Miskolc Mathematical Notes
Vol. 19 (2018), No. 2, pp. 1223–1231

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2018.1163

# DISTANCE $k$-DOMINATION IN SOME CYCLE RELATED GRAPHS

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*Received 12 March, 2014*

*Abstract.* In this paper we determine distance $k-$domination number of graph obtained by duplication of vertices altogether by edges in cycle $C_n$, splitting graph of cycle $C_n$ as well as graph obtained by duplication of edges altogether by vertices in cycle $C_n$.

2010 *Mathematics Subject Classification:* 05C69; 05C12; 05C38; 05C76

*Keywords:* dominating set, domination number, distance domination, splitting graph

## 1. INTRODUCTION

Graph $G = (V(G), E(G))$, we mean simple, finite, connected and undirected graph. The open neighbourhood $N(v)$ of $v \in V(G)$ is the set of all vertices adjacent to $v$. That is, $N(v) = \{u \in V(G)/uv \in E(G)\}$. The closed neighbourhood of $v \in V(G)$ is the set $N[v] = N(v) \cup \{v\}$. The distance $d(u,v)$ between two vertices $u$ and $v$ is the length of the shortest $uv-$path in $G$, if exists, otherwise $d(u,v) = \infty$. The open $k-$neighbourhood $N_k(v)$ of a vertex $v \in V(G)$ is the set of all vertices of $G$ which are different from $v$ and at a distance at most $k$ from $v$ in $G$. That is $N_k(v) = \{u \in V(G)/d(u,v) \leq k\}$. The closed $k-$neighbourhood set is defined as $N_k[v] = N_k(v) \cup \{v\}$. It is obvious that $N(v) = N_1(v)$. A set $D \subseteq V(G)$ is called a dominating set if every vertex in $V(G) - D$ is adjacent to at least one vertex in $D$. For terminology and notation not defined here we follow West [13] and Haynes *et al.* [3]. The concept of distance dominating set was initiated by Slater [7] with special reference to communication network, while the term distance $k-$dominating set was given by Henning *et al.* [5]. For an integer $k \geq 1$, $D \subseteq V(G)$ is a distance $k-$dominating set of $G$, if every vertex in $V(G) - D$ is within the distance $k$ from some vertex $v \in D$. That is, $N_k[D] = V(G)$. The minimum cardinality among all the distance $k-$dominating sets of $G$ is called the distance $k-$domination number of $G$ and it is denoted by $\gamma_k(G)$. It is obvious that $\gamma(G) = \gamma_1(G)$. A distance $k-$dominating set of cardinality $\gamma_k(G)$ is called a $\gamma_k-$set. Many reserchers have explored the concept of distance $k-$domination in graphs. The distance domination number for cartesian products of two paths has been investigated by Klobucar [6] while the distance domination in the context of spanning tree of the graph is discussed by Griggs and Hutchinson [2]. The bounds on the distance two-domination number and the classes of graphs attaining these bounds are reported in Sridharan *et al.* [8]. Tian and Xu [9] have established upper bound for distance $k-$domination for connected graph $G$ and show that $\gamma_k(G) \leq \left\lfloor \frac{n-\Delta+k-1}{k} \right\rfloor$. The same authors in [10] have studied average distance and distance domination number and established an upper bound of average distance in terms of distance domination. Fischermann and Volkmann [1] have characterized the graphs whose distance $n-$domination number is equal to half of their number of vertices, when the diameter is greater or equal to $2n-1$. Vaidya and Kothari [12] have investigated distance $k-$domination number of total graph, shadow graph and middle graph of path $P_n$. The same authors in [11] have investigated distance $k-$domination number for the graphs obtained by graph operations on some standard graphs. For more bibliographic references on distance $k-$domination, the readers are advised to refer a survey article by Henning [4].

## 2. Results

**Proposition 1** ([4]). *Let $k \geq 1$ and $D$ be a distance $k$-dominating set of a graph $G$. Then $D$ is a minimal distance $k$-dominating set of $G$ if and only if each $d \in D$ has at least one of the following two properties hold.*
*(1) There exist a vertex $v \in V(G) - D$ such that $N_k(v) \cap D = \{d\}$.*
*(2) The vertex $d$ is at distance at least $k+1$ from every other vertex $d$ of $D$ in $G$.*

**Definition 1.** Duplication of a vertex $v$ by a new edge $e = v'v''$ of graph $G$ produces a new graph $G'$ such that $N(v') \cap N(v'') = \{v\}$.

