



SOME RELATIONS ON KONHAUSER MATRIX POLYNOMIALS

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Abstract. This paper deals with the study of the generalized hypergeometric matrix function and obtains some of its properties. We rephrase some results from the previous (earlier) works that will be used in this study. We get the hypergeometric matrix function representation, matrix differential equation, generating matrix functions, bilinear generating matrix functions, matrix recurrence relations, finite summation formulas and related results for the Konhauser matrix polynomials given in [34]. Finally, we give some important results involving properties of Mittag-Leffler and Bessel-Maitland matrix functions as applications.

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1. INTRODUCTION

In mathematics, the Konhauser polynomials, introduced by Konhauser [20], are biorthogonal polynomials for the distribution function of the Laguerre polynomials [23, 24]. Special matrix functions are very closed to statistics, Lie group theory and number theory are well known in [6, 10, 22]. In the recent work, matrix polynomials have significant emergent in [1–3, 5, 8, 11, 12, 14–19, 21, 26–31, 33, 35]. Results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials in [4, 13]. Hence we feel that the expansions of some functions in terms of the family of Konhauser matrix polynomials $Z_n^A(x; k)$ may be of interest to their intrinsic mathematical importance and to the fact that these polynomials have applications in physics.

Throughout this paper, for a matrix A in $\mathbb{C}^{N \times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A . The two-norm will be denoted by $\|A\|_2$ and it is defined by (see [17])

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where for a vector x in \mathbb{C}^N , $\|x\|_2 = (x^T x)^{\frac{1}{2}}$ is the Euclidean norm of x . Let us denote the real numbers $M(A)$ and $m(A)$ as in the following

$$M(A) = \max\{Re(z) : z \in \sigma(A)\}; \quad m(A) = \min\{Re(z) : z \in \sigma(A)\}. \quad (1.1)$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z , which are defined in an open set Ω of the complex plane, and A, B are matrices in $\mathbb{C}^{N \times N}$ with $\sigma(A) \subset \Omega$ and $\sigma(B) \subset \Omega$, such that $AB = BA$, then from the properties of the matrix functional calculus in [9], it follows that

$$f(A)g(B) = g(B)f(A). \quad (1.2)$$

Throughout this study, a matrix polynomial of degree n in x means an expression of the form

$$P_n(x) = A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0 \quad (1.3)$$

where x is a real variable or complex variable, A_j , for $0 < j < n$ and $A_n \neq \mathbf{0}$, where $\mathbf{0}$ is the null matrix or zero matrix in $\mathbb{C}^{N \times N}$.

We recall that the reciprocal gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z and thus for any matrix A in $\mathbb{C}^{N \times N}$, $\Gamma^{-1}(A)$ is a well defined matrix. Furthermore, if A is a matrix such that

$$A + nI \quad \text{is an invertible matrix for all integers } n \geq 0 \quad (1.4)$$

where I is the identity matrix in $\mathbb{C}^{N \times N}$, then from [8, 11], it follows that

$$(A)_n = A(A+I) \dots (A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A); \quad n \geq 1; \quad (A)_0 = I. \quad (1.5)$$

If k is large enough so then for $k > \|B\|$, then we will mention to the following relation which existed in Jódar and Cortés [14, 16] in the form

$$\|(B+kI)^{-1}\| \leq \frac{1}{k-\|B\|}; \quad k > \|B\|. \quad (1.6)$$

If $A(k, n)$ and $B(k, n)$ are matrices in $\mathbb{C}^{N \times N}$ for $n \geq 0, k \geq 0$, then in an analogous way to the proof of Lemma 11 [8], it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) &= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{1}{2}n]} A(k, n-2k), \\ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n) &= \sum_{n=0}^{\infty} \sum_{k=0}^n B(k, n-k). \end{aligned} \quad (1.7)$$

Similarly to (1.7), we can write

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{1}{2}n]} A(k, n) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+2k), \\ \sum_{n=0}^{\infty} \sum_{k=0}^n B(k, n) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n+k). \end{aligned} \quad (1.8)$$

The hypergeometric matrix function ${}_2F_1(A, B; C; z)$ has been given in the form

$${}_2F_1(A, B; C; z) = \sum_{k=0}^{\infty} \frac{(A)_k (B)_k [(C)_k]^{-1}}{n!} z^k, \quad (1.9)$$

for matrices A, B and C in $\mathbb{C}^{N \times N}$ such that $C + nI$ is an invertible matrix for all integers $n \geq 0$ and for $|z| < 1$. It has been seen by Jódar and Cortés [14] that the series is absolutely convergent for $|z| = 1$ when

$$m(C) > M(A) + M(B),$$

where $m(P)$ and $M(P)$ in (1.1) for any matrix P in $\mathbb{C}^{N \times N}$.

