Remarks on quasi-metric spaces

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Abstract. In this paper, we prove some properties of quasi-metric spaces and state some fixed point theorems in this setting. As applications, we show that most of recent results on $G$-metric spaces in [3, 10] may be also implied from certain fixed point theorems on metric spaces and quasi-metric spaces.

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1. INTRODUCTION AND PRELIMINARIES

In recent time, many generalized metric spaces were introduced and the fixed point theory in these spaces was investigated. In [13], Mustafa and Sims introduced the concept of a $G$-metric space as a generalized metric space. After that, many fixed point theorems on $G$-metric spaces were stated, see [1,4,5,9,12,14,15] and references therein. But in [8], Jleli and Samet showed that most of the obtained fixed point theorems on $G$-metric spaces may be deduced immediately from fixed point theorems on metric spaces or quasi-metric spaces. The similar results can be found in [2, 17].

Very recently, Karapinar and Agarwal modified some existing results to suggest new fixed point theorems that fit with the nature of a $G$-metric space in [10]. Also, they asserted that for their results the techniques used in [8] and [17] are inapplicable. After that, this idea was continuously developed in [3, 7].

In this paper, we prove some properties of quasi-metric spaces and state some fixed point theorems in this setting. As applications, we show that most of recent results on $G$-metric spaces in [3, 10] may be also implied from certain fixed point theorems on metric spaces and quasi-metric spaces.

First, we recall notions and results which will be useful in what follows.

Definition 1 ([13], Definition 3). Let $X$ be a nonempty set and $G : X \times X \times X \rightarrow [0, \infty)$ be a function such that, for all $x, y, z \in X$,

1. $G(x, y, z) = 0$ if $x = y = z$.
2. $0 < G(x, x, y)$ if $x \neq y \in X$.
3. $G(x, x, y) \leq G(x, y, z)$ if $y \neq z$.

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(4) \(G(x, y, z) = G(x, z, y) = G(y, x, z) = G(y, z, x) = G(z, x, y) = G(z, y, x)\).
(5) \(G(x, y, z) \leq G(x, a, a) + G(a, y, z)\).

Then \(G\) is called a \(G\)-metric on \(X\) and the pair \((X, G)\) is called a \(G\)-metric space.

**Definition 2** ([13], Definition 4). The \(G\)-metric space \((X, G)\) is called symmetric if \(G(x, y, y) = G(x, x, y)\) for all \(x, y \in X\).

**Definition 3** ([13]). Let \((X, G)\) be a \(G\)-metric space and \(\{x_n\}\) be a sequence in \(X\).
(1) For each \(x_0 \in X\) and \(r > 0\), the set
\[B_G(x_0, r) = \{x \in X : G(x_0, x, x) < r\}\]
is called a \(G\)-ball with center \(x_0\) and radius \(r\).
(2) The family of all \(G\)-balls forms a base of a topology \(\tau(G)\) on \(X\), and \(\tau(G)\) is called the \(G\)-metric topology.
(3) \(\{x_n\}\) is called convergent to \(x\) in \(X\) if \(\lim_{n \to \infty} x_n = x\) in the \(G\)-metric topology \(\tau(G)\).
(4) \(\{x_n\}\) is called Cauchy in \(X\) if \(\lim_{n,m,l \to \infty} G(x_n, x_m, x_l) = 0\).
(5) \((X, G)\) is called a complete \(G\)-metric space if every Cauchy sequence is convergent.

**Lemma 1** ([13], Proposition 6). Let \((X, G)\) be a \(G\)-metric space. Then the following statements are equivalent.
(1) \(x_n\) is convergent to \(x\) in \(X\).
(2) \(\lim_{n \to \infty} G(x_n, x_n, x) = 0\).
(3) \(\lim_{n \to \infty} G(x_n, x, x) = 0\).
(4) \(\lim_{n, m \to \infty} G(x_n, x_m, x) = 0\).

**Lemma 2** ([13], Proposition 9). Let \((X, G)\) be a \(G\)-metric space. Then the following statements are equivalent.
(1) \(\{x_n\}\) is a Cauchy sequence.
(2) \(\lim_{n,m,l \to \infty} G(x_n, x_m, x_l) = 0\).

**Definition 4** ([8], Definition 2.1). Let \(X\) be a nonempty set and \(d : X \times X \to [0, +\infty)\) be a function such that, for all \(x, y, z \in X\),
(1) \(d(x, y) = 0\) if and only if \(x = y\).
(2) \(d(x, y) \leq d(x, z) + d(z, y)\).

Then \(d\) is called a quasi-metric and the pair \((X, d)\) is called a quasi-metric space.
Remark 1 ([8]). (1) Every metric is a quasi-metric.

(2) In a quasi-metric space, a sequence \( \{x_n\} \) is Cauchy if and only if it is left-Cauchy and right-Cauchy.

The following examples show that the inversion of Remark 1.(1) does not hold.

Example 1. Let \( X = \mathbb{R} \) and \( d \) be defined by

\[
d(x, y) = \begin{cases} 
x - y & \text{if } x \geq y \\
1 & \text{if } x < y.
\end{cases}
\]

Then \( d \) is a quasi-metric on \( X \) but \( d \) is not a metric on \( X \).

Proof. It is clear that \( d : X \times X \to [0, +\infty) \) and \( d(x, y) = 0 \) if and only if \( x = y \).

For all \( x, y, z \in X \), we consider two following cases.

Case 1. \( x \geq y \). We have \( d(x, y) = x - y \).

If \( z < y \), then \( d(x, z) = x - z \) and \( d(z, y) = 1 \).

If \( y \leq z < x \), then \( d(x, z) = x - z \) and \( d(z, y) = z - y \).

If \( x \leq z \), then \( d(x, z) = 1 \) and \( d(z, y) = z - y \).

So we have \( d(x, y) \leq d(x, z) + d(z, y) \).

Case 2. \( x < y \). We have \( d(x, y) = 1 \).

If \( z < x \), then \( d(x, z) = x - z \) and \( d(z, y) = 1 \).

If \( x \leq z < y \), then \( d(x, z) = 1 \) and \( d(z, y) = 1 \).

If \( y \leq z \), then \( d(x, z) = 1 \) and \( d(z, y) = z - y \).

So we have \( d(x, y) \leq d(x, z) + d(z, y) \).

By the above, \( d \) is a quasi-metric on \( \mathbb{R} \). Since \( d(0, 2) = 1 \neq d(2, 0) = 2 \), \( d \) is not a metric on \( \mathbb{R} \). \( \square \)

Example 2. Let \( X = X_1 \cup X_2 \), \( X_1 \cap X_2 \neq \emptyset \) and \( d \) be defined by

\[
d(x, y) = \begin{cases} 
0 & \text{if } x = y \\
2 & \text{if } x \in X_1, y \in X_2 \\
1 & \text{otherwise}.
\end{cases}
\]
Then $d$ is a quasi-metric on $X$ but $d$ is not a metric on $X$.

For more quasi-metrics which are not metrics, see [16, Example 1.4].

In [8], Jleli and Samet showed that most of the obtained fixed point theorems on $G$-metric spaces may be deduced immediately from fixed point theorems on metric spaces or quasi-metric spaces. The main results in [8] are as follows.