**Theorem 1.** *If $G$ is a graph obtained by duplication of vertices altogether by edges in cycle $C_n$ $(n \leq 2k-1)$ then $\gamma_k(G) = 1$.*

*Proof.* Let $G$ be a graph obtained by duplication of vertices $v_1, v_2, \ldots, v_n$ by edges $u_{2i-1}u_{2i}$ $(1 \leq i \leq n)$ in cycle $C_n$. Then $D = \left\{ v_{\left\lfloor \frac{n}{2} \right\rfloor} \right\}$ is distance $k$-dominating set of $G$ as $n \leq 2k-1$. Hence $\gamma_k(G) = 1$. □

**Theorem 2.** *If $G$ is a graph obtained by duplication of vertices altogether by edges in cycle $C_n$ $(n > 2k-1)$ then*

$$\gamma_k(G) = \begin{cases} \left\lfloor \dfrac{n}{2k-1} \right\rfloor + 1, & \text{for } n \equiv 1, 2, ..., 2k-2 \pmod{2k-1} \\ \dfrac{n}{2k-1} & \text{for } n \equiv 0 \pmod{2k-1} \end{cases}$$

*Proof.* Let $G$ be a graph obtained by duplication of vertices $v_1, v_2, \ldots, v_n$ by edges $u_{2i-1}u_{2i}$ $(1 \leq i \leq n)$ in cycle $C_n$. One can observe that $v_i$'s dominate more vertices than $u_i$'s. Here vertices from $v_{n-(k-1)}$ to $v_{k+1}$, $u_{2n-(2k-3)}$ to $u_{2n}$ and $u_1$

to $u_{2k}$ are dominated by a vertex $v_1$ at a distance $k$. Also $d(v_1, u_{2k}) = k$, and $d(u_{2n-(2k-3)}, v_1) = k$. Hence $2k-1$ consecutive vertices from $u_i$'s dominated by only one vertex $v_1$. Hence

$$\gamma_k(G) \geq \left\lfloor \frac{n}{2k-1} \right\rfloor \tag{2.1}$$

Now depending upon the number of vertices of $C_n$, consider the following subsets,
For $n \equiv 1, 2, \ldots, 2k-2 \pmod{2k-1}$

$$D = \left\{ v_{1+(2k-1)j} / 0 \leq j \leq \left\lfloor \frac{n}{2k-1} \right\rfloor \right\}, |D| = \left\lfloor \frac{n}{2k-1} \right\rfloor + 1,$$

for $n \equiv 0 \pmod{2k-1}$

$$D = \left\{ v_{1+(2k-1)j} / 0 \leq j < \frac{n}{2k-1} \right\}, |D| = \frac{n}{2k-1}.$$

We claim that each $D$ is a distance $k$-dominating set as
For $j \neq 0$,

$$d(v_{1+(2k-1)j}, v_{i+(2k-1)j}) \leq k, \text{ where } -k+1 \leq i \leq k+1, i \neq 1$$

$$d(v_{1+(2k-1)j}, u_{i+(2k-1)j}) \leq k, \text{ where } (3-2k)+(2k-1)j \leq i \leq 2k+(2k-1)j$$

and for $j = 0$,

$$d(v_1, v_i) \leq k, \text{ where } n-k+1 \leq i \leq n, 2 \leq i \leq k+1$$

$$d(v_1, u_i) \leq k, \text{ where } 1 \leq i \leq 2k, 2n-(2k-3) \leq i \leq 2n$$

