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ON DIRECTIONAL DERIVATIVE SETS OF MAX-MIN SET-VALUED MAPS

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Abstract. In this paper, necessary statements are given for a minimum point and a maximum point of the max-min function. Moreover estimations for the directional lower and upper derivative sets of the max-min set-valued map which are used to state a characterization of the directional derivative of the max-min functions are given. Furthermore, a sufficient condition ensuring the existence of the directional derivative of the max-min function is obtained by using the lower differentiability of the max-min set-valued maps.

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1. INTRODUCTION

The directional derivatives of max-min functions were studied in [7]. It is well-known that max-min functions are considered and occur in control theory problems, parametric optimization problems and differential game theory problems [4]. Moreover marginal functions are max-min functions (see [3–6, 10–14]).

In this paper, necessary statements are given for a minimum point and a maximum point of the max-min function. It is well-known that the directional lower and upper derivative sets of max-min set valued maps are used to state a characterization of the directional derivative of max-min function [7]. In this paper, estimations for the directional lower and upper derivative sets of max-min set valued maps are given. Furthermore, a sufficient condition ensuring the existence of the directional derivative of max-min function is obtained by using the lower differentiability of max-min set-valued maps.

In this study, $\mathrm{cl}(\mathbb{R}^m)$ $(\mathrm{comp}(\mathbb{R}^m))$ denotes the set of all nonempty closed (compact) subsets in $\mathbb{R}^m$. Let $a(\cdot):\mathbb{R}^n \to \mathrm{cl}(\mathbb{R}^m)$ be an upper semi-continuous set-valued map. For $(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$ and vector $f\in\mathbb{R}^n$, let us consider the following sets:

$$Da(x,y)\mid(f)=\big\{d\in\mathbb{R}^m:\liminf_{\delta\to+0}\delta^{-1}\,\mathrm{dist}\,(y+\delta d,a(x+\delta f))=0\big\},$$

$$D^*a(x,y)\mid(f)=\big\{v\in\mathbb{R}^m:\lim_{\delta\to+0}\delta^{-1}\,\mathrm{dist}\,(y+\delta d,a(x+\delta f))=0\big\},$$

where $x \in \mathbb{R}^n$, $D \subset \mathbb{R}^n$, $\operatorname{dist}(x, D) = \inf_{d \in D} \|x - d\|$. $Da(x, y) \mid (f)$ ($D^*a(x, y) \mid (f)$) is called the upper (lower) derivative set of the set-valued map $a(\cdot)$ at (x, y) in direction f. Note that the directional upper (lower) derivative set of the set-valued map $a(\cdot)$ is closed and there is a connection between the upper (lower) derivative set of a set-valued map and the upper (lower) contingent cone which is used to investigate several problems in nonsmooth analysis [1,2,8]. It is obvious that $D^*a(x, y) \mid (f) \subset Da(x, y) \mid (f)$. The symbol

$$A = \operatorname{gr} a(\cdot) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in a(x)\}$$

denotes the graph of the set-valued map $a(\cdot)$. Since $a(\cdot)$ is upper semicontinuous, A is a closed set. It is possible to show that $Da(x, y) \mid (f) = D^*a(x, y) \mid (f) = \varnothing$ if $(x, y) \notin A$, $Da(x, y) \mid (f) = D^*a(x, y) \mid (f) = \mathbb{R}^m$ if $(x, y) \in \operatorname{int} A$, where $\operatorname{int} A$ denotes the interior of A.

Let $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be a function. The lower and upper derivative of $f(\cdot)$ at the point x in direction v are denoted by the symbols $\frac{\partial^- f(x)}{\partial v}$ and $\frac{\partial^+ f(x)}{\partial v}$ respectively, and defined by the formulas

$$\frac{\partial^- f(x)}{\partial v} = \liminf_{\delta \to +0} [f(x + \delta v) - f(x)]\delta^{-1},$$

and

$$\frac{\partial^+ f(x)}{\partial v} = \limsup_{\delta \to +0} [f(x + \delta v) - f(x)]\delta^{-1}.$$

If

$$\frac{\partial f(x)}{\partial v} = \lim_{\delta \to +0} [f(x + \delta v) - f(x)]\delta^{-1}$$

exists and is finite, then $f(\cdot)$ is said to be differentiable at the point x in direction v and $\frac{\partial f(x)}{\partial v}$ denotes the derivative of $f(\cdot)$ at the point x in direction v.

