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BOUNDS FOR LAPLACIAN–TYPE GRAPH ENERGIES

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Abstract. Let G be an undirected simple and connected graph with n vertices ($n \geq 3$) and m edges. Denote by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, and $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > \rho_n = 0$, respectively, the Laplacian, signless Laplacian, and normalized Laplacian eigenvalues of G. The Laplacian energy, signless Laplacian energy, and normalized Laplacian energy of G are defined as $LE = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$, $SLE = \sum_{i=1}^{n} \left| \gamma_i - \frac{2m}{n} \right|$, and $NLE = \sum_{i=1}^{n} |\rho_i - 1|$, respectively. Lower bounds for LE, SLE, and NLE are obtained.

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1. Introduction

Let G be an undirected simple and connected graph with n vertices ($n \geq 2$) and m edges, and let $d_1, d_2, \ldots, d_n$ be its vertex degrees.

If the i-th and j-th vertex of the graph G are adjacent, we write $i \sim j$. Then the adjacency matrix $\mathbf{A} = (a_{ij})$ of G is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \quad \text{and} \quad i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of $\mathbf{A}$ form the (ordinary) spectrum of G; for details on the respective spectral theory see [9].

Denote by $\mathbf{D}$ the diagonal matrix of the vertex degrees of G. The Laplacian matrix of G is $\mathbf{L} = \mathbf{D} - \mathbf{A}$ and its eigenvalues are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ (see [3, 16, 25]). In addition, $\mathbf{Q} = \mathbf{D} + \mathbf{A}$ is the signless Laplacian matrix of G and its eigenvalues will be denoted by $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0$ [10, 11].

Because the graph G is assumed to be connected, it has no isolated vertices (i.e., $d_i > 0$ for all $1 \leq i \leq n$) and therefore the matrix $\mathbf{D}^{-1/2}$ is well–defined. Then $\mathbf{L}^* = \mathbf{D}^{-1/2} \mathbf{L} \mathbf{D}^{-1/2}$ is called the normalized Laplacian matrix of the graph G. Its eigenvalues are $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > \rho_n = 0$. For details of the spectral theory of the normalized Laplacian matrix see [8].

It is convenient to write the normalized Laplacian matrix as $\mathbf{I} - \mathbf{R}$, where $\mathbf{R}$ is the so-called Randić matrix [4, 29, 30], whose (i, j)-entry is

$$r_{ij} = \begin{cases} 1/\sqrt{d_i\, d_j} & \text{if } i \neq j \quad \text{and} \quad i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

The (ordinary) energy of the graph G is defined as [23]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| \,. \tag{1.1}$$

Its theory is nowadays well elaborated [23]. Energy–like spectral invariants have been introduced also for other graph matrices [18]. In this paper we are concerned with the Laplacian [21, 23], signless Laplacian [1], and normalized Laplacian (or Randić) energies [5, 20], defined as

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

$$SLE = SLE(G) = \sum_{i=1}^{n} \left| \gamma_i - \frac{2m}{n} \right|$$

$$NLE = NLE(G) = \sum_{i=1}^{n} |\rho_i - 1|$$

respectively. In what follows lower bounds for LE, SLE and NLE are obtained.

Remark 1. In analogy to (1.1), the "Randić energy" is defined as the sum of the absolute values of the eigenvalues of the Randić matrix. It has been shown in [20], that the Randić energy coincides with the normalized signless Laplacian energy.

Remark 2. One could also consider the normalized signless Laplacian matrix, $\mathbf{D}^{-1/2}\mathbf{Q}\mathbf{D}^{-1/2}$ and its "energy" (sum of absolute values of eigenvalues). However, the energy of this matrix is exactly the same as the normalized Laplacian energy, NLE [20]. For the general definition of the energy of a matrix see [28].

The Laplacian, signless Laplacian, and normalized (or Randić) Laplacian spreads of a graph G are defined as $LS(G) = \mu_1 - \mu_{n-1}$, $SLS(G) = \gamma_1 - \gamma_n$, and $NLS(G) = \rho_1 - \rho_{n-1}$, respectively (see [5, 13, 15, 24]).

2. Preliminaries

In this section we recall some results from spectral graph theory, and state a few analytical inequalities needed for our work.