**Theorem 1** (8, Theorem 2.2). Let $(X, G)$ be a $G$-metric space and $d_G : X \times X \to [0, +\infty)$ be defined by $d_G(x, y) = G(x, y, y)$ for all $x, y \in X$. Then we have

1. $(X, d_G)$ is a quasi-metric space.
2. A sequence $\{x_n\}$ is convergent to $x$ in $(X, G)$ if and only if $\{x_n\}$ is convergent to $x$ in $(X, d_G)$.
3. A sequence $\{x_n\}$ is Cauchy in $(X, G)$ if and only if $\{x_n\}$ is Cauchy in $(X, d_G)$.
4. The $G$-metric space $(X, G)$ is complete if and only if the quasi-metric space $(X, d_G)$ is complete.

**Theorem 2** (8, Theorem 2.3). Let $(X, G)$ be a $G$-metric space and $\delta_G : X \times X \to [0, +\infty)$ be defined by $\delta_G(x, y) = \max \{G(x, y, y), G(y, x, x)\}$ for all $x, y \in X$. Then we have

1. $(X, \delta_G)$ is a metric space.
2. A sequence $\{x_n\}$ is convergent to $x$ in $(X, G)$ if and only if $\{x_n\}$ is convergent to $x$ in $(X, \delta_G)$.
3. A sequence $\{x_n\}$ is Cauchy in $(X, G)$ if and only if $\{x_n\}$ is Cauchy in $(X, \delta_G)$.
4. The $G$-metric space $(X, G)$ is complete if and only if the metric space $(X, \delta_G)$ is complete.

**Theorem 3** (8, Theorem 3.2). Let $(X, d)$ be a complete quasi-metric space and $T : X \to X$ be a map such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is continuous with $\varphi^{-1}(\{0\}) = \{0\}$. Then $T$ has a unique fixed point.

Recently, in [16], Rajić proved the following result which is a generalization of Theorem 3.

**Theorem 4** ([16], Theorem 2.2). Let $(X, d)$ be a complete quasi-metric space and $f, g : X \to X$ be two maps such that

$$\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \phi(d(gx, gy)) \tag{1.2}$$

for all $x, y \in X$, where $\psi : [0, +\infty) \to [0, +\infty)$ is continuous, non-decreasing, $\psi^{-1}(0) = \{0\}$, $\phi : [0, +\infty) \to [0, +\infty)$ is continuous and $\phi^{-1}(0) = \{0\}$. If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly...
compatible, that is, \(fgx = gf x\) provided that \(fx = gx\), then \(f\) and \(g\) have a unique common fixed point.

The main results of the paper are presented in Section 2 and Section 3. In Section 2, we prove some properties of the quasi-metric space and its modification. Then, by similar arguments as in metric spaces, we prove some analogues of fixed point theorems in quasi-metric spaces. In Section 3, we show that most of recent fixed point theorems on \(G\)-metric spaces in \([3, 10]\) may be implied from certain fixed point theorems proved in Section 2.

2. REMARKS ON QUASI-METRIC SPACES

Note that every quasi-metric space \((X, d)\) is a topological space with the topology induced by its convergence. Then \(X \times X\) is a topological space with the product topology. The following result shows that the product space \(X \times X\) is also a quasi-metric space.

**Proposition 1.** Let \((X, d_X)\) and \((Y, d_Y)\) be two quasi-metric spaces. Then we have

1. \(d(x, y) = d_X(x_1, y_1) + d_Y(x_2, y_2)\) for all \(x = (x_1, x_2), y = (y_1, y_2) \in X \times Y\) is a quasi-metric on \(X \times Y\).
2. \(\lim_{n \to \infty} (x_n, y_n) = (x, y)\) in \((X \times Y, d)\) if and only if \(\lim_{n \to \infty} x_n = x\) in \((X, d_X)\) and \(\lim_{n \to \infty} y_n = y\) in \((Y, d_Y)\). In particular, the product topology on \(X \times Y\) coincides the topology induced by \(d\).
3. \(\{ (x_n, y_n) \}\) is a Cauchy sequence in \((X \times Y, d)\) if and only if \(\{ x_n \}\) is a Cauchy sequence in \((X, d_X)\) and \(\{ y_n \}\) is a Cauchy sequence in \((Y, d_Y)\).
4. \((X \times Y, d)\) is complete if and only if \((X, d_X)\) and \((Y, d_Y)\) are complete.

**Proof.** (1). For all \(x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in X \times Y\), we have

\[ d(x, z) = d_X(x_1, z_1) + d_Y(x_2, z_2) \leq d_X(x_1, y_1) + d_X(y_1, z_1) + d_Y(x_2, y_2) + d_Y(y_2, z_2) = d_X(x_1, y_1) + d_Y(x_2, y_2) + d_X(y_1, z_1) + d_Y(y_2, z_2) = d(x, y) + d(y, z). \]

By the above, \(d\) is a quasi-metric on \(X \times Y\).

(2). \(\lim_{n \to \infty} (x_n, y_n) = (x, y)\) in \((X \times Y, d)\) if and only if

\[ \lim_{n \to \infty} d((x_n, y_n), (x, y)) = \lim_{n \to \infty} [d_X(x_n, x) + d_Y(y_n, y)] = 0 \]

and

\[ \lim_{n \to \infty} d((x, y), (x_n, y_n)) = \lim_{n \to \infty} [d_X(x, x_n) + d_Y(y, y_n)] = 0. \]
It is equivalent to
\[
\lim_{n \to \infty} d_X(x_n, x) = \lim_{n \to \infty} d_Y(y_n, y) = \lim_{n \to \infty} d_X(x, x_n) = \lim_{n \to \infty} d_Y(y, y_n) = 0.
\]
That is, \( \lim_{n \to \infty} x_n = x \) in \((X, d_X)\) and \( \lim_{n \to \infty} y_n = y \) in \((Y, d_Y)\).

(3). \( \{(x_n, y_n)\} \) is a Cauchy sequence in \((X \times Y, d)\) if and only if
\[
\lim_{n,m \to \infty} d((x_n, y_n), (x_m, y_m)) = \lim_{n,m \to \infty} [d_X(x_n, x_m) + d_Y(y_n, y_m)] = 0.
\]
It is equivalent to
\[
\lim_{n,m \to \infty} d_X(x_n, x_m) = \lim_{n,m \to \infty} d_Y(y_n, y_m) = 0.
\]
That is, \( \{x_n\} \) is a Cauchy sequence in \((X, d_X)\) and \( \{y_n\} \) is a Cauchy sequence in \((Y, d_Y)\).

(4). It is a direct consequence of (2) and (3).

In the proof of [8, Theorem 3.2], Jleli and Samet used the sequential continuity of a quasi-metric \( d \) without proving. From Proposition 1, we see that the sequential continuity and the continuity of \( d \) are equivalent and they are guaranteed by the following proposition.

Proposition 2. Let \((X, d)\) be a quasi-metric space. Then \( d \) is a continuous function.