Therefore for $j \neq 0$,

$$\begin{aligned} N_k(v_{1+(2k-1)j}) = \{ & v_{2+(2k-1)j}, v_{3+(2k-1)j}, \ldots, v_{(k+1)+(2k-1)j}, \ldots, v_{(2k-1)j}, \\ & v_{(2k-1)j-1}, \ldots, v_{(2k-1)j-(k-1)}, u_{1+2(2k-1)j}, u_{2(1+(2k-1)j)}, \\ & \ldots, u_{2k+2((2k-1)j)}, u_{(3-2k)+2((2k-1)j)}, u_{(4-2k)+2((2k-1)j)}, \\ & \ldots, u_{2(2k-1)j} \}. \end{aligned}$$

While for $j = 0$,

$$\begin{aligned} N_k(v_1) = \{ & v_2, v_3, \ldots, v_{k+1}, v_{n-(k-1)}, v_{n-k}, \ldots, v_n, u_1, u_2, \ldots, u_{2k}, u_{2n}, \\ & u_{2n-1}, \ldots, u_{2n-(2k-3)} \}. \end{aligned}$$

Then $N_k[D] = N_k[v_1] \cup N_k[v_{1+(2k-1)j}] = V(G)$.
For some $j = j_1$, $v_{1+(2k-1)j_1} \in D$ and for some $j = j_2$, $v_{1+(2k-1)j_2} \in D$,

$$d(v_{1+(2k-1)j_1}, v_{1+(2k-1)j_2}) = (j_2 - j_1)(2k-1) \geq k+1.$$

Which implies that every vertex $d$ of $D$ is at a distance $k+1$ apart from every other vertex of $D$ in $G$. Thus by Proposition 1 above defined $D$ is a minimal distance

$k-$dominating set of $G$ and by expression 2.1 it is also of minimum cardinality for $n > 2k-1$. Hence

$$\gamma_k(G) = \begin{cases} \left\lfloor \dfrac{n}{2k-1} \right\rfloor + 1, & \text{for } n \equiv 1,2,...,2k-2 \,(\text{mod } 2k-1) \\ \dfrac{n}{2k-1} & \text{for } n \equiv 0 \;(\text{ mod } \; 2k-1) \end{cases}$$

□

**Illustration 1.** *Distance 3−dominating set in graph $G$ obtained by duplication of vertices in cycle $C_{22}$ altogether by edges is shown by solid vertices in* FIGURE *1*.

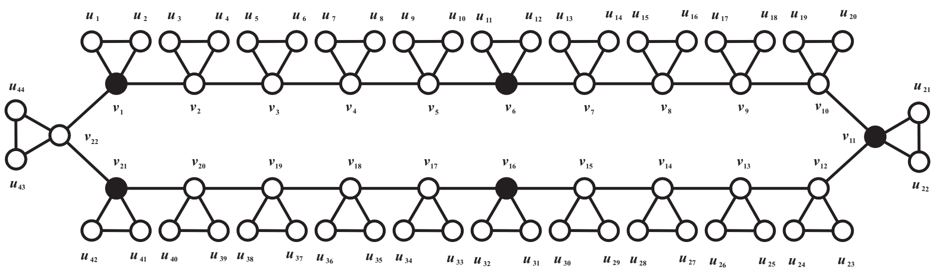

FIGURE 1.

**Definition 2.** For a graph $G$, the splitting graph $S'(G)$ of graph $G$ is obtained by adding a new vertex $v'$ corresponding to each vertex $v$ of $G$ such that $N(v) = N(v')$.

**Theorem 3.** *If $n \leq 2k+1$, $k \neq 1$, then $\gamma_k(S'(C_n)) = 1$.*

*Proof.* Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle $C_n$ and $u_1, u_2, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, \ldots, v_n$ which are added to obtain $S'(C_n)$. Then $D = \left\{ v_{\left\lceil \frac{n}{2} \right\rceil} \right\}$ is distance $k$-dominating set of $S'(C_n)$ as $n \leq 2k+1$. Hence $\gamma_k(S'(C_n)) = 1$. □

**Theorem 4.** *If $n > 2k+1$ then*

$$\gamma_k(S'(C_n)) = \begin{cases} \left\lfloor \dfrac{n}{2k+1} \right\rfloor + 1, & \text{for } n \equiv 1,2,...,2k \;(\text{mod } 2k+1) \\ \dfrac{n}{2k+1} & \text{for } n \equiv 0 \;(\text{ mod } \; 2k+1) \end{cases}$$

