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REMARKS ON QUASI-METRIC SPACES

N. V. DUNG

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Abstract. In this paper, we prove some properties of quasi-metric spaces and state some fixed point theorems in this setting. As applications, we show that most of recent results on G-metric spaces in [3, 10] may be also implied from certain fixed point theorems on metric spaces and quasi-metric spaces.

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1. Introduction and preliminaries

In recent time, many generalized metric spaces were introduced and the fixed point theory in these spaces was investigated. In [13], Mustafa and Sims introduced the concept of a G-metric space as a generalized metric space. After that, many fixed point theorems on G-metric spaces were stated, see [1,4,5,9,12,14,15] and references therein. But in [8], Jleli and Samet showed that most of the obtained fixed point theorems on G-metric spaces may be deduced immediately from fixed point theorems on metric spaces or quasi-metric spaces. The similar results can be found in [2, 17].

Very recently, Karapinar and Agarwal modified some existing results to suggest new fixed point theorems that fit with the nature of a G-metric space in [10]. Also, they asserted that for their results the techniques used in [8] and [17] are inapplicable. After that, this idea was continuously developed in [3, 7].

In this paper, we prove some properties of quasi-metric spaces and state some fixed point theorems in this setting. As applications, we show that most of recent results on G-metric spaces in [3, 10] may be also implied from certain fixed point theorems on metric spaces and quasi-metric spaces.

First, we recall notions and results which will be useful in what follows.

Definition 1 ([13], Definition 3)**.** Let X be a nonempty set and $G : X \times X \times X \longrightarrow [0,\infty)$ be a function such that, for all $x, y, z \in X$,

(1) $G(x,y,z) = 0$ if $x = y = z$.
(2) $0 < G(x,x,y)$ if $x \neq y \in X$.
(3) $G(x,x,y) \leq G(x,y,z)$ if $y \neq z$.

(4) $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(y,z,x) = G(z,x,y) = G(z,y,x)$.
(5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$.

Then G is called a *G-metric* on X and the pair (X,G) is called a *G-metric space*.

Definition 2 ([13], Definition 4)**.** The G-metric space (X,G) is called *symmetric* if $G(x,y,y) = G(x,x,y)$ for all $x,y \in X$.

Definition 3 ([13])**.** Let (X,G) be a G-metric space and $\{x_n\}$ be a sequence in X.

(1) For each $x_0 \in X$ and $r > 0$, the set

$$B_G(x_0,r) = \{x \in X : G(x_0,x,x) < r\}$$

is called a *G-ball with center x_0 and radius r*.

(2) The family of all G-balls forms a base of a topology $\tau(G)$ on X, and $\tau(G)$ is called the *G-metric topology*.

(3) $\{x_n\}$ is called *convergent* to x in X if $\lim\limits_{n\to\infty} x_n = x$ in the G-metric topology $\tau(G)$.

(4) $\{x_n\}$ is called *Cauchy* in X if $\lim\limits_{n,m,l\to\infty} G(x_n,x_m,x_l) = 0$.

(5) (X,G) is called a *complete G-metric* space if every Cauchy sequence is convergent.

Lemma 1 ([13], Proposition 6)**.** *Let (X,G) be a G-metric space. Then the following statements are equivalent.*

(1) *x_n is convergent to x in X.*
(2) $\lim\limits_{n\to\infty} G(x_n,x_n,x) = 0$.
(3) $\lim\limits_{n\to\infty} G(x_n,x,x) = 0$.
(4) $\lim\limits_{n,m\to\infty} G(x_n,x_m,x) = 0$.

Lemma 2 ([13], Proposition 9)**.** *Let (X,G) be a G-metric space. Then the following statements are equivalent.*

(1) *$\{x_n\}$ is a Cauchy sequence.*
(2) $\lim\limits_{n,m\to\infty} G(x_n,x_m,x_m) = 0$.

Definition 4 ([8], Definition 2.1)**.** Let X be a nonempty set and $d : X \times X \longrightarrow [0,+\infty)$ be a function such that, for all $x,y,z \in X$,

(1) $d(x,y) = 0$ if and only if $x = y$.
(2) $d(x,y) \leq d(x,z) + d(z,y)$.

Then d is called a *quasi-metric* and the pair (X,d) is called a *quasi-metric space*.

Definition 5 ([8]). Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X.

(1) $\{x_n\}$ is called *convergent* to $x \in X$, written $\lim_{n\to\infty} x_n = x$, if
$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$
(2) $\{x_n\}$ is called *left-Cauchy* if for each $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m > n(\varepsilon)$.
(3) $\{x_n\}$ is called *right-Cauchy* if for each $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n > n(\varepsilon)$.
(4) $\{x_n\}$ is called *Cauchy* if for each $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > n(\varepsilon)$, that is, $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.
(5) (X, d) is called *complete* if each Cauchy sequence in (X, d) is convergent.

Remark 1 ([8]). (1) Every metric is a quasi-metric.
(2) In a quasi-metric space, a sequence $\{x_n\}$ is Cauchy if and only if it is left-Cauchy and right-Cauchy.

The following examples show that the inversion of Remark 1.(1) does not hold.

Example 1. Let $X = \mathbb{R}$ and d be defined by
$$d(x, y) = \begin{cases} x - y & \text{if } x \geq y \\ 1 & \text{if } x < y. \end{cases}$$
Then d is a quasi-metric on X but d is not a metric on X.

Proof. It is clear that $d : X \times X \longrightarrow [0, +\infty)$ and $d(x, y) = 0$ if and only if $x = y$. For all $x, y, z \in X$, we consider two following cases.
Case 1. $x \geq y$. We have $d(x, y) = x - y$.
If $z < y$, then $d(x, z) = x - z$ and $d(z, y) = 1$.
If $y \leq z < x$, then $d(x, z) = x - z$ and $d(z, y) = z - y$.
If $x \leq z$, then $d(x, z) = 1$ and $d(z, y) = z - y$.
So we have $d(x, y) \leq d(x, z) + d(z, y)$.
Case 2. $x < y$. We have $d(x, y) = 1$.
If $z < x$, then $d(x, z) = x - z$ and $d(z, y) = 1$.
If $x \leq z < y$, then $d(x, z) = 1$ and $d(z, y) = 1$.
If $y \leq z$, then $d(x, z) = 1$ and $d(z, y) = z - y$.
So we have $d(x, y) \leq d(x, z) + d(z, y)$.
By the above, d is a quasi-metric on $\mathbb{R}$. Since $d(0, 2) = 1 \neq d(2, 0) = 2$, d is not a metric on $\mathbb{R}$. □

Example 2. Let $X = X_1 \cup X_2$, $X_1 \cap X_2 \neq \varnothing$ and d be defined by
$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \in X_1, y \in X_2 \\ 1 & \text{otherwise.} \end{cases}$$

Then d is a quasi-metric on X but d is not a metric on X.

For more quasi-metrics which are not metrics, see [16, Example 1.4].

In [8], Jleli and Samet showed that most of the obtained fixed point theorems on G-metric spaces may be deduced immediately from fixed point theorems on metric spaces or quasi-metric spaces. The main results in [8] are as follows.