Proof. Suppose that \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \) in \((X, d)\). We have
\[
d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n).
\]
It implies that
\[
d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y, y_n).
\]
(2.1)
Also, we have
\[
d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y, y_n).
\]
It implies that
\[
d(x, y) - d(x_n, y_n) \leq d(x, x_n) + d(y_n, y).
\]
(2.2)
From (2.1) and (2.2), we have
\[
0 \leq |d(x, y) - d(x_n, y_n)| \leq \max \{d(x_n, x) + d(y, y_n), d(x, x_n) + d(y, y_n)\}. \tag{2.3}
\]
Taking the limit as \( n \to \infty \) in (2.3), we obtain \( \lim_{n \to \infty} |d(x, y) - d(x_n, y_n)| = 0. \) That is, \( \lim_{n \to \infty} d(x_n, y_n) = d(x, y). \) This proves that \( d \) is a continuous function.

The following proposition proves that the topology of each quasi-metric space is metrizable. Then all topological properties of metric spaces hold on quasi-metric spaces.
Remark 3. Let \((X,d)\) be a quasi-metric space and 
\[
\delta_d(x, y) = \max\{d(x, y), d(y, x)\}
\]
for all \(x, y \in X\). Then we have

(1) \((X, \delta_d)\) is a metric space.
(2) A sequence \(\{x_n\}\) is convergent to \(x\) in \((X, d)\) if and only if \(\{x_n\}\) is convergent to \(x\) in \((X, \delta_d)\).
(3) A sequence \(\{x_n\}\) is Cauchy in \((X, d)\) if and only if \(\{x_n\}\) is Cauchy in \((X, \delta_d)\).
(4) The quasi-metric space \((X, d)\) is complete if and only if the metric space \((X, \delta_d)\) is complete.

Proof. (1). See [8], page 3.
(2). We have 
\[
\lim_{n \to \infty} x_n = x \quad \text{in} \quad (X, d) \quad \text{if and only if} \quad \lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0.
\]
It is equivalent to 
\[
\lim_{n \to \infty} \delta_d(x_n, x) = \lim_{n \to \infty} \max\{d(x_n, x), d(x, x_n)\} = 0.
\]
That is, 
\[
\lim_{n \to \infty} x_n = x \quad \text{in} \quad (X, \delta_d).
\]
(3). A sequence \(\{x_n\}\) is Cauchy in \((X, d)\) if and only if 
\[
\lim_{n, m \to \infty} d(x_n, x_m) = \lim_{n, m \to \infty} d(x_m, x_n) = 0.
\]
It is equivalent to 
\[
\lim_{n, m \to \infty} \delta_d(x_n, x_m) = \lim_{n, m \to \infty} \max\{d(x_n, x_m), d(x_m, x_n)\} = 0.
\]
That is, \(\{x_n\}\) is Cauchy in \((X, \delta_d)\).
(4). It is a direct consequence of (2) and (3). \(\square\)

By modifying the notion of \(T\)-orbital completeness in [6], we introduce the notion of weak \(T\)-orbitally completeness as follows.

Definition 6. Let \((X, d)\) be a quasi-metric space and \(T : X \to X\) be a map. Then \(X\) is called weak \(T\)-orbitally complete if \(\{T^n x\}\) is convergent in \(X\) provided that it is a Cauchy sequence in \(X\).

Note that every \(T\)-orbitally complete quasi-metric space is a weak \(T\)-orbitally complete quasi-metric space for all maps \(T : X \to X\). The following example shows that the inversion does not hold, even when \((X, d)\) is a metric space.

Example 3. Let \(X = \{1, 3, \ldots, 2n + 1, \ldots\} \cup \left\{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2n}, \ldots\right\}\) with the usual metric and 
\[
T \frac{1}{2n} = 2n + 1, T(2n - 1) = \frac{1}{2n}
\]
for all \( n \in \mathbb{N} \). Since \( \{T^n x\} \) is not Cauchy for all \( x \in X \), \((X,d)\) is weak \(T\)-orbitally complete. For \( x = 1 \), we have
\[
\{T^n 1 : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{4}, \ldots, 2n + 1, \frac{1}{2n}, \ldots\}.
\]
Since \( \{\frac{1}{2n}\} \) is a Cauchy sequence in \((X,d)\) which is not convergent, \((X,d)\) is not \(T\)-orbitally complete.

Following the proof of [11, Theorem 3.1], we get the following fixed point theorem on quasi-metric spaces.

**Theorem 5.** Let \((X,d)\) be a quasi-metric space and \(T : X \rightarrow X\) be a map such that

1. \(X\) is weak \(T\)-orbitally complete.
2. There exists \(q \in [0,1)\) such that for all \(x, y \in X\),
   \[
d(Tx,Ty) \leq q \max \{d(x,y), d(y,x), d(x,Tx), d(Tx,x), d(y,Ty)\},
   \]
   \[
d(Tx,Ty), d(y,Tx), d(Tx,y), d(Tx,y), d(T^2 x,x), d(T^2 x,Tx),
   \]
   \[
d(Tx,T^2 x), d(T^2 y,y), d(y,T^2 y), d(T^2 x,Ty)\},
   \]
Then we have

1. \(T\) has a unique fixed point \(x^*\) in \(X\).
2. \(\lim_{n \to \infty} T^n x = x^*\) for all \(x \in X\).
3. \(\max \{d(T^n x^*,d(x^*,T^n x)\} \leq q^n \max \{d(x,Tx), d(Tx,x)\} \) for all \(x \in X\) and \(n \in \mathbb{N}\).

**Proof.** (1). For each \(x \in X\) and \(1 \leq i \leq n-1, 1 \leq j \leq n\), we have
\[
d(T^i x, T^{j-1} x) = d(T T^{i-1} x, T T^{j-1} x)
\]
\[
\leq q \max \{d(T^{i-1} x, T^{j-1} x), d(T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T T^{i-1} x),
   d(T T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T T^{j-1} x), d(T^{i-1} x, T T^{j-1} x),
   d(T^{i-1} x, T^{j-1} x), d(T T^{i-1} x, T T^{i-1} x), d(T^{i-1} x, T T^{i-1} x),
   d(T T^{i-1} x, T T^{i-1} x), d(T T^{i-1} x, T T^{i-1} x), d(T T^{i-1} x, T T^{i-1} x),
   d(T^{i-1} x, T T^{i-1} x), d(T T^{i-1} x, T T^{i-1} x), d(T T^{i-1} x, T T^{i-1} x),
   d(T T^{i-1} x, T T^{i-1} x), d(T T^{i-1} x, T T^{i-1} x)\}
\]
\[
= q \max \{d(T^{i-1} x, T^{j-1} x), d(T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T^{i-1} x),
   d(T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T^{i-1} x),
   d(T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T^{i-1} x),
   d(T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T^{i-1} x),
   d(T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T^{i-1} x), d(T^{i-1} x, T^{i-1} x)\}
\]
\[ \leq q \delta [O_T(x, n)] \]

where \( \delta [O_T(x, n)] = \max \{d(T^i x, T^j x) : 0 \leq i \leq n - 1, 0 \leq j \leq n\} \).