Lemma 1 ([3]). *Let G be an undirected simple and connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m$$

where M_1 is the sum of squares of the vertex degrees, usually referred to as the first Zagreb index (see [2, 7, 19]*).*

Lemma 2 ([12]). *Let G be an undirected simple and connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$\frac{M_1}{m} \geq 2\sqrt{\frac{M_1}{n}} \geq \frac{4m}{n} \,. \tag{2.1}$$

Lemma 3 ([31]). *Let G an (n,m)-graph, such that $n \geq 3$ and $m \geq 1$. Then*

$$LE(G) \geq \mu_1 - \mu_{n-1} + \frac{2m}{n} \tag{2.2}$$

with equality if and only if $n = 3$ or for $n \geq 4$ if $\mu_2 = \cdots = \mu_{n-2} = \frac{2m}{n}$.

Lemma 4 ([26]). *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$LS(G) = \mu_1 - \mu_{n-1} \leq \sqrt{\frac{2}{n-1}}\,\sqrt{(n-1)(M_1+2m) - 4m^2} \,. \tag{2.3}$$

Equality holds if and only if $G \cong K_n$.

Lemma 5 ([27]). *Let $a_1, a_2, \ldots, a_n$ be real numbers and $p_1, p_2, \ldots, p_n$ non-negative real numbers with the property $p_1 + p_2 + \cdots + p_n = 1$. Then, for each α, $\alpha \leq 0$ and $\alpha \geq 1$,*

$$\sum_{i=1}^{n} p_i\, a_i^{\alpha} \geq \left(\sum_{i=1}^{n} p_i\, a_i \right)^{\alpha} . \tag{2.4}$$

For the case $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality in (2.4) holds if and only if $\alpha = 0$ or $\alpha = 1$ or $a_1 = a_2 = \cdots = a_n$.

Lemma 6 ([6]). *Let $a_1, a_2, \ldots, a_n$ be real numbers, and assume that there are $r, R \in \mathbb{R}$ such that $-\infty < r \leq a_i \leq R < +\infty$, for each $i = 1, 2, \ldots, n$. Then for any non-negative $p_1, p_2, \ldots, p_n$ with the property $p_1 + p_2 + \cdots + p_n = 1$,*

$$0 \leq \sum_{i=1}^{n} p_i\, a_i^2 - \left(\sum_{i=1}^{n} p_i\, a_i \right)^2 \leq \frac{1}{2}(R-r) \sum_{i=1}^{n} p_i \left| a_i - \sum_{i=1}^{n} p_i\, a_i \right| . \tag{2.5}$$

The constant $\frac{1}{2}$ is sharp.

Lemma 7 ([32]). *Let G be an undirected simple and connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$\sum_{i=1}^{n-1} \rho_i = n \qquad \text{and} \qquad \sum_{i=1}^{n} \rho_i^2 = n + 2R_{-1} \tag{2.6}$$

where $R_{-1} = \sum_{i \sim j} \frac{1}{d_i d_j}$; for details on the graph invariant R_{-1} see [4, 22].

Lemma 8 ([17]). *Let G be an undirected simple and connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n} \gamma_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m$$

where M_1 is the first Zagreb index.

Lemma 9 ([17]). *The signless Laplacian spread has an upper bound*

$$SLS(G) \leq \sqrt{\frac{2[n(M_1 + 2m) - 4m^2]}{n}}\,.$$

Lemma 10 ([14]). *Suppose that G is a graph without isolated vertices. Then*

$$\mu_1 - \mu_{n-1} \geq \frac{2}{n-1}\sqrt{(n-1)(2m + M_1) - 4m^2}\,. \tag{2.7}$$

3. Main results

3.1. *Lower bound for Laplacian energy*

Theorem 1. *Let G be an undirected connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$LE(G) \geq \frac{2m}{n} + \frac{2}{n-1}\sqrt{(n-1)(2m + M_1) - 4m^2}\,. \tag{3.1}$$