Let $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$, $b(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^k)$ be set-valued maps and $\sigma(\cdot, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ be a continuous function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$. The max-min function is denoted by $m(\cdot)$ and is defined by

$$m(x) = \max_{y \in a(x)} \min_{z \in b(x)} \sigma(x, y, z). \tag{1.1}$$

In this paper, we will assume that $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$, $b(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^k)$ are continuous set-valued maps and $\sigma(\cdot, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ is a continuous function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$ and locally Lipschitz on $\mathbb{R}^m \times \mathbb{R}^k$, i. e., for every bounded $D \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$, there exists $L(D) > 0$ such that

$$|\sigma(x, y_1, z_1) - \sigma(x, y_2, z_2)| \le L(D) \, \|(y_1 - y_2, z_1 - z_2)\|$$

for any (x, y_1, z_1), $(x, y_2, z_2) \in D$. Under these conditions $m(\cdot)$ is a continuous function (see [1]). Let

$$Y_*(x) = \big\{(y_*, z_*) \in a(x) \times b(x) : \\ m(x) = \max_{y \in a(x)} \min_{z \in b(x)} \sigma(x,y,z) = \sigma(x, y_*, z_*)\big\}. \quad (1.2)$$

The map $x \mapsto Y_*(x)$ is an upper semicontinuous set-valued map and it is called a max-min set-valued map.

2. Minimization and maximization problems of max-min function

Now we give necessary statements for a minimum point and a maximum point of max-min function.

Theorem 1. *Let the function $m(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be in the form (1.1). Suppose that $x_* \in \mathbb{R}^n$ is a minimum point of max-min function $m(\cdot)$. Then*

$$\inf_{f \in \mathbb{R}^n} \inf_{(y,z) \in Y_*(x_*)} \inf_{(d,n) \in DY_*(x_*,y,z)|(f)} \frac{\partial^+ \sigma(x_*, y, z)}{\partial(f,d,n)} \geq 0.$$

Proof. Let $f \in \mathbb{R}^n$. Since $x_* \in \mathbb{R}^n$ is a minimum point of max-min function $m(\cdot)$, then it follows that $m(x_* + \delta f) \geq m(x_*)$ for all $\delta > 0$. In that case, it follows from here that

$$\frac{\partial^- m(x_*)}{\partial f} = \liminf_{\delta \to +0} \frac{1}{\delta}\left[m(x_* + \delta f) - m(x_*)\right] \geq 0.$$

Hence, from here and of [7, Proposition 7], we have

$$0 \leq \frac{\partial^- m(x_*)}{\partial f} \leq \inf_{(y,z) \in Y_*(x_*)} \inf_{(d,n) \in DY_*(x_*,y,z)|(f)} \frac{\partial^+ \sigma(x_*, y, z)}{\partial(f,d,n)}$$

and hence we obtain the inequality. □

Theorem 2. *Let the function $m(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be in the form (1.1). Suppose that $x_* \in \mathbb{R}^n$ is a maximum point of max-min function $m(\cdot)$ and there exists a $(y,z) \in Y_*(x_*)$ such that $DY_*(x_*, y, z) \mid (f) \neq \varnothing$ for all $f \in \mathbb{R}^n$. Then*

$$\sup_{f \in \mathbb{R}^n} \inf_{(y,z) \in Y_*(x_*)} \inf_{(d,n) \in DY_*(x_*,y,z)|(f)} \frac{\partial^- \sigma(x_*, y, z)}{\partial(f,d,n)} \leq 0.$$

Proof. Let $f \in \mathbb{R}^n$. Since $x_* \in \mathbb{R}^n$ is a maximum point of max-min function $m(\cdot)$, then it follows that $m(x_* + \delta f) \leq m(x_*)$ for all $\delta > 0$. In that case, it follows from here that

$$\frac{\partial^+ m(x_*)}{\partial f} = \limsup_{\delta \to +0} \frac{1}{\delta}\left[m(x_* + \delta f) - m(x_*)\right] \leq 0.$$

Hence, from here and [7, Proposition 8], we have

$$0 \geq \frac{\partial^+ m(x_*)}{\partial f} \geq \inf_{(y,z) \in Y_*(x_*)} \inf_{(d,n) \in DY_*(x_*,y,z)|(f)} \frac{\partial^- \sigma(x_*, y, z)}{\partial(f,d,n)}$$

and hence we obtain the inequality. □

3. DIRECTIONAL DERIVATIVE SETS OF MAX-MIN SET-VALUED MAPS

Now we give the estimations for the directional lower and upper derivative sets of max-min set-valued map which are used to state a characterization of the directional derivative of the max-min functions in [7].