From (2.5), since \( 0 \leq q < 1 \), there exists \( k_n(x) \leq n \) such that
\[ d(x, T^{k_n(x)} x) = \delta [O_T(x, n)] \tag{2.6} \]
or there exists \( k_n(x) \leq n - 1 \) such that
\[ d(T^{k_n(x)} x, x) = \delta [O_T(x, n)]. \tag{2.7} \]

If (2.6) holds, we have
\[
d(x, T^{k_n(x)} x) \leq d(x, T x) + d(T x, T^{k_n(x)} x)
\leq d(x, T x) + q \delta [O_T(x, n)]
= d(x, T x) + q d(x, T^{k_n(x)} x).
\]

It implies that
\[ \delta [O_T(x, n)] = d(x, T^{k_n(x)} x) \leq \frac{1}{1 - q} d(x, T x). \tag{2.8} \]

If (2.7) holds, we have
\[
d(T^{k_n(x)} x, x) \leq d(T^{k_n(x)} x, T x) + d(T x, x)
\leq q \delta [O_T(x, n)] + d(T x, x)
= q d(T^{k_n(x)} x, x) + d(T x, x).
\]

It implies that
\[ \delta [O_T(x, n)] = d(T^{k_n(x)} x, x) \leq \frac{1}{1 - q} d(T x, x). \tag{2.9} \]

For all \( n < m \), it follows from (2.4) and (2.8), (2.9) that
\[ d(T^n x, T^m x) = d(T T^{n-1} x, T^{m-n+1} T^{n-1} x) \]
\[ \leq q \delta [O_T(T^{n-1} x, m - n + 1)]
= q d(T^{n-1} x, T^{k_m-n+1} x) T^{n-1} x)
= q d(T T^{n-2} x, T^{k_m-n+1} x) x + T^{n-2} x)
\]
\[ \leq q^2 \delta [O_T(T^{n-2} x, k_m-n+1 (T^{n-1} x) + 1)]
\leq q^2 \delta [O_T(T^{n-2} x, m - n + 2)]
\leq \ldots
\]
\[ \leq q^n \delta [O_T(x, m)]
\leq \frac{q^n}{1 - q} \max \{d(x, T x), d(T x, x)\}. \]
Since \( \lim_{n \to \infty} q^n = 0 \), by taking the limit as \( n, m \to \infty \) in (2.10), we have
\[
\lim_{n,m \to \infty} d(T^n x, T^m x) = 0. \tag{2.11}
\]
This proves that \( \{T^n x\} \) is a Cauchy sequence in \( X \). Since \( X \) is weak \( T \)-orbitally complete, there exists \( x^* \in X \) such that
\[
\lim_{n \to \infty} d(T^n x, x^*) = \lim_{n \to \infty} d(x^*, T^n x) = 0. \tag{2.12}
\]
Therefore, by using (2.4) again, we have
\[
d(x^*, T x^*) \leq d(x^*, T^n x) + d(T^n x, T x^*)
\leq d(x^*, T^n x) + q \max \{d(T^n x, x^*), d(x^*, T x^*), d(T^n x, T T^n x), d(T T^n x, T^n x), d(x^*, T x^*), d(T^n x, T T^n x), d(T T^n x, T^n x), d(x^*, T x^*), d(T^n x, T^2 T^n x), d(T^2 T^n x, T^n x), d(T^n x, T T^n x), d(T^2 T^n x, T^n x), d(T^n x, T^2 T^n x), d(T^2 T^n x, T^n x), d(x^*, T x^*), d(T^2 T^n x, T x^*)\}
= d(x^*, T^n x) + q \max \{d(T^n x, x^*), d(x^*, T x^*), d(T^n x, T T^n x), d(T T^n x, T^n x), d(x^*, T x^*), d(T^n x, T^2 T^n x), d(T^2 T^n x, T^n x), d(T^n x, T T^n x), d(T^2 T^n x, T^n x), d(x^*, T x^*), d(T^2 T^n x, T x^*)\}.
\[
\text{Taking the limit as } n \to \infty \text{ in (2.13), and using (2.11), (2.12) and Proposition 2, we get } d(x^*, T x^*) \leq q d(x^*, T x^*). \text{ Since } q \in [0, 1], d(x^*, T x^*) = 0, \text{ that is, } x^* = T x^*. \text{ Then } T \text{ has a fixed point.}
\]
Now, we prove the uniqueness of the fixed point of \( T \). Let \( x^*, y^* \) be two fixed points of \( T \). From (2.4), we have
\[
d(x^*, y^*) = d(T x^*, T y^*)
\leq q \max \{d(x^*, y^*), d(x^*, x^*), d(T x^*, x^*), d(T y^*, y^*), d(x^*, T y^*), d(x^*, y^*), d(y^*, T x^*), d(y^*, T^2 x^*), d(T x^*, x^*), d(T^2 x^*, x^*), d(T x^*, T x^*), d(T x^*, T^2 x^*), d(T y^*, y^*), d(y^*, T x^*), d(y^*, T^2 x^*), d(T y^*, y^*)\}
= q \max \{d(x^*, y^*), d(y^*, x^*)\}.
\[
\text{Since } q \in [0, 1], \text{ we get }
\]
\[
d(x^*, y^*) \leq q d(y^*, x^*). \tag{2.14}
\]
Again, from (2.4), we also have
\[
d(y^*, x^*) = d(T y^*, T x^*)
\leq q \max \{d(y^*, x^*), d(x^*, y^*), d(y^*, T y^*), d(T y^*, y^*), d(x^*, T x^*), d(y^*, T x^*)\},
\]
\[ d(x^*, Ty^*), d(x^*, Ty^*), d(T^2 y^*, y^*), d(y^*, T^2 y^*), d(T y^*, T^2 y^*), d(T^2 y^*, x^*), d(x^*, T^2 y^*), d(T y^*, T x^*) \]
\[ = q \max \{d(y^*, x^*), d(x^*, y^*)\}. \]

Since \( q \in [0, 1) \), we get
\[ d(y^*, x^*) \leq q d(x^*, y^*). \] (2.15)

From (2.14) and (2.15), since \( q \in [0, 1) \), we obtain \( d(x^*, y^*) = 0 \). That is, \( x^* = y^* \). Then the fixed point of \( T \) is unique.

(2). It is proved by (2.12).

(3). Taking the limit as \( m \to \infty \) in (2.10) and using Proposition 2, we get
\[ d(T^n x, x^*) \leq q^n \max \{d(x, T x), d(T x, x)\}. \]

Similarly, we have \( d(x^*, T^n x) \leq q^n \max \{d(x, T x), d(T x, x)\} \). Therefore,
\[ \max \{d(T^n x, x^*), d(x^*, T^n x)\} \leq \frac{q^n}{1-q} \max \{d(x, T x), d(T x, x)\}. \]

If \( d \) in Theorem 5 is a metric, then we have the following result. Note that Corollary 1 is a generalization of the following well-known result of Ćirić in [6]. This generalization is proper by [11, Example 3.6].

**Corollary 1** ([11], Theorem 3.1). Let \( (X, d) \) be a metric space and \( T : X \to X \) be a map satisfying the following

1. \( X \) is weak \( T \)-orbitally complete.
2. There exists \( q \in [0, 1) \) such that for all \( x, y \in X \),
\[ d(T x, T y) \leq q \max \{d(x, y), d(x, T x), d(y, T y), d(x, T y)\}, \] (2.16)
\[ d(y, T x), d(T^2 x, x), d(T^2 x, T x), d(T^2 x, y), d(T^2 x, T y) \} \].

