise a problem: what are the best constants α and β such that the double inequality

$$\alpha < \frac{M(a,b)-C(a,b)}{CH(a,b)} < \beta \tag{3.5}$$

holds for all $a,b>0$ with $a \neq b$? The following theorem gives a solution to this problem.

Theorem 1. *The double inequality* (3.5) *holds for all* $a,b>0$ *with* $a\neq b$ *if and only if*

$$\alpha \le \frac{1}{2\ln(1+\sqrt{2})}-1 = -0.4327\dots \quad \textit{and} \quad \beta \ge -\frac{5}{12} = -0.4166\dots$$

Proof. Without loss of generality, we assume that $a>b>0$. Let $x=\frac{a}{b}$. Then $x>1$ and

$$\frac{M(a,b)-C(a,b)}{CH(a,b)} = \frac{\frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}}-\frac{x^2+1}{x+1}}{\frac{(x-1)^2}{x+1}}.$$

Let $t=\frac{x-1}{x+1}$. Then $t\in(0,1)$ and

$$\frac{M(a,b)-C(a,b)}{CH(a,b)} = \frac{\frac{t}{\operatorname{arcsinh} t}-t^2-1}{2t^2}.$$

Let $t=\sinh\theta$ for $\theta\in\big(0,\ln\big(1+\sqrt{2}\big)\big)$. Then

$$\frac{M(a,b)-C(a,b)}{CH(a,b)} = \frac{\frac{\sinh\theta}{\theta}-\sinh^2\theta-1}{2\sinh^2\theta} = \frac{\sinh\theta-\theta}{2\theta\sinh^2\theta}-\frac{1}{2}.$$

In virtue of Lemma 2, Theorem 1 is thus proved. □

Corollary 1. *The double inequality*

$$\alpha CH(a,b)+M(a,b) < C(a,b) < \beta CH(a,b)+M(a,b) \tag{3.6}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha \le \frac{5}{12} = 0.4166\dots$ *and*

$$\beta \ge 1-\frac{1}{2\ln(1+\sqrt{2})} = 0.4327\dots$$

Corollary 2. *The double inequality*

$$\alpha CH(a,b)+M(a,b) < \overline{C}(a,b) < \beta CH(a,b)+M(a,b) \tag{3.7}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha \le \frac{1}{12} = 0.0833\dots$ *and*

$$\beta \ge \frac{2}{3}-\frac{1}{2\ln(1+\sqrt{2})} = 0.0993\dots$$

4. Some new inequalities involving Neuman-Sándor mean

Finally we further establish some new inequalities involving Neuman-Sándor, centroidal, root-square, and other means.

Theorem 2. *The inequality*

$$M(a,b) > \lambda CH(a,b) \tag{4.1}$$

holds for all $a,b>0$ *with* $a \ne b$ *if and only if* $\lambda \le \frac{1}{2\ln(1+\sqrt{2})} = 0.5672\dots$.

Proof. It is clear that

$$\frac{M(a,b)}{CH(a,b)} = \frac{(a-b)(a+b)}{(a-b)^2 2\operatorname{arcsinh}\frac{a-b}{a+b}} = \frac{a+b}{a-b}\frac{1}{2\operatorname{arcsinh}\frac{a-b}{a+b}}.$$

Without loss of generality, we assume that $a>b>0$. Let $x=\frac{a-b}{a+b}$. Then $x\in(0,1)$ and

$$\frac{M(a,b)}{CH(a,b)} = \frac{1}{2x\operatorname{arcsinh} x} \triangleq f(x).$$

Differentiating $f(x)$ yields

$$f'(x) = -\frac{\frac{x}{\sqrt{1+x^2}}+\operatorname{arcsinh} x}{2x^2\operatorname{arcsinh}^2 x} \le 0$$

which means that function $f(x)$ is decreasing for $x\in(0,1)$.