The outline of this paper is as follows: In Section 2, we define generalized hypergeometric matrix function and give the convergence properties, matrix differential equation, matrix recurrence relations and integrals. Section 3, we give the hypergeometric matrix function representation, matrix differential equation, matrix recurrence relations, finite summation formulas, an explicit expression, and a different approach to proof of generating functions of the Konhauser matrix polynomials. Finally in Section 4, we give the definition of the Mittag-Leffler and Bessel-Maitland matrix functions and some properties have also been obtained.

2. GENERALIZED HYPERGEOMETRIC MATRIX FUNCTION

In this section, the generalized hypergeometric matrix function is defined as in the following.

Definition 1. Let us suppose that A_i ; $1 \leq i \leq p$ and B_j ; $1 \leq j \leq q$ are matrices in $\mathbb{C}^{N \times N}$ such that

$$B_j + nI \quad \text{are invertible matrices for all integers } n \geq 0 \quad (2.1)$$

where p and q are finite positive integers. Then we define a generalized hypergeometric matrix function as

$$\begin{aligned} & {}_pF_q(A_1, A_2, \dots, A_p; B_1, B_2, \dots, B_q; z) \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} (A_1)_k (A_2)_k \dots (A_p)_k [(B_1)_k]^{-1} [(B_2)_k]^{-1} \dots [(B_q)_k]^{-1} \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \prod_{i=1}^p (A_i)_k \left[\prod_{j=1}^q (B_j)_k \right]^{-1}. \end{aligned} \quad (2.2)$$

For simplicity, we can write

${}_pF_q(A_1, A_2, \dots, A_p; B_1, B_2, \dots, B_q; z)$ in the form ${}_pF_q$,
 ${}_pF_q(A_1 \pm, A_2, \dots, A_p; B_1, B_2, \dots, B_q; z)$ in the form ${}_pF_q(A_1 \pm)$,
 ${}_pF_q(A_1, A_2 \pm, \dots, A_p; B_1, B_2, \dots, B_q; z)$ in the form ${}_pF_q(A_2 \pm)$,
 ${}_pF_q(A_1, A_2, \dots, A_p \pm; B_1, B_2, \dots, B_q; z)$ in the form ${}_pF_q(A_p \pm)$,
 ${}_pF_q(A_1, A_2, \dots, A_p; B_1 \pm, B_2, \dots, B_q; z)$ in the form ${}_pF_q(B_1 \pm)$,
 ${}_pF_q(A_1, A_2, \dots, A_p; B_1, B_2, \dots, B_q \pm; z)$ in the form ${}_pF_q(B_q \pm)$,
 $\frac{z^k}{k!} (A_1)_k (A_2)_k \dots (A_p)_k [(B_1)_k]^{-1} [(B_2)_k]^{-1} \dots [(B_q)_k]^{-1}$ in the form $U_k(z)$,
and $\frac{1}{k!} (A_1)_k (A_2)_k \dots (A_p)_k [(B_1)_k]^{-1} [(B_2)_k]^{-1} \dots [(B_q)_k]^{-1}$ in the form U_k .

Now, we are going to study the convergence properties of the generalized hypergeometric matrix function. If k is large enough so that $k > \|B_j\|$, then by the perturbation lemma, see [14], it follows

$$\left\| \left(\frac{1}{k} B_j + I \right)^{-1} \right\| \leq \frac{1}{1 - \frac{1}{k} \|B_j\|} = \frac{k}{k - \|B_j\|}; \quad 1 \leq j \leq q$$

and

$$\begin{aligned} \|(B_j + kI)^{-1}\| &= \left\| \frac{1}{k} \left(\frac{1}{k} B_j + I \right)^{-1} \right\| = \frac{1}{k} \left\| \left(\frac{1}{k} B_j + I \right)^{-1} \right\| \\ &\leq \frac{1}{k - \|B_j\|}; \quad k > \|B_j\|; \quad 1 \leq j \leq q. \end{aligned} \quad (2.3)$$

Let's consider the expression

$$T_j(k) = \|B_j^{-1}\| \|(B_j + I)^{-1}\| \dots \|(B_j + (k-1)I)^{-1}\| \quad (2.4)$$

for $k \geq 1$ and $1 \leq j \leq q$.