Then we have

1. \( T \) has a unique fixed point \( x^* \) in \( X \).
2. \( \lim_{n \to \infty} T^n x = x^* \) for all \( x \in X \).
3. \( d(T^n x, x^*) \leq \frac{q^n}{1-q} d(x, T x) \) for all \( x \in X \).

Now, we modify the notion of a quasi-metric space in Definition 4 as follows.

**Definition 7.** Let \( X \) be a nonempty set and \( d : X \times X \to [0, +\infty) \) be a function such that, for all \( x, y, z \in X \),

1. \( d(x, y) = 0 \) if and only if \( x = y \).
2. \( d(x, y) \leq d(x, z) + d(y, z) \) if \( z \neq x \).
Then $d$ is also called a *quasi-metric* and the pair $(X, d)$ is also called a *quasi-metric space*.

Note that if $\varepsilon = x$ in Definition 7.2, then $d(x, y) = d(y, x)$ for all $x, y \in X$. In this case, $d$ deduces a metric on $X$.

The following proposition gives a way to set examples of quasi-metrics in the sense of Definition 7.

**Proposition 4.** Let $(X, G)$ be a $G$-metric space and $T : X \to X$ be a map. For all $x, y \in X$, put

$$d_{T, G}(x, y) = \begin{cases} 0 & \text{if } x = y \\ G(x, Tx, y) & \text{if } x \neq y. \end{cases}$$

If $T$ has no any fixed point, then $d_{T, G}$ is a quasi-metric in the sense of Definition 7 on $X$.

**Proof.** For all $x, y, z \in X$ with $z \neq x$, note that $y \neq Ty$ for all $y \in X$, we have $G(y, z, z) \leq G(z, y, Ty)$ for all $y, z \in X$. Then

$$d_{T, G}(x, y) \leq G(x, Tx, y)$$

$$= G(y, x, Tx)$$

$$\leq G(y, z, z) + G(x, Tx, z)$$

$$\leq G(z, y, Ty) + G(x, Tx, z)$$

$$= d_{T, G}(x, z) + d_{T, G}(y, z).$$

This proves that $d_{T, G}$ is a quasi-metric in the sense of Definition 7 on $X$. □

**Example 4.** Let $X = \{1, 2, 3\}$ and

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } (x, y) = (1, 2) \\ 1 & \text{otherwise.} \end{cases}$$

Then $d$ is a quasi-metric in the sense of Definition 7 on $X$. For $(2, 1), (1, 2), (1, 1) \in X \times X$, we have

$$d(2, 1) + d(1, 2) = 3$$

$$d(2, 1) + d(1, 1) = 1$$

$$d(1, 1) + d(2, 1) = 1.$$ 

This proves that $d(2, 1) + d(1, 2) > d(2, 1) + d(1, 1) + d(1, 1) + d(2, 1)$. So Proposition 1.1 does not hold for quasi-metrics in the sense of Definition 7.

**Proposition 5.** Let $(X, d)$ be a quasi-metric space in the sense of Definition 7. For each $y \in X$, if $\lim_{n \to \infty} x_n = x$, then $\lim_{n \to \infty} d(y, x_n) = d(y, x)$. 


Proof. Case 1. \(y = x\). Then we have
\[
d(y, x) = d(x, x) = 0 = \lim_{n \to \infty} d(x, x_n) = \lim_{n \to \infty} d(y, x_n).
\]

Case 2. \(y \neq x\). If \(x_n = x\) for infinitely many \(n\), then
\[
\lim_{m \to \infty} d(y, x_n) = d(y, x).
\]
So, we may assume that \(x_n \neq x\) for \(n\) large enough. Also, \(y \neq x_n\) for \(n\) large enough. Then we have, for all \(n \in \mathbb{N}\),
\[
d(y, x) \leq d(y, x_n) + d(x, x_n) \leq d(y, x) + d(x_n, x) + d(x, x_n).
\] (2.17)
Taking the limit as \(n \to \infty\) in (2.17), we get \(\lim_{n \to \infty} d(y, x_n) = d(y, x)\).

With some minor changes in the proof of Theorem 5, we have the following result.
Note that these changes mainly relate to Definition 7.(2).

Proposition 6. Let \((X, d)\) be a quasi-metric space in the sense of Definition 7 and \(T : X \to X\) be a map satisfying the following

1. \(X\) is weak \(T\)-orbitally complete.
2. There exists \(q \in [0, 1)\) such that for all \(x, y \in X\),
\[
d(Tx, Ty) \leq q \max\{d(x, y), d(y, x), d(Tx, x), d(Ty, y), d(Ty, x), d(Tx, y), d(Tx, Ty), d(Ty, Ty), d(Ty, T^2y), d(Tx, T^2y), d(T^2y, x), d(x, T^2y), d(Tx, T^2y)\}. \tag{2.18}
\]
Then we have

1. \(T\) has a unique fixed point \(x^*\) in \(X\).
2. \(\lim_{n \to \infty} T^n x = x^*\) for all \(x \in X\).
3. \(\max\{d(T^n x, x^*), d(x^*, T^n x)\} \leq q^n \max\{d(x, Tx), d(Tx, x)\}\) for all \(x \in X\) and \(n \in \mathbb{N}\).

Proof. As in the proof of Theorem 5.(1), there exists \(k_n(x) \leq n\) such that
\[
d(x, T^{k_n(x)} x) = \delta[O_T(x, n)] \tag{2.19}
\]
or there exists \(k_n(x) \leq n - 1\) such that
\[
d(T^{k_n(x)} x, x) = \delta[O_T(x, n)]. \tag{2.20}
\]
If \(Tx = x\), then \(T\) has a fixed point. So we may assume that \(Tx \neq x\). If (2.19) holds, we have
\[
d(x, T^{k_n(x)} x) \leq d(x, Tx) + d(T^{k_n(x)} x, Tx) \\
\leq d(x, Tx) + q\delta[O_T(x, n)]
\]
\[ d(x, Tx) = d(x, T x) + q d(x, T^{k_n}(x)x). \]

It implies that
\[ \delta[O_T(x, n)] = d(x, T^{k_n}(x)x) \leq \frac{1}{1-q} d(x, Tx). \quad (2.21) \]

If (2.20) holds and \( T^{k_n}(x)x = Tx \), we have
\[ \delta[O_T(x, n)] = d(T^{k_n}(x)x, x) = d(Tx, x) \leq \frac{1}{1-q} d(Tx, x). \quad (2.22) \]

So, we may assume that \( T^{k_n}(x)x \neq Tx \). Then
\[ d(T^{k_n}(x)x, x) \leq d(T^{k_n}(x)x, Tx) + d(x, Tx) \]
\[ \leq q \delta[O_T(x, n)] + d(x, Tx) \]
\[ = q d(T^{k_n}(x)x, x) + d(x, Tx). \]

It implies that
\[ \delta[O_T(x, n)] = d(T^{k_n}(x)x, x) \leq \frac{1}{1-q} d(x, Tx). \quad (2.23) \]

As in the proof of Theorem 5.1, we also have
\[ \lim_{n,m \to \infty} d(T^n x, T^m x) = 0 \quad (2.24) \]
and there exists \( x^* \in X \) such that
\[ \lim_{n \to \infty} d(T^n x, x^*) = \lim_{n \to \infty} d(x^*, T^n x) = 0. \quad (2.25) \]

If \( T^{n+1} x = Tx^* \) for infinitely many \( n \), then \( \lim_{n \to \infty} T^{n+1} x = Tx^* = x^* \). Then \( x^* \) is a fixed point of \( T \). So, we may assume that \( T^{n+1} x \neq x^* \) for \( n \) large enough.