Let us take the max-min function $m(\cdot)$ such as:

$$m(x) = \min_{y \in a(x)} \sigma(x,y) \tag{3.1}$$

where $a(\cdot) : \mathbb{R}^n \to \text{comp}(\mathbb{R}^m)$ is a continuous set-valued map and $\sigma(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function on $\mathbb{R}^n \times \mathbb{R}^m$ and locally Lipschitz on $\mathbb{R}^m$, i. e., for every bounded $D \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists $L(D) > 0$ such that

$$|\sigma(x,y_1) - \sigma(x,y_2)| \le L(D) \, \|y_1 - y_2\|$$

for any (x,y_1), $(x,y_2) \in D$. Under these conditions $m(\cdot)$ is a continuous function (see [1]). Then we take

$$Y_*(x) = \left\{ y_* \in a(x) : m(x) = \min_{y \in a(x)} \sigma(x,y) = \sigma(x,y_*) \right\}. \tag{3.2}$$

Theorem 3. *Let the set-valued map $Y_*(\cdot) : \mathbb{R}^n \to \text{comp}(\mathbb{R}^m)$ be in the form (3.2). Suppose that there exists a $y_* \in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a derivable function at the point (x_*,y_*) in direction (f,d) for all $d \in \mathbb{R}^m$. Then*

$$D^*Y_*(x_*,y_*) \mid (f) \subset \left\{ p \in \mathbb{R}^m : \inf_{d \in Da(x_*,y_*)|(f)} \frac{\partial \sigma(x_*,y_*)}{\partial (f,d)} = \frac{\partial \sigma(x_*,y_*)}{\partial (f,p)} \right\}. \tag{3.3}$$

Proof. Let $y_* \in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a derivable function at the point (x_*,y_*) in direction (f,d) for all $d \in \mathbb{R}^m$.

Let $D^*Y_*(x_*,y_*) \mid (f) = \varnothing$. Then the statement (3.3) holds.

Let $Da(x_*,y_*) \mid (f) = \varnothing$. Then

$$\inf_{d \in Da(x_*,y_*)|(f)} \frac{\partial \sigma(x_*,y_*)}{\partial (f,d)} = +\infty.$$

Since $D^*Y_*(x_*,y_*) \mid (f) \subset D^*a(x_*,y_*) \mid (f) \subset Da(x_*,y_*) \mid (f)$, then it follows that statement (3.3) holds.

Now $D^*Y_*(x_*,y_*) \mid (f) \neq \varnothing$. Take $p \in D^*Y_*(x_*,y_*) \mid (f)$. Then from the definition of $D^*Y_*(x_*,y_*) \mid (f)$, there exists a $\delta_* > 0$ such that for all $\delta \in [0,\delta_*]$,

$$y_*(\delta) = y_* + \delta p + o_1(\delta) \in Y_*(x_* + \delta f)$$

where $\|o_1(\delta)\| \, \delta^{-1} \to 0$ as $\delta \to +0$. Since

$$Y_*(x) = \{ y_* \in a(x) : \sigma(x,y) \ge \sigma(x,y_*), \forall y \in a(x) \},$$

then it follows that for any $y \in a(x_* + \delta f)$,

$$\sigma(x_* + \delta f, y) \ge \sigma(x_* + \delta f, y_* + \delta p + o_1(\delta)). \tag{3.4}$$