Theorem 1 ([8], Theorem 2.2). *Let (X,G) be a G-metric space and $d_G : X \times X \longrightarrow [0,+\infty)$ be defined by $d_G(x,y) = G(x,y,y)$ for all $x,y \in X$. Then we have*

(1) *(X,d_G) is a quasi-metric space.*
(2) *A sequence $\{x_n\}$ is convergent to x in (X,G) if and only if $\{x_n\}$ is convergent to x in (X,d_G).*
(3) *A sequence $\{x_n\}$ is Cauchy in (X,G) if and only if $\{x_n\}$ is Cauchy in (X,d_G).*
(4) *The G-metric space (X,G) is complete if and only if the quasi-metric space (X,d_G) is complete.*

Theorem 2 ([8], Theorem 2.3). *Let (X,G) be a G-metric space and $\delta_G : X \times X \longrightarrow [0,+\infty)$ be defined by $\delta_G(x,y) = \max\big\{G(x,y,y), G(y,x,x)\big\}$ for all $x,y \in X$. Then we have*

(1) *(X,δ_G) is a metric space.*
(2) *A sequence $\{x_n\}$ is convergent to x in (X,G) if and only if $\{x_n\}$ is convergent to x in (X,δ_G).*
(3) *A sequence $\{x_n\}$ is Cauchy in (X,G) if and only if $\{x_n\}$ is Cauchy in (X,δ_G).*
(4) *The G-metric space (X,G) is complete if and only if the metric space (X,δ_G) is complete.*

Theorem 3 ([8], Theorem 3.2). *Let (X,d) be a complete quasi-metric space and $T : X \longrightarrow X$ be a map such that*

$$d(Tx,Ty) \le d(x,y) - \varphi\big(d(x,y)\big) \tag{1.1}$$

for all $x,y \in X$, where $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ is continuous with $\varphi^{-1}\big(\{0\}\big) = \{0\}$. Then T has a unique fixed point.

Recently, in [16], Rajić proved the following result which is a generalization of Theorem 3.

Theorem 4 ([16], Theorem 2.2). *Let (X,d) be a complete quasi-metric space and $f,g : X \longrightarrow X$ be two maps such that*

$$\psi\big(d(fx,fy)\big) \le \psi\big(d(gx,gy)\big) - \phi\big(d(gx,gy)\big) \tag{1.2}$$

for all $x,y \in X$, where $\psi : [0,+\infty) \longrightarrow [0,+\infty)$ is continuous, non-decreasing, $\psi^{-1}(0) = \{0\}$, $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ is continuous and $\phi^{-1}(0) = \{0\}$. If the range of g contains the range of f and $f(X)$ or $g(X)$ is a closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly *compatible, that is, $fgx = gfx$ provided that $fx = gx$, then f and g have a unique common fixed point.*

The main results of the paper are presented in Section 2 and Section 3. In Section 2, we prove some properties of the quasi-metric space and its modification. Then, by similar arguments as in metric spaces, we prove some analogues of fixed point theorems in quasi-metric spaces. In Section 3, we show that most of recent fixed point theorems on G-metric spaces in [3, 10] may be implied from certain fixed point theorems proved in Section 2.

2. REMARKS ON QUASI-METRIC SPACES

Note that every quasi-metric space (X,d) is a topological space with the topology induced by its convergence. Then $X\times X$ is a topological space with the product topology. The following result shows that the product space $X\times X$ is also a quasi-metric space.

Proposition 1. *Let (X,d_X) and (Y,d_Y) be two quasi-metric spaces. Then we have*

(1) $d(x,y) = d_X(x_1,y_1)+d_Y(x_2,y_2)$ *for all* $x=(x_1,x_2), y=(y_1,y_2)\in X\times Y$ *is a quasi-metric on* $X\times Y$.

(2) $\lim\limits_{n\to\infty}(x_n,y_n)=(x,y)$ *in* $(X\times Y,d)$ *if and only if* $\lim\limits_{n\to\infty}x_n = x$ *in* (X,d_X) *and* $\lim\limits_{n\to\infty}y_n=y$ *in* (Y,d_Y). *In particular, the product topology on* $X\times Y$ *coincides the topology induced by* d.

(3) $\{(x_n,y_n)\}$ *is a Cauchy sequence in* $(X\times Y,d)$ *if and only if* $\{x_n\}$ *is a Cauchy sequence in* (X,d_X) *and* $\{y_n\}$ *is a Cauchy sequence in* (Y,d_Y).

(4) $(X\times Y,d)$ *is complete if and only if* (X,d_X) *and* (Y,d_Y) *are complete.*

Proof. (1). For all $x=(x_1,x_2), y=(y_1,y_2), z=(z_1,z_2)\in X\times Y$, we have $d(x,y)=0$ if and only if $d_X(x_1,y_1)+d_Y(x_2,y_2)=0$, that is, $d_X(x_1,y_1) = d_Y(x_2,y_2)=0$. It is equivalent to $x_1=y_1$ and $x_2=y_2$, that is, $x=y$.

We also have

$$\begin{aligned}
d(x,z) &= d_X(x_1,z_1)+d_Y(x_2,z_2)\\
&\le d_X(x_1,y_1)+d_X(y_1,z_1)+d_Y(x_2,y_2)+d_Y(y_2,z_2)\\
&= d_X(x_1,y_1)+d_Y(x_2,y_2)+d_X(y_1,z_1)+d_Y(y_2,z_2)\\
&= d(x,y)+d(y,z).
\end{aligned}$$

By the above, d is a quasi-metric on $X\times Y$.

(2). $\lim\limits_{n\to\infty}(x_n,y_n)=(x,y)$ in $(X\times Y,d)$ if and only if

$$\lim_{n\to\infty} d\big((x_n,y_n),(x,y)\big) = \lim_{n\to\infty}\big[d_X(x_n,x)+d_Y(y_n,y)\big]=0$$

and

$$\lim_{n\to\infty} d\big((x,y),(x_n,y_n)\big) = \lim_{n\to\infty}\big[d_X(x,x_n)+d_Y(y,y_n)\big]=0.$$

It is equivalent to

$$\lim_{n\to\infty} d_X(x_n,x) = \lim_{n\to\infty} d_Y(y_n,y) = \lim_{n\to\infty} d_X(x,x_n) = \lim_{n\to\infty} d_Y(y,y_n) = 0.$$

That is, $\lim_{n\to\infty} x_n = x$ in (X,d_X) and $\lim_{n\to\infty} y_n = y$ in (Y,d_Y).

(3). $\{(x_n,y_n)\}$ is a Cauchy sequence in $(X\times Y,d)$ if and only if

$$\lim_{n,m\to\infty} d\big((x_n,y_n),(x_m,y_m)\big) = \lim_{n,m\to\infty}\big[d_X(x_n,x_m)+d_Y(y_n,y_m)\big] = 0.$$

It is equivalent to

$$\lim_{n,m\to\infty} d_X(x_n,x_m) = \lim_{n,m\to\infty} d_Y(y_n,y_m) = 0.$$

That is, $\{x_n\}$ is a Cauchy sequence in (X,d_X) and $\{y_n\}$ is a Cauchy sequence in (Y,d_Y).

(4). It is a direct consequence of (2) and (3). □

In the proof of [8, Theorem 3.2], Jleli and Samet used the sequential continuity of a quasi-metric d without proving. From Proposition 1, we see that the sequential continuity and the continuity of d are equivalent and they are guaranteed by the following proposition.