Therefore, by using (2.18) again, we have
\[ d(Tx^*, x^*) \leq d(Tx^*, T^{n+1} x) + d(x^*, T^{n+1} x) \]
\[ = d(x^*, T^{n+1} x) + d(Tx^*, T^{n+1} x) \]
\[ \leq d(x^*, T^{n+1} x) + q \max \{d(T^n x, x^*), d(x^*, T^n x), d(T^n x, TT^n x), \]
\[ d(TT^n x, T^n x), d(Tx^*, x^*), d(Tx^*, T^n x), d(x^*, TT^n x), d(TT^n x, x^*), \]
\[ d(T^2 T^n x, T^n x), d(T^n x, T^2 T^n x), d(T^2 T^n x, TT^n x), d(TT^n x, T^2 T^n x), \]
\[ d(T^2 T^n x, x^*), d(x^*, T^2 T^n x), d(Tx^*, T^2 T^n x) \} \]
\[ = d(x^*, T^{n+1} x) + q \max \{d(T^n x, x^*), d(x^*, T^n x), d(T^n x, T^{n+1} x), \]
\[ d(T^{n+1} x, T^n x), d(Tx^*, x^*), d(Tx^*, T^n x), d(x^*, T^{n+1} x), d(T^{n+1} x, x^*), \]
\[ d(T^{n+2} x, T^n x), d(T^n x, T^{n+2} x), d(T^{n+2} x, T^{n+1} x), d(T^{n+1} x, T^{n+2} x), \]
Let \( (X, d) \) be a quasi-metric space in the sense of Definition 7 and \( T : X \rightarrow X \) be a map such that \( (X, d) \) is weak \( T \)-orbitally complete and
\[
d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))
\] (2.27)
for all \( x, y \in X \), where \( \varphi : [0, +\infty) \rightarrow [0, +\infty) \) is continuous with \( \varphi^{-1}(\{0\}) = \{0\} \).

Then \( T \) has a unique fixed point.

**Proof.** Let \( x_0 \in X \) and define the sequence \( x_{n+1} = Tx_n \) for all \( n \geq 0 \). From (2.27), we have
\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq d(x_{n-1}, x_n) - \varphi(d(x_{n-1}, x_n))
\] (2.28)
for all \( n \geq 1 \). This proves that \( \{d(x_n, x_{n+1})\} \) is a non-increasing sequence of positive numbers. Then there exists \( r \geq 0 \) such that \( \lim_{n \to \infty} d(x_n, x_{n+1}) = r \). Taking the limit as \( n \to \infty \) in (2.28), we get \( \varphi(r) = 0 \), that is, \( r = 0 \). Then
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
\] (2.29)

Using the same technique, we also have
\[
\lim_{n \to \infty} d(x_{n+1}, x_n) = 0.
\] (2.30)

Now, we will prove that \( \{x_n\} \) is a Cauchy sequence in the quasi-metric space \( (X, d) \), that is, \( \{x_n\} \) is left-Cauchy and right-Cauchy. By (2.29) and (2.30), since the sequences \( \{d(x_{n+1}, x_n)\} \) and \( \{d(x_n, x_{n+1})\} \) are non-increasing, we have \( \{x_n\} \) is a Cauchy sequence if there exists \( n \) such that \( d(x_{n+1}, x_n) = 0 \) or \( d(x_n, x_{n+1}) = 0 \). Then, we may assume that, for all \( n \in \mathbb{N} \),
\[
d(x_{n+1}, x_n) \neq 0 \text{ and } d(x_n, x_{n+1}) \neq 0.
\] (2.31)

Now, suppose to the contrary that \( \{x_n\} \) is not a left-Cauchy sequence. Then there exists \( \varepsilon > 0 \) such that for each \( k \in \mathbb{N} \), there exist \( n > m \geq k \) satisfying \( d(x_n, x_m) \geq \varepsilon \).

Put
\[
n(1) = \min \{n : n > 1 \text{ and there exists } m \text{ with } 1 \leq m < n, d(x_n, x_m) \geq \varepsilon\}
\]
\[
m(1) = \max \{m : 1 \leq m < n(1) \text{ with } d(x_{n(1)}, x_m) \geq \varepsilon\}
\]
\[
n(2) = \min \{n : n > n(1), \text{ there exists } m \text{ with } n(1) \leq m < n, d(x_n, x_m) \geq \varepsilon\}
\]
\[
m(2) = \max \{m : n(1) \leq m < n(2) \text{ with } d(x_{n(2)}, x_m) \geq \varepsilon\}.
\]
Note that \( n(1) < n(2), m(1) < m(2) \) and
\[
d(x_{n(1)} - 1, x_{m(1)}) < \varepsilon, d(x_{n(2)} - 1, x_{m(2)}) < \varepsilon.
\]
Continuing this process, we can find two subsequences \( \{x_{n(k)}\} \) and \( \{x_{m(k)}\} \) of \( \{x_n\} \) such that, for all \( k \in \mathbb{N} \), we have \( n(k) > m(k) > k \) and
\[
d(x_{n(k)}, x_{m(k)}) \geq \varepsilon, d(x_{n(k) - 1}, x_{m(k) - 1}) < \varepsilon. \tag{2.32}
\]
Now, by (2.31) and (2.32), we have
\[
d(x_{n(k) - 1}, x_{m(k) - 1}) \leq d(x_{n(k) - 1}, x_{n(k)}) + d(x_{m(k) - 1}, x_{n(k)})
\]
and
\[
d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k) - 1}, x_{n(k)}) + d(x_{m(k) - 1}, x_{m(k)}) \tag{2.35}
\]
and
\[
d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k) - 1}, x_{n(k)}) + d(x_{m(k) - 1}, x_{m(k)}) \tag{2.36}
\]
Taking the limit as \( k \to \infty \) in (2.35) and using (2.29), (2.30), we get
\[
\lim_{k \to \infty} d(x_{n(k) - 1}, x_{m(k) - 1}) = \varepsilon. \tag{2.37}
\]
Now, from (2.27), for all \( k \in \mathbb{N} \), we have
\[
d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k) - 1}, x_{m(k) - 1}) - \varphi(d(x_{n(k) - 1}, x_{m(k) - 1})). \tag{2.38}
\]
Taking the limit as \( k \to \infty \) in (2.38) and using (2.34), (2.37), we obtain \( \varepsilon \leq \varepsilon - \varphi(\varepsilon) \).
It implies that \( \varepsilon = 0 \). It is a contradiction. Then \( \{x_n\} \) is a left-Cauchy sequence. Similarly, we can show that \( \{x_n\} \) is a right-Cauchy sequence. Then \( \{x_n\} \) is Cauchy.
Since \((X, d)\) is weak \( T\)-orbitally complete, there exists \( x^* \in X \) such that
\[
\lim_{n \to \infty} d(x_n, x^*) = \lim_{n \to \infty} d(x^*, x_n) = 0. \tag{2.39}
\]
From (2.27), for all \( n \in \mathbb{N} \), we have
\[
d(Tx^*, x_{n+1}) = d(Tx^*, Tx_n) \leq d(x^*, x_n) - \varphi(d(x^*, x_n)) \tag{2.40}
\]
Taking the limit as $n \to \infty$ in (2.40) and using (2.39), Proposition 5, we get $d(Tx^*,x^*) = 0$. It implies that $x^* = Tx^*$, that is, $x^*$ is a fixed point of $T$.