Proof. Inequality (3.1) directly follows from inequalities (2.2) and (2.7). □

Corollary 1. *Let G be an undirected graph with n, $n \geq 3$, vertices and m edges. Then*

$$LE(G) \geq \frac{2m}{n} + \frac{2}{n-1}\sqrt{\frac{2m(n(n-1) - 2m)}{n}}\,.$$

Corollary 2. *Let G be an undirected simple and connected k-regular graph with n, $n \geq 3$, vertices and m edges, $1 < k \leq n-1$. Then*

$$LE(G) \geq k + \frac{2}{n-1}\sqrt{nk(n-k-1)}\,.$$

Theorem 2. *Let G be an undirected simple and connected graph with n, $n \geq 3$ vertices and m edges. Then*

$$LE(G) \geq \sqrt{\frac{2}{n-1}}\sqrt{(n-1)(M_1+2m)-4m^2}\,. \tag{3.2}$$

Proof. For $n-1$ and $p_i := \frac{1}{n-1}$, $a_i := \mu_i$, $i = 1,2,\ldots n-1$, $r := \mu_{n-1}$ and $R := \mu_1$, the inequality (2.5) transforms into

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\mu_i^2 - \frac{1}{(n-1)^2}\left(\sum_{i=1}^{n-1}\mu_i\right)^2 \leq \frac{\mu_1-\mu_{n-1}}{2(n-1)}\sum_{i=1}^{n-1}\left|\mu_i - \frac{1}{n-1}\sum_{i=1}^{n-1}\mu_i\right|$$

i.e., based on Lemma 1,

$$(n-1)(M_1+2m)-4m^2 \leq \frac{n-1}{2}(\mu_1-\mu_{n-1})\sum_{i=1}^{n-1}\left|\mu_i - \frac{2m}{n-1}\right|.$$

Since

$$\sum_{i=1}^{n-1}\left|\mu_i - \frac{2m}{n-1}\right| \leq \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| = LE(G)$$

using inequality (2.3), from the above inequality we obtain (3.2). □

Using Lemma 2, we arrive at the following (n,m)-type lower bound for the Laplacian energy:

Corollary 3. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$LE(G) \geq \sqrt{\frac{4m(n(n-1)-2m)}{n(n-1)}}\,. \tag{3.3}$$

Corollary 4. *Let G be an undirected simple and connected k-regular graph with n, $n \geq 3$, vertices and m edges,* $1 < k \leq n-1$. *Then*

$$LE(G) > \sqrt{\frac{2nk(n-k-1)}{n-1}}\,.$$

Remark 3. Since for undirected k-regular graphs, $LE = E$, the inequality in Corollary 4 provides a lower bound also for the ordinary energy.

Inequalities (3.1) and (3.2) are incomparable. Thus, for example, if $G \cong K_n$, then inequality (3.1) is stronger than (3.2), but if $G \cong K_{1,n-1}$, $n \geq 8$, then the opposite is valid.

3.2. *Lower bound for signless Laplacian energy*

Theorem 3. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$SLE(G) \geq \sqrt{\frac{2(n(M_1+2m)-4m^2)}{n}} \,. \tag{3.4}$$

Proof. For $p_i := \frac{1}{n}$, $a_i = \gamma_i$, $i = 1,2,\ldots,n$, $r = \gamma_n$ and $R = \gamma_1$, the inequality (2.5) becomes

$$\frac{1}{n}\sum_{i=1}^{n}\gamma_i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n}\gamma_i\right)^2 \leq \frac{\gamma_1-\gamma_n}{2n}\sum_{i=1}^{n}\left|\gamma_i - \frac{2m}{n}\right|.$$

Bearing in mind Lemma 8, the above inequality becomes

$$n(M_1+2m)-4m^2 \leq \frac{n}{2}SLS(G)\times SLE(G)\,.$$

By Lemma 9 and the above inequality, we obtain (3.4). □

Bearing in mind Lemma 2 and inequality (3.4), we arrive at a lower bound for $SLE(G)$ depending only on the parameter m.

Corollary 5. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$SLE(G) \geq 2\sqrt{m}\,.$$

Corollary 6. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges, which is k-regular, $1 < k \leq n$. Then*

$$SLE(G) \geq \sqrt{2nk}\,.$$

3.3. *Lower bound for normalized Laplacian energy*

Theorem 4. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Let, as before, $R_{-1} = \sum_{i\sim j}\frac{1}{d_i\,d_j}$. Then*

$$NLS(G) \leq \sqrt{\frac{2}{n-1}}\sqrt{2(n-1)R_{-1}-n}\,. \tag{3.5}$$

Equality holds if and only if $G \cong K_n$.