Proposition 2. *Let (X,d) be a quasi-metric space. Then d is a continuous function.*

Proof. Suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in (X,d). We have

$$d(x_n,y_n) \le d(x_n,x)+d(x,y)+d(y,y_n).$$

It implies that

$$d(x_n,y_n)-d(x,y) \le d(x_n,x)+d(y,y_n). \tag{2.1}$$

Also, we have

$$d(x,y) \le d(x,x_n)+d(x_n,y_n)+d(y_n,y).$$

It implies that

$$d(x,y)-d(x_n,y_n) \le d(x,x_n)+d(y_n,y). \tag{2.2}$$

From (2.1) and (2.2), we have

$$0 \le |d(x,y)-d(x_n,y_n)| \le \max\big\{d(x_n,x)+d(y,y_n),\, d(x,x_n)+d(y_n,y)\big\}. \tag{2.3}$$

Taking the limit as $n\to\infty$ in (2.3), we obtain $\lim_{n\to\infty} |d(x,y)-d(x_n,y_n)| = 0$. That is, $\lim_{n\to\infty} d(x_n,y_n) = d(x,y)$. This proves that d is a continuous function. □

The following proposition proves that the topology of each quasi-metric space is metrizable. Then all topological properties of metric spaces hold on quasi-metric spaces.

Proposition 3. *Let (X,d) be a quasi-metric space and*

$$\delta_d(x,y) = \max\big\{d(x,y), d(y,x)\big\}$$

for all $x,y \in X$. Then we have

(1) *(X,δ_d) is a metric space.*
(2) *A sequence $\{x_n\}$ is convergent to x in (X,d) if and only if $\{x_n\}$ is convergent to x in (X,δ_d).*
(3) *A sequence $\{x_n\}$ is Cauchy in (X,d) if and only if $\{x_n\}$ is Cauchy in (X,δ_d).*
(4) *The quasi-metric space (X,d) is complete if and only if the metric space (X,δ_d) is complete.*

Proof. (1). See [8], page 3.

(2). We have $\lim_{n\to\infty} x_n = x$ in (X,d) if and only if

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0.$$

It is equivalent to

$$\lim_{n\to\infty} \delta_d(x_n,x) = \lim_{n\to\infty} \max\big\{d(x_n,x), d(x,x_n)\big\} = 0.$$

That is, $\lim_{n\to\infty} x_n = x$ in (X,δ_d).

(3). A sequence $\{x_n\}$ is Cauchy in (X,d) if and only if

$$\lim_{n,m\to\infty} d(x_n,x_m) = \lim_{n,m\to\infty} d(x_m,x_n) = 0.$$

It is equivalent to

$$\lim_{n,m\to\infty} \delta_d(x_n,x_m) = \lim_{n,m\to\infty} \max\big\{d(x_n,x_m), d(x_m,x_n)\big\} = 0.$$

That is, $\{x_n\}$ is Cauchy in (X,δ_d).

(4). It is a direct consequence of (2) and (3). □

By modifying the notion of T-orbital completeness in [6], we introduce the notion of weak T-orbital completeness as follows.

Definition 6. Let (X,d) be a quasi-metric space and $T : X \longrightarrow X$ be a map. Then X is called *weak T-orbitally complete* if $\{T^n x\}$ is convergent in X provided that it is a Cauchy sequence in X.

Note that every T-orbitally complete quasi-metric space is a weak T-orbitally complete quasi-metric space for all maps $T : X \longrightarrow X$. The following example shows that the inversion does not hold, even when (X,d) is a metric space.

Example 3. Let $X = \{1,3,\dots,2n+1,\dots\} \cup \big\{\frac{1}{2},\frac{1}{4}\dots,\frac{1}{2n},\dots\big\}$ with the usual metric and

$$T\frac{1}{2n} = 2n+1, T(2n-1) = \frac{1}{2n}$$

for all $n \in \mathbb{N}$. Since $\{T^n x\}$ is not Cauchy for all $x \in X$, (X,d) is weak T-orbitally complete. For $x = 1$, we have

$$\{T^n 1 : n \in \mathbb{N}\} = \big\{1, \frac{1}{2}, 3, \frac{1}{4}, \dots, 2n+1, \frac{1}{2n}, \dots\big\}.$$

Since $\{\frac{1}{2n}\}$ is a Cauchy sequence in (X,d) which is not convergent, (X,d) is not T-orbitally complete.

Following the proof of [11, Theorem 3.1], we get the following fixed point theorem on quasi-metric spaces.

Theorem 5. *Let (X,d) be a quasi-metric space and $T : X \longrightarrow X$ be a map such that*

(1) *X is weak T-orbitally complete.*
(2) *There exists $q \in [0,1)$ such that for all $x, y \in X$,*

$$\begin{aligned} d(Tx,Ty) \leq q \max\big\{&d(x,y), d(y,x), d(x,Tx), d(Tx,x), d(y,Ty), \\ &d(x,Ty), d(y,Tx), d(Tx,y), d(T^2x,x), d(x,T^2x), d(T^2x,Tx), \\ &d(Tx,T^2x), d(T^2x,y), d(y,T^2x), d(T^2x,Ty)\big\}. \end{aligned} \tag{2.4}$$

Then we have

(1) *T has a unique fixed point x^* in X.*
(2) *$\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*
(3) *$\max\big\{d(T^n x, x^*), d(x^*, T^n x)\big\} \leq \frac{q^n}{1-q} \max\big\{d(x,Tx), d(Tx,x)\big\}$ for all $x \in X$ and $n \in \mathbb{N}$.*

Proof. (1). For each $x \in X$ and $1 \leq i \leq n-1$, $1 \leq j \leq n$, we have

$$\begin{aligned} &d(T^i x, T^j x) = d(TT^{i-1}x, TT^{j-1}x) \\ &\leq q \max\big\{d(T^{i-1}x, T^{j-1}x), d(T^{j-1}x, T^{i-1}x), d(T^{i-1}x, TT^{i-1}x), \\ &\quad d(TT^{i-1}x, T^{i-1}x), d(T^{j-1}x, TT^{j-1}x), d(T^{i-1}x, TT^{j-1}x), d(T^{j-1}x, TT^{i-1}x), \\ &\quad d(TT^{i-1}x, T^{j-1}x), d(T^2T^{i-1}x, T^{i-1}x), d(T^{i-1}x, T^2T^{i-1}x), \\ &\quad d(T^2T^{i-1}x, TT^{i-1}x), d(TT^{i-1}x, T^2T^{i-1}x), d(T^2T^{i-1}x, T^{j-1}x), \\ &\quad d(T^{j-1}x, T^2T^{i-1}x), d(T^2T^{i-1}x, TT^{j-1}x)\big\} \\ &= q \max\big\{d(T^{i-1}x, T^{j-1}x), d(T^{j-1}x, T^{i-1}x), d(T^{i-1}x, T^i x), d(T^i x, T^{i-1}x), \\ &\quad d(T^{j-1}x, T^j x), d(T^{i-1}x, T^j x), d(T^{j-1}x, T^i x), d(T^i x, T^{j-1}x), \\ &\quad d(T^{i+1}x, T^{i-1}x), d(T^{i-1}x, T^{i+1}x), d(T^{i+1}x, T^i x), d(T^i x, T^{i+1}x), \\ &\quad d(T^{i+1}x, T^{j-1}x), d(T^{j-1}x, T^{i+1}x), d(T^{i+1}x, T^j x)\big\} \end{aligned} \tag{2.5}$$

$\leq q\delta\big[O_T(x,n)\big]$

where $\delta\big[O_T(x,n)\big] = \max\big\{d(T^i x, T^j x) : 0 \leq i \leq n-1, 0 \leq j \leq n\big\}$.