The uniqueness of the fixed point is easy to see.

Similar as the proof of [16, Theorem 2.2] and the proof of Proposition 7, we get the following result.

**Proposition 8.** Let $(X,d)$ be a weak $T$-orbitally complete quasi-metric space in the sense of Definition 7 and let $T : X \to X$ be a map such that

$$
\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y))
$$

(2.41)

where $\psi, \varphi : [0, +\infty) \to [0, +\infty)$, $\psi$ is continuous and non-decreasing, $\varphi$ is lower semi-continuous, and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then $T$ has a unique fixed point.

3. APPLICATIONS TO RECENT FIXED POINT RESULTS IN $G$-METRIC SPACES

In this section, we show that most of recent results on $G$-metric spaces in [3, 10] may be also implied from certain fixed point theorems in metric spaces and quasi-metric spaces mentioned in Section 2. Notice that the authors of [10] forgot the assumption of completeness in [10, Theorems 3.1 & 3.2].

**Corollary 2 ([10], Theorem 3.1).** Let $(X,G)$ be a complete $G$-metric space and $T : X \to X$ be a map such that

$$
G(Tx,Ty,Tz) \leq kM(x,y,z)
$$

(3.1)

for all $x, y, z \in X$, where $k \in \left[0, \frac{1}{2}\right]$ and

$$
M(x,y,z) = \max \left\{ G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty),
G(y,Tx,y), G(x,Tx,z), G(z,T^2x,Tz),
G(Tx,T^2x,Tz), G(z,Tx,Ty), G(x,y,z),
G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz),
G(z,Tx,Tx), G(x,Ty,Ty), G(y,Tz,Tz),
G(y,Tx,Tx), G(x,Ty,Ty), G(y,Ty,Ty) \right\}
$$

Then $T$ has a unique fixed point.

**Proof.** Let $d_G$ be the quasi-metric in Theorem 1. By choosing $z = y$ and using the axioms (G4) and (G5) in Definition 1, we have

$$
M(x,y,y) = \max \left\{ G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty),
G(y,Tx,Ty), G(x,Tx,y), G(y,T^2x,Ty),
G(Tx,T^2x,Ty), G(y,Tx,Ty), G(x,y,y),
G(x,Tx,Tx), G(y,Ty,Ty), G(y,Ty,Ty),
G(y,Tx,Tx), G(x,Ty,Ty), G(y,Ty,Ty) \right\}
$$
Then (3.1) becomes
\[
\begin{align*}
&= \max \left\{ G(x, T x, y), G(y, T y, T^2 x), G(T x, y, T^2 x),
\right. \\
&\quad \quad \quad \quad G(y, T y, T x),
\left. \quad G(x, y, y), G(y, y, T y), G(y, T y, T y)
\right\}
\end{align*}
\]
\[
\leq \max \left\{ G(x, T x, T x) + G(T x, T x, y), G(y, T y, T y) + G(T y, T y, T^2 x),
\right. \\
&\quad \quad \quad \quad G(y, T y, T x),
\left. \quad G(x, T x, T x) + G(T x, T x, y),
\right\}
\end{align*}
\]
\[
\begin{align*}
&= \max \left\{ d_G(x, T x) + d_G(y, T x), d_G(y, T y) + d_G(T^2 x, T y),
\right. \\
&\quad \quad \quad \quad d_G(T x, T y) + d_G(T x, T y, T^2 x),
\left. \quad d_G(x, T x) + d_G(y, T x),
\right\}
\end{align*}
\]
\[
\leq 2 \max \left\{ d_G(x, T x), d_G(y, T x), d_G(T^2 x, T y), d_G(T x, T y),
\right. \\
&\quad \quad \quad \quad d_G(x, y),
\left. \quad d_G(x, T x),
\right\}
\end{align*}
\]

Then (3.1) becomes
\[
\begin{align*}
d_G(T x, T y) \leq 2k \max \left\{ d_G(x, T x), d_G(y, T x), d_G(y, T y), d_G(T^2 x, T y),
\right. \\
&\quad \quad d_G(T x, T y), d_G(x, y),
\left. \quad d_G(x, T y)
\right\}
\end{align*}
\]

Since 0 ≤ 2k < 1, we have
\[
\begin{align*}
d_G(T x, T y) \leq 2k \max \left\{ d_G(x, y), d_G(x, T x), d_G(y, T y), d_G(x, T y),
\right. \\
&\quad \quad d_G(y, T x),
\left. \quad d_G(T^2 x, T y)
\right\}
\end{align*}
\]

By Theorem 5, we see that T has a unique fixed point. □

Remark 2. The authors of [10] claimed that the proof of [10, Theorem 3.2] is the mimic of [10, Theorem 3.1]. But, by redoing the proof of [10, Theorem 3.1], we see that the equality (23) in the proof of [10, Theorem 3.1] becomes
\[
G(x^*, T x^*, T x^*) \leq k G(x^*, T x^*, T x^*)
\]
and the equality (25) in the proof of [10, Theorem 3.1] becomes
\[
G(t^*, t^*, x^*) \leq k G(t^*, t^*, x^*)
\]
In general, the second inequalities do not hold if k ∈ [0, 1).
Corollary 3 ([10], Theorem 3.3). Let \((X, G)\) be a complete \(G\)-metric space and \(T : X \to X\) be a map such that
\[
\psi \left( G(Tx, T^2x, Ty) \right) \leq G(x, Tx, y) - \varphi(G(x, Tx, y))
\] (3.2)
for all \(x, y \in X\), where \(\varphi : [0, +\infty) \to [0, +\infty)\) is continuous with \(\varphi^{-1}(\{0\}) = 0\). Then \(T\) has a unique fixed point.