Choose any $d \in Da(x_*, y_*) \mid (f)$. Then from the definition of $Da(x_*, y_*) \mid (f)$, there exists a sequence $y_k \in a(x_* + \delta_k f)$ where $\delta_k > 0$ and $\delta_k \to +0$ as $k \to \infty$, such that

$$y_k = y_* + \delta_k d + o_2(\delta_k)$$

where $\|o_2(\delta_k)\| \delta_k^{-1} \to 0$ as $k \to \infty$. In that case from (3.4) since $\delta_k \to +0$, then it follows that there exists a $k_0 \in \mathbb{N}$ such that $\delta_k \in [0, \delta_*]$ for all $k \geq k_0$ and

$$\begin{aligned}\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_*, y_*)& \\ \geq \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*).&\end{aligned}$$

Since the function $\sigma(\cdot,\cdot)$ is locally Lipschitzian on $\mathbb{R}^m$, then it follows that for $k = 1, 2, \ldots$, there exists $L_1 > 0$ and $L_2 > 0$ such that

$$\begin{aligned}&| \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_* + \delta_k f, y_* + \delta_k p) | \leq L_1 \|o_1(\delta_k)\|, \\ &| \sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_* + \delta_k f, y_* + \delta_k d) | \leq L_2 \|o_2(\delta_k)\|.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\frac{\partial \sigma(x_*, y_*)}{\partial(f,p)} &= \lim_{\delta \to +0} [\sigma(x_* + \delta f, y_* + \delta p) - \sigma(x_*, y_*)]\, \delta^{-1} \\
&= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p) - \sigma(x_*, y_*)]\, \delta_k^{-1} \\
&= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p) - \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) \\
&\quad + \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&\leq \lim_{k \to \infty} [L_1 \|o_1(\delta_k)\| + \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)]\, \delta_k^{-1} \\
&\leq \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_*, y_*)]\, \delta_k^{-1} \\
&= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_* + \delta_k f, y_* + \delta_k d) \\
&\quad + \sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&\leq \lim_{k \to \infty} L_2 \|o_2(\delta_k)\|\, \delta_k^{-1} + \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&= \lim_{\delta \to +0} [\sigma(x_* + \delta f, y_* + \delta d) - \sigma(x_*, y_*)]\, \delta^{-1} \\
&= \frac{\partial \sigma(x_*, y_*)}{\partial(f,d)}.
\end{aligned}$$

Thus $\frac{\partial\sigma(x_*,y_*)}{\partial(f,p)} \le \frac{\partial\sigma(x_*,y_*)}{\partial(f,d)}$ for any $d \in Da(x_*,y_*) \mid (f)$. It follows from here that

$$\inf_{d\in Da(x_*,y_*)|(f)} \frac{\partial\sigma(x_*,y_*)}{\partial(f,d)} \ge \frac{\partial\sigma(x_*,y_*)}{\partial(f,p)}.$$

Since

$$D^*Y_*(x_*,y_*) \mid (f) \subset DY_*(x_*,y_*) \mid (f) \subset Da(x_*,y_*) \mid (f),$$

then for any $p \in D^*Y_*(x_*,y_*) \mid (f)$, $p \in Da(x_*,y_*) \mid (f)$ and it follows that the relation

$$\inf_{d\in Da(x_*,y_*)|(f)} \frac{\partial\sigma(x_*,y_*)}{\partial(f,d)} \le \frac{\partial\sigma(x_*,y_*)}{\partial(f,p)}$$

is satisfied. In that case, it follows from the above two inequalities that the statement holds. □

Theorem 4. *Let the set-valued map $Y_*(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ be in the form (3.2). Suppose that there exists a $y_* \in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a derivable function at the point (x_*,y_*) in the direction (f,d) for all $d \in \mathbb{R}^m$. Then*

$$DY_*(x_*,y_*) \mid (f) \subset \left\{ p \in \mathbb{R}^m : \inf_{d\in D^*a(x_*,y_*)|(f)} \frac{\partial\sigma(x_*,y_*)}{\partial(f,d)} \ge \frac{\partial\sigma(x_*,y_*)}{\partial(f,p)} \right\}. \quad (3.5)$$

Proof. Let $y_* \in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a derivable function at the point (x_*,y_*) in direction (f,d) for all $d \in \mathbb{R}^m$.

Let $DY_*(x_*,y_*) \mid (f) = \varnothing$. Then statement (3.5) holds.