*Proof.* Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle $C_n$ and $u_1, u_2, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, \ldots, v_n$ which are added to obtain $S'(C_n)$. One can observe that $v_i$'s dominate more vertices than $u_i$'s at a distance $k$. In graph $S'(C_n)$

vertices $v_{n-(k-1)}$ to $v_{k+1}$ and $u_{n-(k-1)}$ to $u_{n+k}$ are dominated by a vertex $v_1$ at a distance $k$. Also $d(v_{n-(k-1)}, v_{k+1}) = 2k+1$, and $d(u_{n-(k-1)}, u_{k+1}) = 2k+1$. Hence $2k+1$ consecutive vertices from $v_i$'s and $(2k+1)$ consecutive vertices from $u_i$'s dominated by only one vertex at a distance $k$. Which implies that

$$\gamma_k\left(S'(C_n)\right) \geq \left\lfloor \frac{n}{2k+1} \right\rfloor \tag{2.2}$$

Now depending upon the number of vertices of $C_n$, consider the following subsets,
For $n \equiv 1, 2, \ldots, 2k \pmod{2k+1}$

$$D = \left\{ v_{1+(2k+1)j} / 0 \leq j \leq \left\lfloor \frac{n}{2k+1} \right\rfloor \right\}, |D| = \left\lfloor \frac{n}{2k+1} \right\rfloor + 1,$$

for $n \equiv 0 \pmod{2k+1}$

$$D = \left\{ v_{1+(2k+1)j} / 0 \leq j < \frac{n}{2k+1} \right\}, |D| = \frac{n}{2k+1}.$$

We claim that each $D$ is a distance $k$-dominating set as
for $j \neq 0$,

$$d(v_{1+(2k+1)j}, v_{i+(2k+1)j}) \leq k, \text{ where } -k+1 \leq i \leq k+1, i \neq 1$$

$$d(v_{1+(2k+1)j}, u_{i+(2k+1)j}) \leq k, \text{ where } -k+1 \leq i \leq k+1$$

while for $j = 0$

$$d(v_1, v_i) \leq k, \text{ where } n-k+1 \leq i \leq n,\ 1 \leq i \leq k+1$$

$$d(v_1, u_i) \leq k, \text{ where } 1 \leq i \leq k+1,\ n-k+1 \leq i \leq n$$

Therefore $j \neq 0$,

$$\begin{aligned} N_k(v_{1+(2k+1)j}) = \{&v_{2+(2k+1)j}, v_{3+(2k+1)j}, \ldots, v_{(k+1)+(2k+1)j}, v_{(2k+1)j},\\ &v_{(2k+1)j-1}, \ldots, v_{(2k+1)j-(k-1)}, u_{1+(2k+1)j}, u_{2+(2k+1)j}, \ldots,\\ &u_{(k+1)+(2k+1)j}, u_{(2k+1)j}, u_{(2k+1)j-1}, \ldots, u_{(2k+1)j-(k-1)}\}. \end{aligned}$$

While for $j = 0$,

$$\begin{aligned} N_k(v_1) = \{&v_2, v_3, \ldots, v_{k+1}, v_n, v_{n-1}, v_{n-(-k+1)}, u_1, u_2, \ldots, u_{k+1}, u_{n-k+1},\\ &u_{n-k+2}, \ldots, u_n\}. \end{aligned}$$

Then $N_k[D] = N_k[v_1] \cup N_k[v_{1+(2k+1)j}] = V(S'(C_n))$.
For some $j = j_1$, $v_{1+(2k+1)j_1} \in D$ and for some $j = j_2$, $v_{1+(2k+1)j_2} \in D$,

$$d(v_{1+(2k+1)j_1}, v_{1+(2k+1)j_2}) = (j_2 - j_1)(2k+1) \geq k+1$$

This implies that every vertex $d$ of $D$ is at a distance $k+1$ apart from every other vertex of $D$ in $S'(C_n)$. Thus by Proposition 1 above defined $D$ is a minimal distance