Proof. It is easy to see that \(T\) has at most one fixed point. Suppose to the contrary that \(T\) has no any fixed point. Let \(d_{T,G}\) be defined as in Proposition 4. Then, \(d_{T,G}\) is a quasi-metric in the sense of Definition 7 on \(X\). We prove that \((X, d_{T,G})\) is a weak \(T\)-orbitally complete quasi-metric space. Let \(\{x_n\}\) be a Cauchy sequence in \((X, d_{T,G})\) where \(x_0 \in X\) and \(x_{n+1} = Tx_n\) for all \(n \in \mathbb{N}\). We have
\[
\lim_{n,m \to \infty} d_{T,G}(x_n, x_m) = 0.
\]
We may assume that \(x_n \neq x_m\) for all \(n \neq m \in \mathbb{N}\). Then
\[
0 \leq \lim_{n,m \to \infty} G(x_n, x_m, x_m)
\]
\[
\leq \lim_{n,m \to \infty} G(x_n, x_{n+1}, x_m)
\]
\[
= \lim_{n,m \to \infty} G(x_n, Tx_n, x_m)
\]
\[
= \lim_{n,m \to \infty} d_{T,G}(x_n, x_m)
\]
\[
= 0.
\]
It implies that \(\lim_{n,m \to \infty} G(x_n, x_m, x_m) = 0\). By Lemma 2, \(\{x_n\}\) is a Cauchy sequence in \((X, G)\). Since \((X, G)\) is complete, there exists \(x^* \in X\) such that \(\lim_{n \to \infty} x_n = x^*\) in \((X, G)\). Since \(x_n \neq x_m\) for all \(n \neq m \in \mathbb{N}\), we may assume that \(x_n \neq x^*\) for all \(n \in \mathbb{N}\). Therefore,
\[
\lim_{n \to \infty} d_{T,G}(x_n, x^*) = \lim_{n \to \infty} G(x_n, Tx_n, x^*) = \lim_{n \to \infty} G(x_n, x_{n+1}, x^*) = 0. \quad (3.3)
\]
We also have
\[
d_{T,G}(x^*, x_n) = G(x^*, Tx^*, x_n)
\]
\[
\leq G(x^*, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tx^*, x_n)
\]
\[
= G(x^*, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+1}, Tx^*)
\]
\[
= G(x^*, x_{n+1}, x_{n+1}) + G(Tx_{n-1}, T^2x_{n-1}, Tx^*)
\]
\[
\leq G(x^*, x_{n+1}, x_{n+1}) + G(x_{n-1}, Tx_{n-1}, x^*) - \varphi(G(x_{n-1}, Tx_{n-1}, x^*))
\]
\[
\leq G(x^*, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_n, x^*) - \varphi(G(x_{n-1}, x_n, x^*)).
\]
Taking the limit as \(n \to \infty\) in (3.4) and using Lemma 1, we obtain
\[
\lim_{n \to \infty} d_{T,G}(x^*, x_n) = 0. \quad (3.5)
\]
From (3.3) and (3.5), we get \( \lim_{n \to \infty} x_n = x^* \) in \((X, d_{T,G})\). Then \((X, d_{T,G})\) is weak \(T\)-orbitally complete. Note that (3.2) becomes
\[
\psi(d_{T,G}(Tx,Ty)) \leq d_{T,G}(x,y) - \varphi(d_{T,G}(x,y))
\]
Therefore, by using Proposition 7, we conclude that \(T\) has a fixed point. It is a contradiction.

By the above, \(T\) has a unique fixed point. \(\square\)

**Corollary 4** ([3], Theorem 2.3). Let \((X,G)\) be a complete \(G\)-metric space and \(T : X \to X\) be a map such that
\[
\psi(G(Tx,T^2x,Ty)) \leq \psi(G(x,Tx,y)) - \varphi(G(x,Tx,y))
\]
for all \(x,y \in X\), where \(\psi : [0, +\infty) \to [0, +\infty)\) is non-decreasing and continuous, and \(\varphi : [0, +\infty) \to [0, +\infty)\) is lower semi-continuous and \(\varphi(t) = \varphi(t) = 0\) if and only if \(t = 0\). Then \(T\) has a unique fixed point.

**Proof.** It is easy to see that \(T\) has at most one fixed point. Suppose to the contrary that \(T\) has no any fixed point. Using \(d_{T,G}\) as in the proof of Corollary 3, then \(d_{T,G}\) is a quasi-metric in the sense of Definition 7 on \(X\). We prove that \((X,d_{T,G})\) is a weak \(T\)-orbitally complete quasi-metric space. Let \(\{x_n\}\) be a Cauchy sequence in \((X,d_{T,G})\) where \(x_0 \in X\) and \(x_{n+1} = Tx_n\) for all \(n \in \mathbb{N}\). We have \(\lim_{n,m \to \infty} d_{T,G}(x_n,x_m) = 0\).

We may assume that \(x_n \neq x_m\) for all \(n \neq m \in \mathbb{N}\). Then
\[
0 \leq \lim_{n,m \to \infty} G(x_n,x_m,x_m) \\
\leq \lim_{n,m \to \infty} G(x_n,x_{n+1},x_m) \\
= \lim_{n,m \to \infty} G(x_n,Tx_n,x_m) \\
= \lim_{n,m \to \infty} d_{T,G}(x_n,x_m) \\
= 0.
\]
It implies that \(\lim_{n,m \to \infty} G(x_n,x_m,x_m) = 0\). By Lemma 2, \(\{x_n\}\) is a Cauchy sequence in \((X,G)\). Since \((X,G)\) is complete, there exists \(x^* \in X\) such that \(\lim_{n \to \infty} x_n = x^*\) in \((X,G)\). Since \(x_n \neq x_m\) for all \(n \neq m \in \mathbb{N}\), we may assume that \(x_n \neq x^*\) for all \(n \in \mathbb{N}\). Therefore,
\[
\lim_{n \to \infty} d_{T,G}(x_n,x^*) = \lim_{n \to \infty} G(x_n,Tx_n,x^*) = \lim_{n \to \infty} G(x_n,x_{n+1},x^*) = 0. \tag{3.7}
\]
We also have
\[
d_{T,G}(x^*,x_n) = G(x^*,Tx^*,x_n) \leq G(x^*,x_{n+1},x_{n+1}) + G(x_{n+1},Tx^*,x_n) \\
= G(x^*,x_{n+1},x_{n+1}) + G(x_n,x_{n+1},Tx^*)
\]

We have
Taking the limit as $n \to \infty$ in (3.8) and using Lemma 1, we get
\[
\lim_{n \to \infty} d_{T,G}(x^*, x_n) = 0.
\] (3.9)
From (3.7) and (3.9), we get $\lim_{n \to \infty} x_n = x^*$ in $(X, d_{T,G})$. Then $(X, d_{T,G})$ is weak $T$-orbitally complete. Note that (3.6) becomes
\[
\psi(d_{T,G}(T x, T y)) \leq \psi(d_{T,G}(x, y) - d_{T,G}(x, y)).
\]
Therefore, by using Proposition 8, we conclude that $T$ has a fixed point. It is a contradiction.

By the above, $T$ has a unique fixed point. □

Remark 3. By using $d_{T,G}$ as in the proof of Corollary 3, we see that the inequality (30) in [3, Theorem 3.1] becomes
\[
d_{T,G}(T x, T y) \geq \alpha d_{T,G}(x, y).
\]
By similar arguments, we get analogues of the results in [18] for expansive maps on quasi-metric spaces and then we get [3, Theorem 3.1]. Also, similar arguments to the above may be possible for results in [7]. Note that for a complete $G$-metric space $(X, G)$ with $|X| \geq 2$ and $T : X \to X$ being the identify map, all assumptions of [3, Theorem 3.2] hold but $T$ has more than one fixed point. This shows that the uniqueness of fixed points in [3, Theorem 3.2] is a gap.

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