Let $D^*a(x_*,y_*) \mid (f) = \varnothing$. Then

$$+\infty = \inf_{d\in D^*a(x_*,y_*)|(f)} \frac{\partial\sigma(x_*,y_*)}{\partial(f,d)} > \frac{\partial\sigma(x_*,y_*)}{\partial(f,p)}$$

and it follows that the statement (3.5) holds.

Now let $DY_*(x_*,y_*) \mid (f) \ne \varnothing$ and $D^*a(x_*,y_*) \mid (f) \ne \varnothing$. Let us take $p \in DY_*(x_*,y_*) \mid (f)$. Then from the definition of $DY_*(x_*,y_*) \mid (f)$, there exists a sequence $y_k \in Y_*(x_* + \delta_k f)$, where $\delta_k > 0$ and $\delta_k \to +0$ as $k \to \infty$ such that

$$y_k = y_* + \delta_k p + o_1(\delta_k)$$

where $\|o_1(\delta_k)\|/\delta_k \to 0$ as $k \to \infty$. Since

$$Y_*(x) = \{y_* \in a(x) : \ \sigma(x,y) \ge \sigma(x,y_*), \forall y \in a(x)\},$$

then it follows that for any $y \in a(x_* + \delta_k f)$,

$$\sigma(x_* + \delta_k f, y) \ge \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)). \quad (3.6)$$

Choose any $d \in D^*a(x_*,y_*) \mid (f)$. Then from the definition of $D^*a(x_*,y_*) \mid (f)$, there exists a $\delta_* > 0$ such that for all $\delta \in [0,\delta_*]$,

$$y(\delta) = y_* + \delta d + o_2(\delta) \in a(x_* + \delta f)$$

where $\|o_2(\delta)\|\,\delta^{-1} \to 0$ as $\delta \to +0$. In that case from (3.6) since $\delta_k \to +0$, then it follows that there exists a $k_0 \in \mathbb{N}$ such that $\delta_k \in [0, \delta_*]$ for all $k \geq k_0$ and

$$\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_*, y_*) \\ \geq \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*).$$

Since the function $\sigma(\cdot,\cdot)$ is locally Lipschitz on $\mathbb{R}^m$, then it follows that for $k = 1, 2, \ldots,$ there exist $L_1 > 0$ and $L_2 > 0$ such that

$$|\,\sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k) - \sigma(x_* + \delta_k f, y_* + \delta_k p)\,| \leq L_1 \|o_1(\delta_k)\|,$$
$$|\,\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_* + \delta_k f, y_* + \delta_k d)\,| \leq L_2 \|o_2(\delta_k)\|.$$

Consequently, we have

$$\begin{aligned}
\frac{\partial\sigma(x_*, y_*)}{\partial(f, p)} &= \lim_{\delta\to+0} [\sigma(x_* + \delta f, y_* + \delta p) - \sigma(x_*, y_*)]\,\delta^{-1} \\
&= \lim_{k\to\infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p) - \sigma(x_*, y_*)]\,\delta_k^{-1} \\
&= \lim_{k\to\infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p) - \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) \\
&\quad + \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&\leq \lim_{k\to\infty} [L_1 \|o_1(\delta_k)\| + \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&= \lim_{k\to\infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)]\,\delta_k^{-1} \\
&\leq \lim_{k\to\infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_*, y_*)]\,\delta_k^{-1} \\
&= \lim_{k\to\infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_* + \delta_k f, y_* + \delta_k d) \\
&\quad + \sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&\leq \lim_{k\to\infty} L_2 \|o_2(\delta_k)\|\,\delta_k^{-1} + \lim_{k\to\infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&= \lim_{k\to\infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&= \lim_{\delta\to+0} [\sigma(x_* + \delta f, y_* + \delta d) - \sigma(x_*, y_*)]\,\delta^{-1} \\
&= \frac{\partial\sigma(x_*, y_*)}{\partial(f, d)}.
\end{aligned}$$