$k-$dominating set of $G$ and by expression 2.2 it is also of minimum cardinality for $n > 2k-1$. Hence

$$\gamma_k(S'(C_n)) = \begin{cases} \left\lfloor \dfrac{n}{2k+1} \right\rfloor + 1, & \text{for } n \equiv 1, 2, ..., 2k \pmod{2k+1} \\ \dfrac{n}{2k+1} & \text{for } n \equiv 0 \pmod{2k+1} \end{cases}$$

□

**Illustration 2.** *Distance* $4-$*dominating set in* $S'(C_{20})$ *is shown by solid vertices in* FIGURE 2.

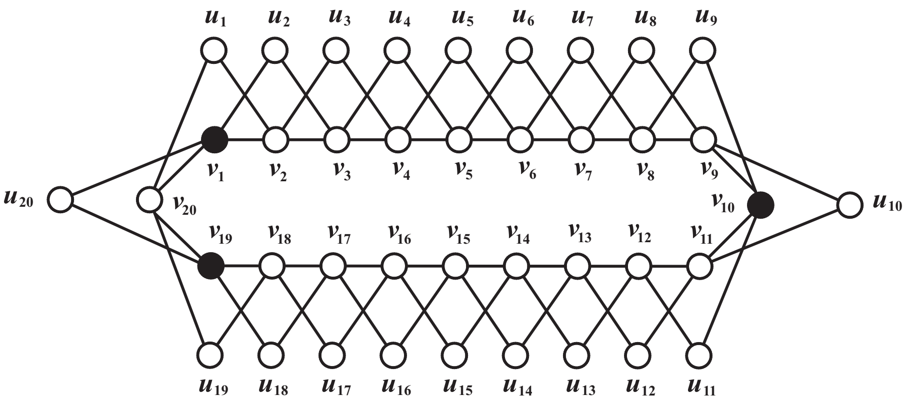


FIGURE 2.

**Definition 3.** Duplication of an edge $e = uv$ by a new vertex $w$ in a graph $G$ produces a new graph $G'$ such that $N(w) = \{u, v\}$.

**Theorem 5.** *If G is a graph obtained by duplication of edges altogether by vertices in cycle* $C_n$
$(n \leq 2k)$ *then* $\gamma_k(G) = 1$.

*Proof.* Let $G$ be a graph obtained by duplication of edges $v_i v_{i+1}$ altogether by vertices $u_i$, $(1 \leq i < n)$ in cycle $C_n$. Then $\left\{ v_{\left\lfloor \frac{n}{2} \right\rfloor} \right\}$ is distance $k$-dominating set of $G$ as $n \leq 2k$. Hence $\gamma_k(G) = 1$. □

**Theorem 6.** *If G is a graph obtained by duplication of edges altogether by vertices in cycle $C_n$ $(n > 2k)$ then*

$$\gamma_k(G) = \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + 1, & \text{for } n \equiv 1,2,...,2k-1 \,(\text{mod } 2k) \\ \frac{n}{2k} & \text{for } n \equiv 0 (\text{ mod } 2k) \end{cases}$$

*Proof.* Let G be a graph obtained by duplication of edges $v_i v_{i+1}$ altogether by vertices $u_i$ $(1 \leq i < n)$ in cycle $C_n$. One can observe that $v_i$'s dominate more vertices than $u_i$'s at a distance $k$. In graph $G$ vertices $v_{n-k}$ to $v_{k+1}$ and $u_{n-(k-1)}$ to $u_k$ are dominated by a vertex $v_1$ at a distance $k$. Also $d(u_{n-(k-1)}, v_1) = k$ and $d(v_1, u_k) = k$. Hence $2k$ consecutive vertices of $u_i$'s are dominated by only one vertex at a distance $k$. Hence

$$\gamma_k(G) \geq \left\lfloor \frac{n}{2k} \right\rfloor \tag{2.3}$$

Now depending upon the number of vertices of $C_n$, consider the following subsets.
For $n \equiv 1, 2, \ldots, 2k-1$ ( mod $2k$)

$$D = \left\{ v_{1+(2k)j} / 0 \leq j \leq \left\lfloor \frac{n}{2k} \right\rfloor \right\}, |D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1,$$

for $n \equiv 0$ ( mod $2k$)

$$D = \left\{ v_{1+(2k)j} / 0 \leq j < \frac{n}{2k} \right\}, |D| = \frac{n}{2k}.$$