From (1.5), we obtain

$$\|(A_i)_k\| \leq (\|A_i\|)_k; \quad 1 \leq i \leq p. \quad (2.5)$$

For $k > \|B_j\|$, by using (2.3), (2.4) and (2.5) can be written

$$\begin{aligned} & \left\| \frac{z^k}{k!} (A_1)_k (A_2)_k \dots (A_p)_k [(B_1)_k]^{-1} [(B_2)_k]^{-1} \dots [(B_q)_k]^{-1} \right\| \\ & \leq \frac{1}{k!} |z^k| \| (A_1)_k \| \| (A_2)_k \| \dots \| (A_p)_k \| T_1(k) T_2(k) \dots T_q(k) \\ & \leq \frac{1}{k!} |z^k| (\|A_1\|)_k (\|A_2\|)_k \dots (\|A_p\|)_k T_1(k) T_2(k) \dots T_q(k). \end{aligned} \quad (2.6)$$

Now, we will investigate the convergence of the following series

$$\sum_{k=0}^{\infty} \frac{1}{k!} |z^k| (\|A_1\|)_k (\|A_2\|)_k \dots (\|A_p\|)_k T_1(k) T_2(k) \dots T_q(k).$$

By the ratio test and using the relation (1.6), one gets

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left| \frac{(\|A_1\|)_{k+1} (\|A_2\|)_{k+1} \dots (\|A_p\|)_{k+1} T_1(k+1) T_2(k+1) \dots T_q(k+1) k! z^{k+1}}{(\|A_1\|)_k (\|A_2\|)_k \dots (\|A_p\|)_k T_1(k) T_2(k) \dots T_q(k) (k+1)! z^k} \right| \\ & \leq \lim_{k \rightarrow \infty} \left| \frac{(\|A_1\| + k) \dots (\|A_p\| + k) (\|B_1 + kI\|)^{-1} \dots (\|B_q + kI\|)^{-1} z^{k+1}}{(k+1) z^k} \right| \\ & \leq \lim_{k \rightarrow \infty} \frac{(\|A_1\| + k) (\|A_2\| + k) \dots (\|A_p\| + k)}{(k - \|B_1\|) (k - \|B_2\|) \dots (k - \|B_q\|) (k+1)} |z|. \end{aligned}$$

The last limit shows that

- (1) If $p \leq q$, then the power series (2.2) converges for all finite z .
- (2) If $p = q + 1$, then the power series (2.2) is convergent for $|z| < 1$ and diverges for $|z| > 1$.
- (3) If $p > q + 1$, then the power series (2.2) diverges for all $z, z \neq 0$.

In analogous to Theorem 3 in [14], we can state the following:

Theorem 1. (1) If $p = q + 1$, then the power series (2.2) is absolutely convergent for $|z| = 1$ when

$$\sum_{j=1}^q m(B_j) > \sum_{j=1}^p M(A_j). \quad (2.7)$$

- (2) If $p = q + 1$, then the power series (2.2) is conditionally convergent for $|z| = 1, z \neq 1$ when

$$\sum_{j=1}^p M(A_j) - 1 < \sum_{j=1}^q m(B_j) \leq \sum_{j=1}^p M(A_j). \quad (2.8)$$

- (3) If $p = q + 1$, then the power series (2.2) diverges for $|z| = 1$ when

$$\sum_{j=1}^q m(B_j) \leq \sum_{j=1}^p M(A_j) - 1 \quad (2.9)$$

where $M(A_j)$ and $m(B_j)$ are defined in (1.1).