Thus, $\frac{\partial\sigma(x_*, y_*)}{\partial(f, p)} \leq \frac{\partial\sigma(x_*, y_*)}{\partial(f, d)}$ for any $d \in D^*a(x_*, y_*) \mid (f)$. It follows from here that

$$\inf_{d\in D^*a(x_*, y_*)|(f)} \frac{\partial\sigma(x_*, y_*)}{\partial(f, d)} \geq \frac{\partial\sigma(x_*, y_*)}{\partial(f, p)}$$

and hence the statement holds. □

Corollary 1. *Let the set-valued map $Y_*(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ be in the form (3.2). Suppose that there exists a $y_* \in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a differentiable function at the point (x_*, y_*). Then*

$$D^* Y_*(x_*, y_*) \mid (f) \subset \left\{ p \in \mathbb{R}^m : \inf_{d \in Da(x_*,y_*)|(f)} \left\langle \frac{\partial \sigma(x_*,y_*)}{\partial y}, d \right\rangle = \left\langle \frac{\partial \sigma(x_*,y_*)}{\partial y}, p \right\rangle \right\},$$

where the symbol $\langle \cdot,\cdot \rangle$ denotes the inner product.

Proof. Since $\sigma(\cdot,\cdot)$ is a differentiable function at the point (x_*, y_*), then it is a derivable function at the point (x_*, y_*) in any direction $(f,d) \in \mathbb{R}^n \times \mathbb{R}^m$ and

$$\frac{\partial \sigma(x_*, y_*)}{\partial (f,d)} = \left\langle \frac{\partial \sigma(x_*, y_*)}{\partial x}, f \right\rangle = \left\langle \frac{\partial \sigma(x_*, y_*)}{\partial y}, d \right\rangle.$$

Then from Theorem 3, we obtain the corollary. □

Corollary 2. *Let the set-valued map $Y_*(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ be in the form (3.2). Suppose that there exists a $y_* \in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a differentiable function at the point (x_*, y_*). Then*

$$DY_*(x_*, y_*) \mid (f) \subset \left\{ p \in \mathbb{R}^m : \inf_{d \in D^*a(x_*,y_*)|(f)} \left\langle \frac{\partial \sigma(x_*,y_*)}{\partial y}, d \right\rangle \geq \left\langle \frac{\partial \sigma(x_*,y_*)}{\partial y}, p \right\rangle \right\},$$

where the symbol $\langle \cdot,\cdot \rangle$ denotes the inner product.

Proof. It is similar to that of Theorem 4. □

Now, we give an example for the above theorems.

Example 1. Take a constant vector $l \in \mathbb{R}^m$. $\sigma(\cdot,\cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is defined by

$$(x, y) \to \sigma(x, y) = \langle l, y \rangle$$

and $a(\cdot)$ is defined by

$$x \to a(x) = \{y \in \mathbb{R}^m : b(x, y) \geq 0\}$$

where $b(\cdot,\cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is continuous differentiable such that $a(x)$ is bounded for all $x \in \mathbb{R}^m$ and $\frac{\partial b(x,y)}{\partial y} \neq 0$ for any $y \in L(x)$ where

$$L(x) = \{y \in \mathbb{R}^m : b(x, y) = 0\}.$$

Then $a(x) : \mathbb{R}^m \to \mathrm{comp}(\mathbb{R}^m)$ and

$$Y_*(x) \subset \partial a(x) \subset L(x)$$

where $\partial a(x)$ is a boundary of $a(x)$. It is shown that

$$Y_*(x) \subset K(x) \tag{3.7}$$

where

$$K(x) = \{y_0 \in a(x) : \langle l, d \rangle \geq 0 \text{ for all } d \in T_{a(x)}(y_0)\}, \tag{3.8}$$

and

$$T_{a(x)}(y_0) = \Big\{ d \in \mathbb{R}^m : \exists \delta_k > 0 \ (\delta_k \to +0 \text{ as } k \to \infty), \quad \exists y_k \in a(x) \ni \ d = \lim_{k\to\infty} \frac{y_k - y_0}{\delta_k} \Big\},$$

where $T_{a(x)}(y_0)$ is an upper contingent cone of $a(x)$ at y_0.