From (2.5), since $0 \leq q < 1$, there exists $k_n(x) \leq n$ such that

$$d(x, T^{k_n(x)}x) = \delta\big[O_T(x,n)\big] \tag{2.6}$$

or there exists $k_n(x) \leq n-1$ such that

$$d(T^{k_n(x)}x, x) = \delta\big[O_T(x,n)\big]. \tag{2.7}$$

If (2.6) holds, we have

$$\begin{aligned} d(x, T^{k_n(x)}x) &\leq d(x, Tx) + d(Tx, T^{k_n(x)}x) \\ &\leq d(x, Tx) + q\delta\big[O_T(x,n)\big] \\ &= d(x, Tx) + qd(x, T^{k_n(x)}x). \end{aligned}$$

It implies that

$$\delta\big[O_T(x,n)\big] = d(x, T^{k_n(x)}x) \leq \frac{1}{1-q} d(x, Tx). \tag{2.8}$$

If (2.7) holds, we have

$$\begin{aligned} d(T^{k_n(x)}x, x) &\leq d(T^{k_n(x)}x, Tx) + d(Tx, x) \\ &\leq q\delta\big[O_T(x,n)\big] + d(Tx, x) \\ &= qd(T^{k_n(x)}x, x) + d(Tx, x). \end{aligned}$$

It implies that

$$\delta\big[O_T(x,n)\big] = d(T^{k_n(x)}x, x) \leq \frac{1}{1-q} d(Tx, x). \tag{2.9}$$

For all $n < m$, it follows from (2.4) and (2.8), (2.9) that

$$\begin{aligned} d(T^n x, T^m x) &= d(TT^{n-1}x, T^{m-n+1}T^{n-1}x) \\ &\leq q\delta\big[O_T(T^{n-1}x, m-n+1)\big] \\ &= qd(T^{n-1}x, T^{k_{m-n+1}(T^{n-1}x)}T^{n-1}x) \\ &= qd(TT^{n-2}x, T^{k_{m-n+1}(T^{n-1}x)+1}T^{n-2}x) \\ &\leq q^2\delta\big[O_T(T^{n-2}x, k_{m-n+1}(T^{n-1}x)+1)\big] \\ &\leq q^2\delta\big[O_T(T^{n-2}x, m-n+2)\big] \\ &\leq \ldots \\ &\leq q^n\delta\big[O_T(x,m)\big] \\ &\leq \frac{q^n}{1-q}\max\{d(x,Tx), d(Tx,x)\}. \end{aligned} \tag{2.10}$$

Since $\lim_{n\to\infty} q^n = 0$, by taking the limit as $n,m \to \infty$ in (2.10), we have

$$\lim_{n,m\to\infty} d(T^n x, T^m x) = 0. \tag{2.11}$$

This proves that $\{T^n x\}$ is a Cauchy sequence in X. Since X is weak T-orbitally complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(T^n x, x^*) = \lim_{n\to\infty} d(x^*, T^n x) = 0. \tag{2.12}$$

Therefore, by using (2.4) again, we have

$$\begin{aligned} d(x^*, Tx^*) &\le d(x^*, T^{n+1}x) + d(T^{n+1}x, Tx^*) \\ &= d(x^*, T^{n+1}x) + d(TT^n x, Tx^*) \\ &\le d(x^*, T^{n+1}x) + q \max\{d(T^n x, x^*), d(x^*, T^n x), d(T^n x, TT^n x), \\ &\quad d(TT^n x, T^n x), d(x^*, Tx^*), d(T^n x, Tx^*), d(x^*, TT^n x), d(TT^n x, x^*), \\ &\quad d(T^2T^n x, T^n x), d(T^n x, T^2T^n x), d(T^2T^n x, TT^n x), d(TT^n x, T^2T^n x), \\ &\quad d(T^2T^n x, x^*), d(x^*, T^2T^n x), d(T^2T^n x, Tx^*)\} \\ &= d(x^*, T^{n+1}x) + q \max\{d(T^n x, x^*), d(x^*, T^n x), d(T^n x, T^{n+1}x), \\ &\quad d(T^{n+1}x, T^n x), d(x^*, Tx^*), d(T^n x, Tx^*), d(x^*, T^{n+1}x), d(T^{n+1}x, x^*), \\ &\quad d(T^{n+2}x, T^n x), d(T^n x, T^{n+2}x), d(T^{n+2}x, T^{n+1}x), d(T^{n+1}x, T^{n+2}x), \\ &\quad d(T^{n+2}x, x^*), d(x^*, T^{n+2}x), d(T^{n+2}x, Tx^*)\}. \end{aligned} \tag{2.13}$$

Taking the limit as $n \to \infty$ in (2.13), and using (2.11), (2.12) and Proposition 2, we get $d(x^*, Tx^*) \le q d(x^*, Tx^*)$. Since $q \in [0,1)$, $d(x^*, Tx^*) = 0$, that is, $x^* = Tx^*$. Then T has a fixed point.

Now, we prove the uniqueness of the fixed point of T. Let x^*, y^* be two fixed points of T. From (2.4), we have

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\le q \max\{d(x^*, y^*), d(y^*, x^*), d(x^*, Tx^*), d(Tx^*, x^*), d(y^*, Ty^*), d(x^*, Ty^*), \\ &\quad d(y^*, Tx^*), d(y^*, Tx^*), d(T^2x^*, x^*), d(x^*, T^2x^*), d(T^2x^*, Tx^*), \\ &\quad d(Tx^*, T^2x^*), d(T^2x^*, y^*), d(y^*, T^2x^*), d(T^2x^*, Ty^*)\} \\ &= q \max\{d(x^*, y^*), d(y^*, x^*)\}. \end{aligned}$$

Since $q \in [0,1)$, we get

$$d(x^*, y^*) \le q d(y^*, x^*). \tag{2.14}$$

Again, from (2.4), we also have

$$\begin{aligned} d(y^*, x^*) &= d(Ty^*, Tx^*) \\ &\le q \max\{d(y^*, x^*), d(x^*, y^*), d(y^*, Ty^*), d(Ty^*, y^*), d(x^*, Tx^*), d(y^*, Tx^*), \end{aligned}$$

$$d(x^*,Ty^*), d(x^*,Ty^*), d(T^2y^*,y^*), d(y^*,T^2y^*), d(T^2y^*,Ty^*),$$
$$d(Ty^*,T^2y^*), d(T^2y^*,x^*), d(x^*,T^2y^*), d(T^2y^*,Tx^*)\big\}$$
$$= q\max\big\{d(y^*,x^*), d(x^*,y^*)\big\}.$$

Since $q \in [0,1)$, we get

$$d(y^*,x^*) \le q d(x^*,y^*). \tag{2.15}$$

From (2.14) and (2.15), since $q \in [0,1)$, we obtain $d(x^*,y^*) = 0$. That is, $x^* = y^*$. Then the fixed point of T is unique.

(2). It is proved by (2.12).

(3). Taking the limit as $m \to \infty$ in (2.10) and using Proposition 2, we get

$$d(T^n x, x^*) \le \frac{q^n}{1-q}\max\{d(x,Tx), d(Tx,x)\}.$$

Similarly, we have $d(x^*,T^n x) \le \dfrac{q^n}{1-q}\max\{d(x,Tx), d(Tx,x)\}$. Therefore, $\max\big\{d(T^n x,x^*), d(x^*,T^n x)\big\} \le \dfrac{q^n}{1-q}\max\big\{d(x,Tx), d(Tx,x)\big\}$. □

If d in Theorem 5 is a metric, then we have the following result. Note that Corollary 1 is a generalization of the following well-known result of Ćirić in [6]. This generalization is proper by [11, Example 3.6].