Consider the differential operator $\theta = z \frac{d}{dz}$, $D_z = \frac{d}{dz}$, $\theta z^k = k z^k$. For these matrices which commute with one another, thus we have

$$\begin{aligned} \theta \prod_{j=1}^q (\theta I + B_j - I) {}_pF_q &= \sum_{k=1}^{\infty} \frac{k z^k}{k!} \prod_{j=1}^q (kI + B_j - I) \prod_{i=1}^p (A_i)_k \left[\prod_{j=1}^q (B_j)_k \right]^{-1} \\ &= \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!} \prod_{i=1}^p (A_i)_k \left[\prod_{j=1}^q (B_j)_{k-1} \right]^{-1}. \end{aligned}$$

Replace k by $k+1$, we have

$$\begin{aligned} \theta \prod_{j=1}^q (\theta I + B_j - I) {}_pF_q &= \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!} \prod_{i=1}^p (A_i)_{k+1} \left[\prod_{j=1}^q (B_j)_k \right]^{-1} \\ &= z \prod_{i=1}^p (\theta I + A_i) {}_pF_q. \end{aligned}$$

Thus, we have shown that ${}_pF_q$ is a solution of the following matrix differential equation

$$\left[\theta \prod_{j=1}^q (\theta I + B_j - I) - z \prod_{i=1}^p (\theta I + A_i) \right] {}_pF_q = \mathbf{0} \quad (2.10)$$

where $\mathbf{0}$ is the null matrix in $\mathbb{C}^{N \times N}$. These results are summarized below.

Theorem 2. Let A_i ; $1 \leq i \leq p$ and B_j ; $1 \leq j \leq q$ be matrices in $\mathbb{C}^{N \times N}$ satisfying (2.1) and all matrices are commutative. The generalized hypergeometric matrix function satisfies the matrix differential equation in (2.10).

The solution is valid for all finite z when $p \leq q$. If $p = q + 1$, the solution is valid in $|z| < 1$. We are not concerning ourselves with the case $p > q + 1$, when the series for ${}_pF_q$ has a zero radius of convergence unless the series terminates.

If $p < q + 1$, the matrix differential equation (2.10) of the preceding section is a linear matrix differential equation of order $(q + 1)$. We have, in the neighborhood of the origin, one solution,

$$\begin{aligned} w_0 &= {}_pF_q = {}_pF_q(A_1, A_2, \dots, A_p; B_1, B_2, \dots, B_q; z) \\ &= \sum_{k=0}^{\infty} \frac{z^k}{n!} (A_1)_k (A_2)_k \dots (A_p)_k [(B_1)_k]^{-1} [(B_2)_k]^{-1} \dots [(B_q)_k]^{-1} \end{aligned}$$

Naturally we wish to obtain q other solutions near $z = 0$.

Let D_0 be the complex plane cut along the negative real axis, and let us denote $z^{I-B_m} = \exp((I - B_m) \log z)$ where \log represents the principal logarithm [25]. We

shall now verify that w_m satisfies equation (2.10) for all commutative matrices. Let us seek a solution, as before, for m any one of the integers $1, 2, 3, \dots, q$, of the form

$$\begin{aligned} w_m &= z^{I-B_m} {}_pF_q(A_1 - B_m + I, A_2 - B_m + I, \dots, A_p - B_m + I, B_1 - B_m + I, \dots, \\ &\quad B_{m-1} - B_m + I, 2I - B_m, B_{m+1} - B_m + I, \dots, B_q - B_m + I; z) \\ &= \sum_{k=0}^{\infty} z^{(k+1)I-B_m} \left[\prod_{i=1}^p (A_i - B_m + I)_k \right] \left[\prod_{j=1}^q (B_j - B_m + I)_k \right]^{-1} \left[(2I - B_m)_k \right]^{-1}; \\ &\quad |z| < 1, z \in D_0. \end{aligned} \tag{2.11}$$

Whenever, in addition to the above restrictions, B_j hasn't any positive integer eigenvalue ($j = 1, 2, \dots, q$), then the linear combination (2.11) is the general solution of equation (2.10) around $z = 0$. Note also that if $p < q$, then the series for w_m ; $m = 0, 1, 2, \dots, q$, converges for all finite z and that for $p = q + 1$, the series for w_m converges for $|z| < 1$. From (2.11), we get

$$\begin{aligned} &\theta \prod_{j=1}^q (\theta I + B_j - I) w_m \\ &= \sum_{k=0}^{\infty} z^{(k+1)I-B_m} \left((k+1)I - B_m \right) \left[\prod_{j=1}^q ((k+1)I - B_m + B_j - I) \right] \\ &\quad \left[\prod_{i=1}^p (A_i - B_m + I)_k \right] \left[\prod_{j=1}^q (B_j - B_m + I)_k \right]^{-1} \left[(2I - B_m)_k \right]^{-1}. \end{aligned}$$

Now $(B_j - B_m + kI)$ is the last factor in the product $(B_j - B_m + I)_k$ and $(kI + I - B_m)$ is the last factor in the product $(2I - B_m)_k$.