Let $y_* \in Y_*(x)$. Then $y_* \in \partial a(x)$ and $b(x, y_*) = 0$. Since $\frac{\partial b(x,y_*)}{\partial y} \neq 0$, we get

$$T_{a(x)}(y_0) = \left\{ d \in \mathbb{R}^m : \left\langle \frac{\partial b(x, y_*)}{\partial y}, d \right\rangle \geq 0 \right\} \tag{3.9}$$

and from [7, Corollary 2],

$$\begin{aligned} Da(x, y_*) \mid (f) &= D^* a(x, y_*) \mid (f) \\ &= \left\{ d \in \mathbb{R}^m : \left\langle \frac{\partial b(x, y_*)}{\partial x}, f \right\rangle + \left\langle \frac{\partial b(x, y_*)}{\partial y}, d \right\rangle \geq 0 \right\}. \end{aligned} \tag{3.10}$$

Since $y_* \in Y_*(x)$, then it follows from (3.7) that $y_* \in K(x)$ and since $y_* \in K(x)$, then from (3.8), $\langle l, d \rangle \geq 0$ for all $d \in T_{a(x)}(y_*)$. Then from (3.9) we get

$$\langle l, d \rangle \geq 0 \text{ for all } d \in \mathbb{R}^m \text{ such that } \left\langle \frac{\partial b(x, y_*)}{\partial y}, d \right\rangle \geq 0.$$

This yields $l = \alpha(x) \frac{\partial b(x,y_*)}{\partial y}$ with $\alpha(x) > 0$, and follows that

$$Y_*(x) \subset \left\{ y_* \in \partial a(x) : l = \alpha(x) \frac{\partial b(x, y_*)}{\partial y}, \ \alpha(x) > 0 \right\}. \tag{3.11}$$

Therefore, Theorems 3 and 4 and statements (3.10), (3.11) yield

$$\begin{aligned} D^* Y_*(x, y_*) \mid (f) &\subset \left\{ p \in \mathbb{R}^m : \min_{d \in Da(x,y_*)|(f)} \langle l, d \rangle = \langle l, p \rangle \right\} \\ &= \left\{ p \in \mathbb{R}^m : \langle l, p \rangle = -\alpha(x) \left\langle \frac{\partial b(x, y_*)}{\partial x}, f \right\rangle \right\}, \\ DY_*(x, y_*) \mid (f) &\subset \left\{ p \in \mathbb{R}^m : \min_{d \in D^* a(x,y_*)|(f)} \langle l, d \rangle \geq \langle l, p \rangle \right\} \\ &= \left\{ p \in \mathbb{R}^m : \langle l, p \rangle \leq -\alpha(x) \left\langle \frac{\partial b(x, y_*)}{\partial x}, f \right\rangle \right\}. \end{aligned}$$

4. Directional differentiability of max-min function

In [7], the directional lower and upper derivatives of max-min function are investigated by using the directional lower and upper derivative sets of max-min set-valued map.

Now, in this paper, a sufficient condition ensuring the existence of the directional derivative of the max-min function is obtained by using the lower differentiability of max-min set-valued maps. Hence by using the lower different, a characterization of the upper and lower directional derivatives of max-min functions is obtained. For $A \subset \mathbb{R}^n$ and $C \subset \mathbb{R}^n$, we put $\beta(A, C) = \sup_{a \in A} d(a, C)$, $d_H(A, C) = \max\{\beta(A, C), \beta(C, A)\}$. It is known that $(\operatorname{comp}(\mathbb{R}^n), d_H(\cdot, \cdot))$ is a metric space.

Definition 1. The set-valued map $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ is said to be lower differentiable at the point $x \in \mathbb{R}^n$ in direction $f \in \mathbb{R}^n$, if there exist $G^-(x, f) \in \operatorname{comp}(\mathbb{R}^m)$, $G^+(x, f) \in \operatorname{comp}(\mathbb{R}^m)$ such that

$$\lim_{\delta \to +0} \frac{1}{\delta} \beta(a(x) + \delta G^+(x, f), a(x + \delta f) + \delta G^-(x, f)) = 0.$$

In that case the pair $(G^-(x, f), G^+(x, f))$ is said to be lower differential of the set-valued map $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ at the point $x \in \mathbb{R}^n$ in the direction $f \in \mathbb{R}^n$ [9].

In [9], the following two propositions are given.

Proposition 1. *Let the set-valued map $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ be lower differentiable at the point $x \in \mathbb{R}^n$ in direction $f \in \mathbb{R}^n$. Then $Da(x, y) \mid (f) \neq \varnothing$ for every $y \in a(x)$.*

Proposition 2. *Let the set-valued map $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ be lower differentiable at the point $x \in \mathbb{R}^n$ in direction $f \in \mathbb{R}^n$ and the pair $(0, G^+(x, f))$ be its lower differential. Then $D^*a(x, y) \mid (f) \neq \varnothing$ for every $y \in a(x)$.*

In [7], the following two propositions characterizing upper and lower directional derivatives of max-min function $m(\cdot)$ are proved.