Corollary 1 ([11], Theorem 3.1)**.** *Let (X,d) be a metric space and $T : X \longrightarrow X$ be a map satisfying the following*

(1) *X is weak T-orbitally complete.*
(2) *There exists $q \in [0,1)$ such that for all $x,y \in X$,*

$$d(Tx,Ty) \le q\max\big\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), d(T^2x,x), d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\big\}. \tag{2.16}$$

Then we have

(1) *T has a unique fixed point x^* in X.*
(2) *$\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*
(3) *$d(T^n x, x^*) \le \dfrac{q^n}{1-q} d(x,Tx)$ for all $x \in X$.*

Now, we modify the notion of a quasi-metric space in Definition 4 as follows.

Definition 7. Let X be a nonempty set and $d : X \times X \longrightarrow [0,+\infty)$ be a function such that, for all $x,y,z \in X$,

(1) $d(x,y) = 0$ if and only if $x = y$.
(2) $d(x,y) \le d(x,z) + d(y,z)$ if $z \ne x$.

Then d is also called a *quasi-metric* and the pair (X,d) is also called a *quasi-metric space*.

Note that if $z = x$ in Definition 7.(2), then $d(x,y) = d(y,x)$ for all $x,y \in X$. In this case, d deduces a metric on X.

The following proposition gives a way to set examples of quasi-metrics in the sense of Definition 7.

Proposition 4. *Let (X,G) be a G-metric space and $T : X \longrightarrow X$ be a map. For all $x,y \in X$, put*

$$d_{T,G}(x,y) = \begin{cases} 0 & \text{if } x = y \\ G(x,Tx,y) & \text{if } x \neq y. \end{cases}$$

If T has no any fixed point, then $d_{T,G}$ is a quasi-metric in the sense of Definition 7 on X.

Proof. For all $x,y,z \in X$ with $z \neq x$, note that $y \neq Ty$ for all $y \in X$, we have $G(y,z,z) \leq G(z,y,Ty)$ for all $y,z \in X$. Then

$$\begin{aligned} d_{T,G}(x,y) &\leq G(x,Tx,y) \\ &= G(y,x,Tx) \\ &\leq G(y,z,z) + G(x,Tx,z) \\ &\leq G(z,y,Ty) + G(x,Tx,z) \\ &= d_{T,G}(x,z) + d_{T,G}(y,z). \end{aligned}$$

This proves that $d_{T,G}$ is a quasi-metric in the sense of Definition 7 on X. □

Example 4. Let $X = \{1,2,3\}$ and

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } (x,y) = (1,2) \\ 1 & \text{otherwise.} \end{cases}$$

Then d is a quasi-metric in the sense of Definition 7 on X. For $(2,1),(1,2),(1,1) \in X \times X$, we have

$$\begin{aligned} d(2,1) + d(1,2) &= 3 \\ d(2,1) + d(1,1) &= 1 \\ d(1,1) + d(2,1) &= 1. \end{aligned}$$

This proves that $d(2,1) + d(1,2) > d(2,1) + d(1,1) + d(1,1) + d(2,1)$. So Proposition 1.(1) does not hold for quasi-metrics in the sense of Definition 7.

Proposition 5. *Let (X,d) be a quasi-metric space in the sense of Definition 7. For each $y \in X$, if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} d(y,x_n) = d(y,x)$.*

Proof. **Case 1.** $y = x$. Then we have

$$d(y,x) = d(x,x) = 0 = \lim_{n\to\infty} d(x,x_n) = \lim_{n\to\infty} d(y,x_n).$$

Case 2. $y \neq x$. If $x_n = x$ for infinitely many n, then

$$\lim_{m\to\infty} d(y,x_n) = d(y,x).$$

So, we may assume that $x_n \neq x$ for n large enough. Also, $y \neq x_n$ for n large enough. Then we have, for all $n \in \mathbb{N}$,

$$d(y,x) \le d(y,x_n) + d(x,x_n) \le d(y,x) + d(x_n,x) + d(x,x_n). \tag{2.17}$$

Taking the limit as $n \to \infty$ in (2.17), we get $\lim_{n\to\infty} d(y,x_n) = d(y,x)$. □

With some minor changes in the proof of Theorem 5, we have the following result. Note that these changes mainly relate to Definition 7.(2).

Proposition 6. *Let (X,d) be a quasi-metric space in the sense of Definition 7 and $T : X \longrightarrow X$ be a map satisfying the following*

(1) *X is weak T-orbitally complete.*
(2) *There exists $q \in [0,1)$ such that for all $x,y \in X$,*

$$\begin{aligned} &d(Tx,Ty) \\ &\le q \max\big\{d(x,y), d(y,x), d(Tx,x), d(y,Ty), d(Ty,y), d(x,Ty), d(Ty,x), \\ &\quad d(Tx,y), d(T^2y,y), d(y,T^2y), d(T^2y,Ty), d(Ty,T^2y), d(T^2y,x), \\ &\quad d(x,T^2y), d(Tx,T^2y)\big\}. \end{aligned} \tag{2.18}$$

Then we have

(1) *T has a unique fixed point x^* in X.*
(2) *$\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*
(3) *$\max\big\{d(T^n x, x^*), d(x^*, T^n x)\big\} \le \dfrac{q^n}{1-q} \max\big\{d(x,Tx), d(Tx,x)\big\}$ for all $x \in X$ and $n \in \mathbb{N}$.*

Proof. As in the proof of Theorem 5.(1), there exists $k_n(x) \le n$ such that

$$d(x, T^{k_n(x)}x) = \delta\big[O_T(x,n)\big] \tag{2.19}$$

or there exists $k_n(x) \le n-1$ such that

$$d(T^{k_n(x)}x, x) = \delta\big[O_T(x,n)\big]. \tag{2.20}$$

If $Tx = x$, then T has a fixed point. So we may assume that $Tx \neq x$. If (2.19) holds, we have

$$\begin{aligned} d(x, T^{k_n(x)}x) &\le d(x,Tx) + d(T^{k_n(x)}x, Tx) \\ &\le d(x,Tx) + q\delta\big[O_T(x,n)\big] \end{aligned}$$

$$= d(x,Tx) + qd(x,T^{k_n(x)}x).$$

It implies that

$$\delta\big[O_T(x,n)\big] = d(x,T^{k_n(x)}x) \leq \frac{1}{1-q} d(x,Tx). \tag{2.21}$$

If (2.20) holds and $T^{k_n(x)}x = Tx$, we have

$$\delta\big[O_T(x,n)\big] = d(T^{k_n(x)}x,x) = d(Tx,x) \leq \frac{1}{1-q} d(Tx,x). \tag{2.22}$$

So, we may assume that $T^{k_n(x)}x \neq Tx$. Then

$$\begin{aligned} d(T^{k_n(x)}x,x) &\leq d(T^{k_n(x)}x,Tx) + d(x,Tx) \\ &\leq q\delta\big[O_T(x,n)\big] + d(x,Tx) \\ &= qd(T^{k_n(x)}x,x) + d(x,Tx). \end{aligned}$$

It implies that

$$\delta\big[O_T(x,n)\big] = d(T^{k_n(x)}x,x) \leq \frac{1}{1-q} d(x,Tx). \tag{2.23}$$