Hence,

$$\begin{aligned} \theta \prod_{j=1}^q (\theta I + B_j - I) w_m &= \sum_{k=1}^{\infty} z^{(k+1)I-B_m} \left[\prod_{i=1}^p (A_i - B_m + I)_k \right] \\ &\quad \left[\prod_{j=1}^q (B_j - B_m + I)_{k-1} \right]^{-1} \left[(2I - B_m)_{k-1} \right]^{-1}. \end{aligned}$$

Next, with a shift of index from k to $(k + 1)$, we get

$$\begin{aligned}
 & \theta \prod_{j=1}^q (\theta I + B_j - I) w_m \\
 &= \sum_{k=0}^{\infty} z^{(k+2)I-B_m} \left[\prod_{i=1}^p (A_i - B_m + I)_{k+1} \right] \left[\prod_{j=1}^q (B_j - B_m + I)_k \right]^{-1} \left[(2I - B_m)_k \right]^{-1} \\
 &= z \sum_{k=0}^{\infty} z^{(k+1)I-B_m} \left[\prod_{i=1}^p (A_i - B_m + (k+1)I) \right] \left[\prod_{i=1}^p (A_i - B_m + I)_k \right] \\
 & \quad \left[\prod_{j=1}^q (B_j - B_m + I)_k \right]^{-1} \left[(2I - B_m)_k \right]^{-1}.
 \end{aligned}$$

Therefore, we have

$$\theta \prod_{j=1}^q (\theta I + B_j - I) w_m = z \prod_{i=1}^p (\theta I + A_i) w_m \quad (2.12)$$

so that, w_m ; $m = 1, 2, \dots, q$, satisfies the matrix differential equation of (2.11) for $z \in D_0$ and $|z| < 1$.

Assume A_i ($i = 1, 2, \dots, p$) and B_j ($j = 1, 2, \dots, q$) haven't any integer eigenvalues are these matrices which commute with one another. The relation $A_i(A_i + I)_k = (A_i + kI)(A_i)_k$ together with the definitions of the matrix contiguous relations, yield the formulas:

$$\begin{aligned}
 {}_pF_q(A_1 +) &= \sum_{k=0}^{\infty} \frac{z^k}{n!} (A_1 + I)_k (A_2)_k \dots (A_p)_k [(B_1)_k]^{-1} [(B_2)_k]^{-1} \dots [(B_q)_k]^{-1} \\
 &= \sum_{k=0}^{\infty} \frac{z^k}{n!} (A_1 + kI) \left(A_1 \right)^{-1} (A_1)_k (A_2)_k \dots (A_p)_k [(B_1)_k]^{-1} [(B_2)_k]^{-1} \dots [(B_q)_k]^{-1} \\
 &= \sum_{k=0}^{\infty} (A_1 + kI) \left(A_1 \right)^{-1} U_k(z).
 \end{aligned} \quad (2.13)$$

Similarly, we get

$$\begin{aligned}
 {}_pF_q(A_i+) &= \left(A_i\right)^{-1} \sum_{k=0}^{\infty} (A_i + kI) U_k(z), \\
 {}_pF_q(A_i-) &= (A_i - I) \sum_{k=0}^{\infty} \left(A_i + (k-1)I\right)^{-1} U_k(z), \\
 {}_pF_q(B_j+) &= (B_j) \sum_{k=0}^{\infty} \left(B_j + kI\right)^{-1} U_k(z), \\
 {}_pF_q(B_j-) &= \left(B_j - I\right)^{-1} \sum_{k=0}^{\infty} (B_j + (k-1)I) U_k(z).
 \end{aligned} \tag{2.14}$$