It is apparent that

$$\lim_{x\to1^-} f(x) = \frac{1}{2\ln(1+\sqrt{2})}.$$

The proof of Theorem 2 is thus complete. □

Theorem 3. *The double inequality*

$$\alpha Q(a,b)+(1-\alpha)M(a,b) < \overline{C}(a,b) < \beta Q(a,b)+(1-\beta)M(a,b) \tag{4.2}$$

holds for all $a,b>0$ *with* $a\ne b$ *if and only if* $\alpha \le \frac{1}{2}$ *and*

$$\beta \ge \frac{3-4\ln(1+\sqrt{2})}{3[1-\sqrt{2}\ln(1+\sqrt{2})]} = 0.7107\dots$$

Proof. It is sufficient to show

$$\alpha < \frac{\overline{C}(a,b)-M(a,b)}{Q(a,b)-M(a,b)} < \beta.$$

Without loss of generality, we assume that $a>b>0$. Let $x=\frac{a}{b}$. Then $x>1$ and

$$\frac{\overline{C}(a,b)-M(a,b)}{Q(a,b)-M(a,b)} = \frac{\frac{2(x^2+x+1)}{3(x+1)} - \frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}}}{\sqrt{\frac{x^2+1}{2}} - \frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}}}.$$

Let $t=\frac{x-1}{x+1}$. Then $t\in(0,1)$ and

$$\frac{\overline{C}(a,b)-M(a,b)}{Q(a,b)-M(a,b)}=\frac{\frac{t^2}{3}+1-\frac{t}{\operatorname{arcsinh}t}}{\sqrt{1+t^2}-\frac{t}{\operatorname{arcsinh}t}}.$$

Let $t=\sinh\theta$ for $\theta\in\big(0,\ln\big(1+\sqrt{2}\big)\big)$. Then

$$\frac{\overline{C}(a,b)-M(a,b)}{Q(a,b)-M(a,b)}=\frac{\frac{\sinh^2\theta}{3}+1-\frac{\sinh\theta}{\theta}}{\cosh\theta-\frac{\sinh\theta}{\theta}}.$$

By Lemma 3, we obtain Theorem 3. □

Theorem 4. *The double inequality*

$$\alpha C(a,b)+(1-\alpha)M(a,b)<Q(a,b)<\beta C(a,b)+(1-\beta)M(a,b) \tag{4.3}$$

holds for all $a,b>0$ with $a\neq b$ if and only if

$$\alpha\le\frac{\sqrt{2}\ln(1+\sqrt{2})-1}{2\ln(1+\sqrt{2})-1}=0.3231\dots \quad \textit{and} \quad \beta\ge\frac{2}{5}.$$

Proof. The double inequalities (4.3) is the same as

$$\alpha<\frac{Q(a,b)-M(a,b)}{C(a,b)-M(a,b)}<\beta.$$

Without loss of generality, we assume that $a>b>0$. Let $x=\frac{a}{b}$. Then $x>1$ and

$$\frac{Q(a,b)-M(a,b)}{C(a,b)-M(a,b)}=\frac{\sqrt{\frac{x^2+1}{2}}-\frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}}}{\frac{x^2+1}{x+1}-\frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}}}.$$

Let $t=\frac{x-1}{x+1}$. Then $t\in(0,1)$ and

$$\frac{Q(a,b)-M(a,b)}{C(a,b)-M(a,b)}=\frac{\sqrt{1+t^2}-\frac{t}{\operatorname{arcsinh}t}}{1+t^2-\frac{t}{\operatorname{arcsinh}t}}.$$

Let $t=\sinh\theta$ for $\theta\in\big(0,\ln\big(1+\sqrt{2}\big)\big)$. Then

$$\frac{Q(a,b)-M(a,b)}{C(a,b)-M(a,b)}=\frac{\cosh\theta-\frac{\sinh\theta}{\theta}}{1+\sinh^2\theta-\frac{\sinh\theta}{\theta}}.$$

According to Lemma 4, the proof of Theorem 3 is complete. □

Remark 1. This paper is a slightly revised version of the preprint [6].

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