As in the proof of Theorem 5.(1), we also have

$$\lim_{n,m\to\infty} d(T^n x, T^m x) = 0 \tag{2.24}$$

and there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(T^n x, x^*) = \lim_{n\to\infty} d(x^*, T^n x) = 0. \tag{2.25}$$

If $T^{n+1}x = Tx^*$ for infinitely many n, then $\lim\limits_{n\to\infty} T^{n+1}x = Tx^* = x^*$. Then x^* is a fixed point of T. So, we may assume that $T^{n+1}x \neq x^*$ for n large enough. Therefore, by using (2.18) again, we have

$$\begin{aligned} & d(Tx^*,x^*) \leq d(Tx^*,T^{n+1}x) + d(x^*,T^{n+1}x) \qquad (2.26) \\ & = d(x^*,T^{n+1}x) + d(Tx^*,TT^n x) \\ & \leq d(x^*,T^{n+1}x) + q\max\big\{d(T^n x,x^*), d(x^*,T^n x), d(T^n x,TT^n x), \\ & \quad d(TT^n x,T^n x), d(Tx^*,x^*), d(Tx^*,T^n x), d(x^*,TT^n x), d(TT^n x,x^*), \\ & \quad d(T^2T^n x,T^n x), d(T^n x,T^2T^n x), d(T^2T^n x,TT^n x), d(TT^n x,T^2T^n x), \\ & \quad d(T^2T^n x,x^*), d(x^*,T^2T^n x), d(Tx^*,T^2T^n x)\big\} \\ & = d(x^*,T^{n+1}x) + q\max\big\{d(T^n x,x^*), d(x^*,T^n x), d(T^n x,T^{n+1}x), \\ & \quad d(T^{n+1}x,T^n x), d(Tx^*,x^*), d(Tx^*,T^n x), d(x^*,T^{n+1}x), d(T^{n+1}x,x^*), \\ & \quad d(T^{n+2}x,T^n x), d(T^n x,T^{n+2}x), d(T^{n+2}x,T^{n+1}x), d(T^{n+1}x,T^{n+2}x), \end{aligned}$$

$$d(T^{n+2}x, x^*), d(x^*, T^{n+2}x), d(Tx^*, T^{n+2}x)\big\}.$$

Taking the limit as $n \to \infty$ in (2.26), and using (2.24), (2.25) and Proposition 5, we get $d(Tx^*, x^*) \le q d(Tx^*, x^*)$. Since $q \in [0, 1)$, $d(Tx^*, x^*) = 0$, that is, $Tx^* = x^*$. Then T has a fixed point.

The remaining is similar as the proof of Theorem 5. □

With some minor changes in the proof of Theorem 3, we get the following result. Also, these changes mainly relate to Definition 7.(2).

Proposition 7. *Let (X, d) be a quasi-metric space in the sense of Definition 7 and $T : X \longrightarrow X$ be a map such that (X, d) is weak T-orbitally complete and*

$$d(Tx, Ty) \le d(x, y) - \varphi\big(d(x, y)\big) \tag{2.27}$$

for all $x, y \in X$, where $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is continuous with $\varphi^{-1}\big(\{0\}\big) = \{0\}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and define the sequence $x_{n+1} = Tx_n$ for all $n \ge 0$. From (2.27), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \le d(x_{n-1}, x_n) - \varphi\big(d(x_{n-1}, x_n)\big) \tag{2.28}$$

for all $n \ge 1$. This proves that $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive numbers. Then there exists $r \ge 0$ such that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = r$. Taking the limit as $n \to \infty$ in (2.28), we get $\varphi(r) = 0$, that is, $r = 0$. Then

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.29}$$

Using the same technique, we also have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.30}$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence in the quasi-metric space (X, d), that is, $\{x_n\}$ is left-Cauchy and right-Cauchy. By (2.29) and (2.30), since the sequences $\{d(x_{n+1}, x_n)\}$ and $\{d(x_n, x_{n+1})\}$ are non-increasing, we have $\{x_n\}$ is a Cauchy sequence if there exists n such that $d(x_{n+1}, x_n) = 0$ or $d(x_n, x_{n+1}) = 0$. Then, we may assume that, for all $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) \ne 0 \text{ and } d(x_n, x_{n+1}) \ne 0. \tag{2.31}$$

Now, suppose to the contrary that $\{x_n\}$ is not a left-Cauchy sequence. Then there exists $\varepsilon > 0$ such that for each $k \in \mathbb{N}$, there exist $n > m \ge k$ satisfying $d(x_n, x_m) \ge \varepsilon$. Put

$$n(1) = \min\big\{n : n > 1 \text{ and there exists } m \text{ with } 1 \le m < n, d(x_n, x_m) \ge \varepsilon\big\}$$
$$m(1) = \max\big\{m : 1 \le m < n(1) \text{ with } d(x_{n(1)}, x_m) \ge \varepsilon\big\}$$
$$n(2) = \min\big\{n : n > n(1), \text{ there exists } m \text{ with } n(1) \le m < n, d(x_n, x_m) \ge \varepsilon\big\}$$
$$m(2) = \max\big\{m : n(1) \le m < n(2) \text{ with } d(x_{n(2)}, x_m) \ge \varepsilon\big\}.$$

Note that $n(1) < n(2)$, $m(1) < m(2)$ and

$$d(x_{n(1)-1}, x_{m(1)}) < \varepsilon, d(x_{n(2)-1}, x_{m(2)}) < \varepsilon.$$

Continuing this process, we can find two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$, we have $n(k) > m(k) > k$ and

$$d(x_{n(k)}, x_{m(k)}) \geq \varepsilon, d(x_{n(k)-1}, x_{m(k)}) < \varepsilon. \tag{2.32}$$

Now, by (2.31) and (2.32), we have

$$\begin{aligned}
\varepsilon &\leq d(x_{n(k)}, x_{m(k)}) \\
&\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{n(k)-1}) \\
&\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{m(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) \\
&\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{m(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\
&\quad + d(x_{m(k)-1}, x_{m(k)}) \\
&< d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{m(k)-1}) + \varepsilon + d(x_{m(k)-1}, x_{m(k)}).
\end{aligned} \tag{2.33}$$

Taking the limit as $k \to \infty$ in (2.33) and using (2.29), (2.30), we get

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.34}$$

Also, by (2.31), we have

$$\begin{aligned}
d(x_{n(k)-1}, x_{m(k)-1}) &\leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{m(k)-1}, x_{n(k)}) \\
&\leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{m(k)-1}, x_{m(k)}) + d(x_{n(k)}, x_{m(k)})
\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}
d(x_{n(k)}, x_{m(k)}) &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{n(k)-1}) \\
&\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{m(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}).
\end{aligned} \tag{2.36}$$

Taking the limit as $k \to \infty$ in (2.35) and (2.36) and using (2.29), (2.30), (2.34), we get

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{2.37}$$

Now, from (2.27), for all $k \in \mathbb{N}$, we have

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)-1}, x_{m(k)-1}) - \varphi\big(d(x_{n(k)-1}, x_{m(k)-1})\big). \tag{2.38}$$

Taking the limit as $k \to \infty$ in (2.38) and using (2.34), (2.37), we obtain $\varepsilon \leq \varepsilon - \varphi\big(\varepsilon\big)$. It implies that $\varepsilon = 0$. It is a contradiction. Then $\{x_n\}$ is a left-Cauchy sequence. Similarly, we can show that $\{x_n\}$ is a right-Cauchy sequence. Then $\{x_n\}$ is Cauchy. Since (X, d) is weak T-orbitally complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} d(x^*, x_n) = 0. \tag{2.39}$$

From (2.27), for all $n \in \mathbb{N}$, we have

$$d(Tx^*, x_{n+1}) = d(Tx^*, Tx_n) \leq d(x^*, x_n) - \varphi\big(d(x^*, x_n)\big) \tag{2.40}$$

Taking the limit as $n \to \infty$ in (2.40) and using (2.39), Proposition 5, we get $d(Tx^*, x^*) = 0$. It implies that $x^* = Tx^*$, that is, x^* is a fixed point of T.