For all integers $m \geq 1$, we deduce that

$$\begin{aligned}
 {}_pF_q(A_i + mI) &= \prod_{r=1}^m \left(A_i + (r-1)I\right)^{-1} \sum_{k=0}^{\infty} \prod_{r=1}^m (A_i + (k+r-1)I) U_k(z), \\
 {}_pF_q(A_i - mI) &= \prod_{r=1}^m (A_i - rI) \sum_{k=0}^{\infty} \prod_{r=1}^m \left(A_i + (k-r)I\right)^{-1} U_k(z), \\
 {}_pF_q(B_j + mI) &= \prod_{r=1}^m (B_j + (r-1)I) \sum_{k=0}^{\infty} \prod_{r=1}^m \left(B_j + (k+r-1)I\right)^{-1} U_k(z)
 \end{aligned}$$

and

$${}_pF_q(B_j - mI) = \prod_{r=1}^m \left(B_j - rI\right)^{-1} \sum_{k=0}^{\infty} \prod_{r=1}^m (B_j + (k-r)I) U_k(z).$$

Using the differential operator $\theta = z \frac{d}{dz}$, we see that

$$(\theta I + A_i) {}_pF_q = \sum_{k=0}^{\infty} (A_i + kI) U_k(z).$$

Hence, with the aid of (2.14),

$$(\theta I + A_i) {}_pF_q = A_i {}_pF_q(A_i+); i = 1, 2, \dots, p. \tag{2.15}$$

Similarly, we have

$$(\theta I + B_j - I) {}_pF_q = (B_j - I) {}_pF_q(B_j-); j = 1, 2, \dots, q. \tag{2.16}$$

The result is the set of simple relations,

$$(A_1 - A_i) {}_pF_q = A_1 {}_pF_q(A_1+) - A_i {}_pF_q(A_i+); i = 2, 3, \dots, p \tag{2.17}$$

and

$$(A_1 - B_j + I) {}_pF_q = A_1 {}_pF_q(A_1 +) - (B_j - I) {}_pF_q(B_j -); j = 1, 2, \dots, q. \quad (2.18)$$

It is well known that

$$D_z {}_pF_q = A_1 \dots A_p \left(B_1 \right)^{-1} \dots \left(B_q \right)^{-1} {}_pF_q(A_1 + I, \dots, A_p + I; B_1 + I, \dots, B_q + I; z); D_z = \frac{d}{dz} \quad (2.19)$$

and in general

$$D_z^k {}_pF_q = (A_1)_k (A_2)_k \dots (A_p)_k \left[(B_1)_k \right]^{-1} \left[(B_2)_k \right]^{-1} \dots \left[(B_q)_k \right]^{-1} {}_pF_q(A_1 + kI, \dots, A_p + kI; B_1 + kI, \dots, B_q + kI; z). \quad (2.20)$$

2.1. Integrals involving generalized Hypergeometric matrix function

The integral representation Eq. (45) in [14] of the generalized hypergeometric matrix function can be extended to obtain the following result:

Theorem 3. Let A_i ; $1 \leq i \leq p$ and B_j ; $1 \leq j \leq q$ be matrices in $\mathbb{C}^{N \times N}$ such that $B_j + kI$ are invertible matrices for all integers $k \geq 0$. Suppose that the matrices A_1 , B_1 and $B_1 - A_1$ are positive stable matrices. If $p \leq q + 1$, then for $|z| < 1$ and $A_i B_j = B_j A_i$, it follows that

$${}_pF_q(A_1, A_2, \dots, A_p; B_1, B_2, \dots, B_q; z) = \Gamma^{-1}(A_1) \Gamma^{-1}(B_1 - A_1) \Gamma(B_1) \int_0^1 t^{A_1 - I} (1-t)^{B_1 - A_1 - I} {}_{p-1}F_{q-1}(A_2, \dots, A_p; B_2, \dots, B_q; zt) dt. \quad (2.21)$$

The proof is similar to the proof of Theorem 5 in [14].

These integral representation can be generalized through the following theorems.

Theorem 4. Let A_i ; $1 \leq i \leq p$ and B_j ; $1 \leq j \leq q$ be matrices in $\mathbb{C}^{N \times N}$ such that $B_j + kI$ are invertible matrices for all integers $k \geq 0$. Suppose that A_i , B_j and $B_j - A_i$ are positive stable matrices. If $p \leq q + 1$, for $|z| < 1$

$$\begin{aligned} & {}_pF_q(A_1, A_2, \dots, A_p; B_1, B_2, \dots, B_q; z) \\ &= \Gamma^{-1}(A_i) \Gamma^{-1}(B_j - A_i) \Gamma(B_j) \int_0^1 t^{A_i - I} (1-t)^{B_j - A_i - I} \\ & \quad \times {}_{p-1}F_{q-1}\left(\begin{matrix} A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_p; \\ B_1, \dots, B_{j-1}, B_{j+1}, \dots, B_q \end{matrix}; zt\right). \end{aligned} \quad (2.22)$$