Now we claim that each $D$ is a distance $k$-dominating set as
for $j \neq 0$,

$$d(v_{1+2kj}, v_{i+2kj}) \leq k, \text{ where } -k+1 \leq i \leq k+1, i \neq 1,$$

$$d(v_{1+2kj}, u_{i+2kj}) \leq k, \text{ where } -k+1 \leq i \leq k$$

while for $j = 0$,

$$d(v_1, v_i) \leq k, \text{ where } n-k+1 \leq i \leq n, 2 \leq i \leq k+1$$

$$d(v_1, u_i) \leq k, \text{ where } n-k+1 \leq i \leq n, 1 \leq i \leq k$$

Therefore for $j \neq 0$,

$$N_k(v_{1+2kj}) = \{v_{2+2kj}, v_{3+2kj}, \ldots, v_{(k+1)+2kj}, v_{2kj}, v_{2kj-1}, \ldots, v_{2kj-(k-1)}, u_{1+2kj}, u_{2+2kj}, \ldots, u_{k+2kj}, u_{2kj}, u_{2kj-1}, \ldots, u_{2kj-(k-1)}\}.$$

While for $j = 0$,

$$N_k(v_1) = \{v_2, v_3, \ldots, v_{k+1}, v_{n-k+1}, v_{n-k+2}, \ldots, v_n, u_1, u_2, \ldots, u_k, u_{n-k+1}, u_{n-k+2}, \ldots, u_n\}.$$

This implies that $N_k[D] = N_k[v_1] \cup N_k[v_{1+(2k)j}] = V(G)$.
For some $j = j_1$, $v_{1+(2k)j_1} \in D$ and for some $j = j_2$, $v_{1+(2k)j_2} \in D$,

$$d(v_{1+(2k)j_1}, v_{1+(2k)j_2}) = (j_2 - j_1)2k \geq k+1$$

Which implies that every vertex $d$ of $D$ is at a distance $k+1$ apart from every other vertex of $D$ in $G$. Thus by Proposition 1 above defined $D$ is a minimal distance $k-$dominating set of $G$ and by expression 2.3 it is also of minimum cardinality for $n > 2k$. Hence

$$\gamma_k(G) = \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + 1, & \text{for} n \equiv 1, 2, ..., 2k-1 \,(\text{mod } 2k) \\ \frac{n}{2k} & \text{for} n \equiv 0 (\text{ mod } 2k) \end{cases}$$

□

**Illustration 3.** *Distance 2−dominating set in graph G obtained by duplication of edges in $C_{18}$ altogether by vertices is shown by solid vertices in* FIGURE *3*.

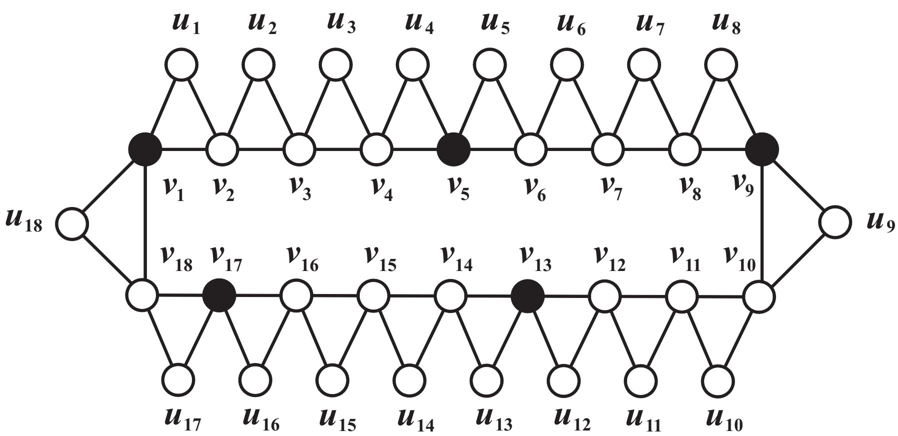


FIGURE 3.

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