The uniqueness of the fixed point is easy to see. □

Similar as the proof of [16, Theorem 2.2] and the proof of Proposition 7, we get the following result.

Proposition 8. *Let (X, d) be a weak T-orbitally complete quasi-metric space in the sense of Definition 7 and let $T : X \longrightarrow X$ be a map such that*

$$\psi\big(d(Tx, Ty)\big) \leq \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big) \tag{2.41}$$

where $\psi, \varphi : [0, +\infty) \longrightarrow [0, +\infty)$, ψ is continuous and non-decreasing, φ is lower semi-continuous, and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

3. Applications to recent fixed point results in G-metric spaces

In this section, we show that most of recent results on G-metric spaces in [3, 10] may be also implied from certain fixed point theorems in metric spaces and quasi-metric spaces mentioned in Section 2. Notice that the authors of [10] forgot the assumption of completeness in [10, Theorems 3.1 & 3.2].

Corollary 2 ([10], Theorem 3.1)**.** *Let (X, G) be a complete G-metric space and $T : X \longrightarrow X$ be a map such that*

$$G(Tx, Ty, Tz) \leq kM(x, y, z) \tag{3.1}$$

for all $x, y, z \in X$, where $k \in \big[0, \frac{1}{2}\big)$ and

$$M(x,y,z) = \max\left\{\begin{array}{l} G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty), \\ G(y,Tx,Ty), G(x,Tx,z), G(z,T^2x,Tz), \\ G(Tx,T^2x,Tz), G(z,Tx,Ty), G(x,y,z), \\ G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz), \\ G(z,Tx,Tx), G(x,Ty,Ty), G(y,Tz,Tz). \end{array}\right\}$$

Then T has a unique fixed point.

Proof. Let d_G be the quasi-metric in Theorem 1. By choosing $z = y$ and using the axioms (G4) and (G5) in Definition 1, we have

$$M(x,y,y) = \max\left\{\begin{array}{l} G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty), \\ G(y,Tx,Ty), G(x,Tx,y), G(y,T^2x,Ty), \\ G(Tx,T^2x,Ty), G(y,Tx,Ty), G(x,y,y), \\ G(x,Tx,Tx), G(y,Ty,Ty), G(y,Ty,Ty), \\ G(y,Tx,Tx), G(x,Ty,Ty), G(y,Ty,Ty) \end{array}\right\}$$

$$= \max \left\{ \begin{array}{l} G(x,Tx,y), G(y,Ty,T^2x), G(Tx,Ty,T^2x), \\ G(y,Ty,Tx), G(x,Tx,y), G(y,Ty,T^2x), \\ G(T^2x,Tx,Ty), G(y,Ty,Tx), G(x,y,y), \\ G(x,Tx,Tx), G(y,Ty,Ty), G(y,Ty,Ty), \\ G(y,Tx,Tx), G(x,Ty,Ty), G(y,Ty,Ty) \end{array} \right\}$$

$$\leq \max \left\{ \begin{array}{l} G(x,Tx,Tx)+G(Tx,Tx,y), G(y,Ty,Ty)+G(Ty,Ty,T^2x), \\ G(Tx,Ty,Ty)+G(Ty,Ty,T^2x), \\ G(y,Ty,Ty)+G(Ty,Ty,Tx), G(x,Tx,Tx)+G(Tx,Tx,y), \\ G(y,Ty,Ty)+G(Ty,Ty,T^2x), \\ G(T^2x,Tx,Tx)+G(Tx,Tx,Ty), \\ G(y,Ty,Ty)+G(Ty,Ty,Tx), \\ G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), G(y,Ty,Ty), \\ G(y,Tx,Tx), G(x,Ty,Ty), G(y,Ty,Ty) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} d_G(x,Tx)+d_G(y,Tx), d_G(y,Ty)+d_G(T^2x,Ty), \\ d_G(Tx,Ty)+d_G(T^2x,Ty), d_G(y,Ty)+d_G(Tx,Ty), \\ d_G(x,Tx)+d_G(y,Tx), d_G(y,Ty)+d_G(T^2x,Ty), \\ d_G(T^2x,Tx)+d_G(Ty,Tx), d_G(y,Ty)+d_G(Tx,Ty), \\ d_G(x,y), d_G(x,Tx), d_G(y,Ty), d_G(y,Ty), d_G(y,Tx), \\ d_G(x,Ty), d_G(y,Ty) \end{array} \right\}$$

$$\leq 2\max \big\{ d_G(x,Tx), d_G(y,Tx), d_G(y,Ty), d_G(T^2x,Ty), d_G(Tx,Ty), d_G(x,y), d_G(x,Ty) \big\}.$$

Then (3.1) becomes

$$d_G(Tx,Ty) \leq 2k\max \big\{ d_G(x,Tx), d_G(y,Tx), d_G(y,Ty), d_G(T^2x,Ty), d_G(Tx,Ty), d_G(x,y), d_G(x,Ty) \big\}.$$

Since $0 \leq 2k < 1$, we have

$$d_G(Tx,Ty) \leq 2k\max \big\{ d_G(x,y), d_G(x,Tx), d_G(y,Ty), d_G(x,Ty), d_G(y,Tx), d_G(T^2x,Ty) \big\}.$$

By Theorem 5, we see that T has a unique fixed point. □

Remark 2. The authors of [10] claimed that the proof of [10, Theorem 3.2] is the mimic of [10, Theorem 3.1]. But, by redoing the proof of [10, Theorem 3.1], we see that the equality (23) in the proof of [10, Theorem 3.1] becomes

$$G(x^*,Tx^*,Tx^*) \leq kG(x^*,Tx^*,Tx^*) \text{ or } G(x^*,Tx^*,Tx^*) \leq kG(x^*,x^*,Tx^*)$$

and the equality (25) in the proof of [10, Theorem 3.1] becomes

$$G(t^*,t^*,x^*) \leq kG(t^*,t^*,x^*) \text{ or } G(t^*,t^*,x^*) \leq kG(t^*,x^*,x^*).$$

In general, the second inequalities do not hold if $k \in [0,1)$.

Corollary 3 ([10], Theorem 3.3)**.** *Let (X, G) be a complete G-metric space and $T : X \longrightarrow X$ be a map such that*

$$\psi\big(G(Tx, T^2x, Ty)\big) \leq G(x, Tx, y) - \varphi\big(G(x, Tx, y)\big) \tag{3.2}$$

for all $x, y \in X$, where $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is continuous with $\varphi^{-1}(\{0\}) = 0$. Then T has a unique fixed point.