Proof. By extending Eq. (35) in [14], we have

$$(A_i)_n (B_j)_n^{-1} = \Gamma^{-1}(A_i) \Gamma^{-1}(B_j - A_i) \Gamma(B_j) \int_0^1 t^{A_i + (n-1)I} (1-t)^{B_j - A_i - I} dt$$

where $A_i B_j = B_j A_i$. Also we have

$$\begin{aligned} & {}_p F_q \left(\begin{matrix} A_1, A_2, \dots, A_p; \\ B_1, B_2, \dots, B_q; \end{matrix} z \right) \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} (A_1)_k \dots (A_i)_k \dots (A_p)_k [(B_1)_k]^{-1} \dots [(B_j)_k]^{-1} \dots [(B_q)_k]^{-1} \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} (A_1)_k \dots (A_{i-1})_k (A_{i+1})_k \dots (A_p)_k [(B_1)_k]^{-1} \dots [(B_{j-1})_k]^{-1} [(B_{j+1})_k]^{-1} \\ &\quad \dots [(B_q)_k]^{-1} \times \Gamma^{-1}(A_i) \Gamma^{-1}(B_j - A_i) \Gamma(B_j) \int_0^1 t^{A_i + (n-1)I} (1-t)^{B_j - A_i - I} dt \\ &= \Gamma^{-1}(A_i) \Gamma^{-1}(B_j - A_i) \Gamma(B_j) \int_0^1 t^{A_i - I} (1-t)^{B_j - A_i - I} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(zt)^k}{k!} (A_1)_k \dots (A_{i-1})_k (A_{i+1})_k \dots (A_p)_k \\ &\quad [(B_1)_k]^{-1} \dots [(B_{j-1})_k]^{-1} [(B_{j+1})_k]^{-1} \dots [(B_q)_k]^{-1} dt \\ &= \Gamma^{-1}(A_i) \Gamma^{-1}(B_j - A_i) \Gamma(B_j) \int_0^1 t^{A_i - I} (1-t)^{B_j - A_i - I} {}_{p-1} F_{q-1} dt, \end{aligned}$$

$$\text{where } {}_{p-1} F_{q-1} = {}_{p-1} F_{q-1} \left(\begin{matrix} A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_p; \\ B_1, \dots, B_{j-1}, B_{j+1}, \dots, B_q; \end{matrix} zt \right).$$

□

Now, we are going to evaluate some matrix integrals involving generalized Hypergeometric matrix function. Let us consider the integral representation

$$J = \int_0^1 {}_p F_q \left(\begin{matrix} A_1, A_2, \dots, A_p; \\ B_1, B_2, \dots, B_q; \end{matrix} \alpha x^m (t-x)^s \right) x^{C-I} (t-x)^{D-I} dx$$

where C , D and $C + D$ are positive stable matrices, $CD = DC$, and $m, s \in \mathbb{N}$ and $B_j C = C B_j$, $B_j D = D B_j$ for $j = 1, 2, \dots, m$.

To evaluate the integral representation, let us substitute $x = ty$, which gives

$$\begin{aligned}
 J &= \int_0^1 {}_pF_q(A_1, A_2, \dots, A_p; B_1, B_2, \dots, B_q; \alpha t^{m+s} y^m (1-y)^s) \\
 &\quad t^{C+D-I} y^{C-I} (1-y)^{D-I} dy \\
 &= \int_0^1 \sum_{k=0}^{\infty} \frac{t^{k(m+s)} \alpha^k}{k!} \prod_{i=1}^p (A_i)_k \left[\prod_{j=1}^q (B_j)_k \right]^{-1} \\
 &\quad y^{C+(mk-1)I} (1-y)^{D+(sk-1)I} t^{C+D-I} dy \\
 &= \sum_{k=0}^{\infty} \frac{\alpha^k t^{k(m+s)}}{k!} \prod_{i=1}^p (A_i)_k \left[\prod_{j=1}^q (B_j)_k \right]^{-1} B(C + mkI, D + skI) t^{C+D-I}.
 \end{aligned}$$