Proof. According to (2.6) we have that

$$\begin{aligned}(n-1)(n+2R_{-1})-n^2 &= (n-1)\sum_{i=1}^{n-1}\rho_i^2 - \left(\sum_{i=1}^{n-1}\rho_i\right)^2 \\ &= \sum_{1\leq i<j\leq n-1}(\rho_i-\rho_j)^2\,. \end{aligned} \tag{3.6}$$

By Lemma 5, i.e., by inequality (2.4), for $n = 2$ and $\alpha = 2$, we get

$$(\rho_1 - \rho_i)^2 + (\rho_i - \rho_{n-1})^2 \geq \frac{1}{2}(\rho_1 - \rho_{n-1})^2 \tag{3.7}$$

for each $i = 2, 3, \dots, n-2$. Then,

$$\begin{aligned}\sum_{1 \leq i < j \leq n-1} (\rho_i - \rho_j)^2 &\geq \sum_{i=2}^{n-2} [(\rho_1 - \rho_i)^2 + (\rho_i - \rho_{n-1})^2] + (\rho_1 - \rho_{n-1})^2 \\ &\geq \frac{n-3}{2}(\rho_1 - \rho_{n-1})^2 + (\rho_1 - \rho_{n-1})^2 \\ &= \frac{n-1}{2}(\rho_1 - \rho_{n-1})^2\end{aligned}$$

which combined with (3.6) yields

$$(n-1)(n + 2R_{-1}) - n^2 = 2(n-1)R_{-1} - n \geq \frac{n-1}{2}(\rho_1 - \rho_{n-1})^2$$

from which the inequality (3.5) follows.

Equality in (3.7) holds if and only if $\rho_1 = \rho_2 = \cdots = \rho_{n-1}$. Therefore, equality in (3.5) holds if and only if $G \cong K_n$. This completes the proof of Theorem 4. □

Corollary 7. *Let G be an undirected simple nad connected k-regular graph, $1 < k \leq n-1$, with n, $n \geq 3$, vertices and m edges. Then*

$$NLS(G) \leq \sqrt{\frac{2n(n-k-1)}{(n-1)k}}\,.$$

Equality holds if and only if $k = n-1$, i.e., $G \cong K_n$.

We now state a theorem, analogous to Theorem 2, which provides a lower bound for NLE in terms of parameters n and R_{-1}.

Theorem 5. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$NLE(G) \geq \sqrt{\frac{2}{n-1}}\sqrt{2(n-1)R_{-1} - n}\,. \tag{3.8}$$

Proof. For $n := n-1$, $p_i := \frac{1}{n-1}$, $a_i := \rho_i$, $i = 1, 2, \dots, n-1$, $r = \rho_{n-1}$ and $R = \rho_1$, inequality (2.5) becomes

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\rho_i^2 - \frac{1}{(n-1)^2}\left(\sum_{i=1}^{n-1}\rho_i\right)^2 \leq \frac{\rho_1 - \rho_{n-1}}{2(n-1)}\sum_{i=1}^{n-1}\left|\rho_i - \frac{1}{n-1}\sum_{i=1}^{n-1}\rho_i\right|.$$

Having in mind Lemma 7, the above inequality transforms into

$$(n-1)(n + 2R_{-1}) - n^2 \leq \frac{n-1}{2} NLS(G)\sum_{i=1}^{n-1}\left|\rho_i - \frac{n}{n-1}\right|\,. \tag{3.9}$$

Since

$$\sum_{i=1}^{n-1}\left|\rho_i-\frac{n}{n-1}\right|\leq\sum_{i=1}^{n}|\rho_i-1|$$

according to (3.9) we obtain

$$(n-1)(n+2R_{-1})-n^2\leq\frac{n-1}{2}NLS(G)NLE(G)\,. \tag{3.10}$$

Combining (3.5) and (3.10) we arrive at (3.8). □

Remark 4. For a k-regular graph, $R_{-1}=m/k^2=n/(2k)$. Since for k-regular graphs, $NLE=\frac{1}{k}E=\frac{1}{k}LE$, inequality (3.8) is equivalent to the result proven in Corollary 4.

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