Proof. It is easy to see that T has at most one fixed point. Suppose to the contrary that T has no any fixed point. Let $d_{T,G}$ be defined as in Proposition 4. Then, $d_{T,G}$ is a quasi-metric in the sense of Definition 7 on X. We prove that $(X, d_{T,G})$ is a weak T-orbitally complete quasi-metric space. Let $\{x_n\}$ be a Cauchy sequence in $(X, d_{T,G})$ where $x_0 \in X$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. We have

$$\lim_{n,m\to\infty} d_{T,G}(x_n, x_m) = 0.$$

We may assume that $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$. Then

$$\begin{aligned}
0 &\leq \lim_{n,m\to\infty} G(x_n, x_m, x_m) \\
&\leq \lim_{n,m\to\infty} G(x_n, x_{n+1}, x_m) \\
&= \lim_{n,m\to\infty} G(x_n, Tx_n, x_m) \\
&= \lim_{n,m\to\infty} d_{T,G}(x_n, x_m) \\
&= 0.
\end{aligned}$$

It implies that $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$. By Lemma 2, $\{x_n\}$ is a Cauchy sequence in (X, G). Since (X, G) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ in (X, G). Since $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$, we may assume that $x_n \neq x^*$ for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty} d_{T,G}(x_n, x^*) = \lim_{n\to\infty} G(x_n, Tx_n, x^*) = \lim_{n\to\infty} G(x_n, x_{n+1}, x^*) = 0. \tag{3.3}$$

We also have

$$\begin{aligned}
&d_{T,G}(x^*, x_n) = G(x^*, Tx^*, x_n) \\
&\leq G(x^*, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tx^*, x_n) \\
&= G(x^*, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, Tx^*) \\
&= G(x^*, x_{n+1}, x_{n+1}) + G(Tx_{n-1}, T^2x_{n-1}, Tx^*) \\
&\leq G(x^*, x_{n+1}, x_{n+1}) + G(x_{n-1}, Tx_{n-1}, x^*) - \varphi\big(G(x_{n-1}, Tx_{n-1}, x^*)\big) \\
&\leq G(x^*, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_n, x^*) - \varphi\big(G(x_{n-1}, x_n, x^*)\big).
\end{aligned} \tag{3.4}$$

Taking the limit as $n \to \infty$ in (3.4) and using Lemma 1, we obtain

$$\lim_{n\to\infty} d_{T,G}(x^*, x_n) = 0. \tag{3.5}$$

From (3.3) and (3.5), we get $\lim_{n\to\infty} x_n = x^*$ in $(X, d_{T,G})$. Then $(X, d_{T,G})$ is weak T-orbitally complete. Note that (3.2) becomes

$$\psi\big(d_{T,G}(Tx,Ty)\big) \le d_{T,G}(x,y) - \varphi\big(d_{T,G}(x,y)\big).$$

Therefore, by using Proposition 7, we conclude that T has a fixed point. It is a contradiction.

By the above, T has a unique fixed point. □

Corollary 4 ([3], Theorem 2.3)**.** *Let (X,G) be a complete G-metric space and $T : X \longrightarrow X$ be a map such that*

$$\psi\big(G(Tx,T^2x,Ty)\big) \le \psi\big(G(x,Tx,y)\big) - \varphi\big(G(x,Tx,y)\big) \tag{3.6}$$

for all $x, y \in X$, where $\psi : [0,+\infty) \longrightarrow [0,+\infty)$ is non-decreasing and continuous, $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ is lower semi-continuous and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof. It is easy to see that T has at most one fixed point. Suppose to the contrary that T has no any fixed point. Using $d_{T,G}$ as in the proof of Corollary 3, then $d_{T,G}$ is a quasi-metric in the sense of Definition 7 on X. We prove that $(X, d_{T,G})$ is a weak T-orbitally complete quasi-metric space. Let $\{x_n\}$ be a Cauchy sequence in $(X, d_{T,G})$ where $x_0 \in X$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. We have $\lim_{n,m\to\infty} d_{T,G}(x_n, x_m) = 0$. We may assume that $x_n \ne x_m$ for all $n \ne m \in \mathbb{N}$. Then

$$\begin{aligned}0 &\le \lim_{n,m\to\infty} G(x_n,x_m,x_m)\\ &\le \lim_{n,m\to\infty} G(x_n,x_{n+1},x_m)\\ &= \lim_{n,m\to\infty} G(x_n,Tx_n,x_m)\\ &= \lim_{n,m\to\infty} d_{T,G}(x_n,x_m)\\ &= 0.\end{aligned}$$

It implies that $\lim_{n,m\to\infty} G(x_n,x_m,x_m) = 0$. By Lemma 2, $\{x_n\}$ is a Cauchy sequence in (X,G). Since (X,G) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ in (X,G). Since $x_n \ne x_m$ for all $n \ne m \in \mathbb{N}$, we may assume that $x_n \ne x^*$ for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty} d_{T,G}(x_n,x^*) = \lim_{n\to\infty} G(x_n,Tx_n,x^*) = \lim_{n\to\infty} G(x_n,x_{n+1},x^*) = 0. \tag{3.7}$$

We also have

$$\begin{aligned}d_{T,G}(x^*,x_n) &= G(x^*,Tx^*,x_n)\\ &\le G(x^*,x_{n+1},x_{n+1}) + G(x_{n+1},Tx^*,x_n)\\ &= G(x^*,x_{n+1},x_{n+1}) + G(x_n,x_{n+1},Tx^*)\end{aligned} \tag{3.8}$$

$$= G(x^*, x_{n+1}, x_{n+1}) + G(Tx_{n-1}, T^2x_{n-1}, Tx^*)$$
$$\le G(x^*, x_{n+1}, x_{n+1}) + \psi\big(G(x_{n-1}, Tx_{n-1}, x^*)\big) - \varphi\big(G(x_{n-1}, Tx_{n-1}, x^*)\big)$$
$$\le G(x^*, x_{n+1}, x_{n+1}) + \psi\big(G(x_{n-1}, x_n, x^*)\big) - \varphi\big(G(x_{n-1}, x_n, x^*)\big).$$

Taking the limit as $n \to \infty$ in (3.8) and using Lemma 1, we get

$$\lim_{n\to\infty} d_{T,G}(x^*, x_n) = 0. \tag{3.9}$$

From (3.7) and (3.9), we get $\lim_{n\to\infty} x_n = x^*$ in $(X, d_{T,G})$. Then $(X, d_{T,G})$ is weak T-orbitally complete. Note that (3.6) becomes

$$\psi\big(d_{T,G}(Tx, Ty)\big) \le \psi\big(d_{T,G}(x, y)\big) - \varphi\big(d_{T,G}(x, y)\big).$$

Therefore, by using Proposition 8, we conclude that T has a fixed point. It is a contradiction.

By the above, T has a unique fixed point. □

Remark 3. By using $d_{T,G}$ as in the proof of Corollary 3, we see that the inequality (30) in [3, Theorem 3.1] becomes

$$d_{T,G}(Tx, Ty) \ge \alpha d_{T,G}(x, y).$$

By similar arguments, we get analogues of the results in [18] for expansive maps on quasi-metric spaces and then we get [3, Theorem 3.1]. Also, similar arguments to the above may be possible for results in [7]. Note that for a complete G-metric space (X, G) with $|X| \ge 2$ and $T : X \longrightarrow X$ being the identify map, all assumptions of [3, Theorem 3.2] hold but T has more than one fixed point. This shows that the uniqueness of fixed points in [3, Theorem 3.2] is a gap.

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Author's address

N. V. Dung
Dong Thap University, Faculty of Mathematics and Information Technology Teacher Education, 783 Pham Huu Lau St., Cao Lanh City, Dong Thap Province, Viet Nam
E-mail address: nvdung@dthu.edu.vn