Proposition 3. *For all $x \in \mathbb{R}^n$ and $f \in \mathbb{R}^n$*

$$\frac{\partial^- m(x)}{\partial f} \leq \inf_{(y,z) \in Y_*(x)} \inf_{(d,n) \in DY_*(x,y,z)|(f)} \frac{\partial^+ \sigma(x, y, z)}{\partial(f, d, n)},$$

$$\frac{\partial^+ m(x)}{\partial f} \leq \inf_{(y,z) \in Y_*(x)} \inf_{(d,n) \in D^*Y_*(x,y,z)|(f)} \frac{\partial^+ \sigma(x, y, z)}{\partial(f, d, n)}.$$

Proposition 4. *Let $x \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ and there exists $(y_*, z_*) \in Y_*(x)$ such that $DY_*(x, y_*, z_*) \mid (f) \neq \varnothing$. Then*

$$\frac{\partial^+ m(x)}{\partial f} \geq \inf_{(y,z) \in Y_*(x)} \inf_{(d,n) \in DY_*(x,y,z)|(f)} \frac{\partial^- \sigma(x, y, z)}{\partial(f, d, n)}.$$

Moreover if there exists $(y^*, z^*) \in Y_*(x)$ such that $D^*Y_*(x, y^*, z^*) \mid (f) \neq \varnothing$ then

$$\frac{\partial^- m(x)}{\partial f} \geq \inf_{(y,z) \in Y_*(x)} \inf_{(d,n) \in D^*Y_*(x,y,z)|(f)} \frac{\partial^- \sigma(x, y, z)}{\partial(f, d, n)}.$$

Now, by using the lower differentiability of max-min set-valued maps, we give a characterization of the upper and lower directional derivatives of $m(\cdot)$ which follows from Propositions 1 – 4.

Theorem 5. *Suppose that the set-valued map $Y_*(\cdot) : \mathbb{R}^n \to \text{comp}(\mathbb{R}^m \times \mathbb{R}^k)$ is lower differentiable at the point $x \in \mathbb{R}^n$ in the direction $f \in \mathbb{R}^n$. Then*

$$\frac{\partial^+ m(x)}{\partial f} \le \inf_{(y,z)\in Y_*(x)} \inf_{(d,n)\in D^* Y_*(x,y,z)|(f)} \frac{\partial^+ \sigma(x,y,z)}{\partial(f,d,n)},$$

$$\frac{\partial^+ m(x)}{\partial f} \ge \inf_{(y,z)\in Y_*(x)} \inf_{(d,n)\in D Y_*(x,y,z)|(f)} \frac{\partial^- \sigma(x,y,z)}{\partial(f,d,n)}.$$

Theorem 6. *Suppose that the set-valued map $Y_*(\cdot) : \mathbb{R}^n \to \text{comp}(\mathbb{R}^m \times \mathbb{R}^k)$ is lower differentiable at the point $x \in \mathbb{R}^n$ in direction $f \in \mathbb{R}^n$ and $(0, G^+(x, f))$ is its lower differential. Let the function $\sigma(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ be directional differentiable at the point (x, y, z) in direction $(f, d, n) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$ for any $(y, z) \in Y_*(x)$ and $(d, n) \in \mathbb{R}^m \times \mathbb{R}^k$. Then*

$$\frac{\partial m(x)}{\partial f} = \inf_{(y,z)\in Y_*(x)} \inf_{(d,n)\in D Y_*(x,y,z)|(f)} \frac{\partial \sigma(x,y,z)}{\partial(f,d,n)}.$$

5. Conclusions

Necessary statements are given for a minimum point and a maximum point of the max-min function. The estimations for the directional lower and upper derivative sets of max-min set-valued map which are used to state a characterization of the directional derivative of max-min function are given. Moreover, by using the lower differentiability of max-min set-valued maps, sufficient condition ensuring the existence of the directional derivative of the max-min function is obtained.

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