We can write

$$\begin{aligned}
 B(C + mkI, D + skI) &= \Gamma(C + mkI) \Gamma(D + skI) \Gamma^{-1}(C + D + k(m+s)I) \\
 &= \Gamma(C) \Gamma(D) \Gamma^{-1}(C + D) (C)_{mk} (D)_{sk} \left[(C + D)_{(m+s)k} \right]^{-1}.
 \end{aligned}$$

Note that we have used here the following elementary result:

$$(A)_{km} = k^{km} \left(\frac{A}{k} \right)_m \left(\frac{A+I}{k} \right)_m \cdots \left(\frac{A+(k-1)I}{k} \right)_m. \quad (2.23)$$

By using (2.23), we have

$$\begin{aligned}
 B(C + mkI, D + skI) &= \frac{m^{mk} s^{sk}}{(m+s)^{(m+s)k}} B(C, D) \prod_{u=1}^m \left(\frac{C + (u-1)I}{m} \right)_k \\
 &\quad \times \prod_{w=1}^s \left(\frac{D + (w-1)I}{s} \right)_k \left[\prod_{v=1}^{m+s} \left(\frac{C + D + (v-1)I}{m+s} \right)_k \right]^{-1}.
 \end{aligned}$$

Hence we get

$$\begin{aligned}
 J &= \sum_{k=0}^{\infty} \frac{m^{mk} s^{sk} \alpha^k t^{k(m+s)}}{k! (m+s)^{(m+s)k}} \\
 &\quad \times \prod_{i=1}^p (A_i)_k \prod_{u=1}^m \left(\frac{C + (u-1)I}{m} \right)_k \prod_{w=1}^s \left(\frac{D + (w-1)I}{s} \right)_k \\
 &\quad \times \left[\prod_{j=1}^q (B_j)_k \right]^{-1} \left[\prod_{v=1}^{m+s} \left(\frac{C + D + (v-1)I}{m+s} \right)_k \right]^{-1} t^{C+D-I} B(C, D) \\
 &= {}_{p+m+s}F_{q+m+s} t^{C+D-I} B(C, D),
 \end{aligned}$$

where

$${}_{p+m+s}F_{q+m+s} = {}_{p+m+s}F_{q+m+s} \left(\begin{matrix} A_1, \dots, A_p, \frac{C}{m}, \dots, \frac{C+(m-1)I}{m}, \frac{D}{m}, \dots, \frac{D+(s-1)I}{m}; & m^m s^s \alpha t^{(m+s)} \\ B_1, \dots, B_q, \frac{C+D}{m+s}, \dots, \frac{C+D+(m+s-1)I}{m+s}; & (m+s)^{(m+s)} \end{matrix} \right).$$

Theorem 5. Let $A_i; 1 \leq i \leq p$ and $B_j; 1 \leq j \leq q$ be matrices in $\mathbb{C}^{N \times N}$ such that $B_j + kI$ are invertible matrices for all integers $k \geq 0$. Suppose that C, D and $C + D$ are positive stable matrices such that $CD = DC, B_j C = C B_j, B_j D = D B_j$ and $m, s \in \mathbb{N}$. If $p \leq q + 1$, then the following integral representation is hold

$$\begin{aligned} & \int_0^1 {}_pF_q \left(\begin{matrix} A_1, A_2, \dots, A_p; \\ B_1, B_2, \dots, B_q; \end{matrix} \alpha x^m (t-x)^s \right) x^{C-I} (t-x)^{D-I} dx \\ &= {}_{p+m+s}F_{q+m+s} \left(\begin{matrix} A_1, \dots, A_p, \frac{C}{m}, \dots, \frac{C+(m-1)I}{m}, \frac{D}{m}, \dots, \frac{D+(s-1)I}{m}; & m^m s^s \alpha t^{(m+s)} \\ B_1, \dots, B_q, \frac{C+D}{m+s}, \dots, \frac{C+D+(m+s-1)I}{m+s}; & (m+s)^{(m+s)} \end{matrix} \right) \\ & t^{C+D-I} B(C, D). \end{aligned} \quad (2.24)$$

Corollary 1. Let $A_i; 1 \leq i \leq p$ and $B_j; 1 \leq j \leq q$ be matrices in $\mathbb{C}^{N \times N}$ such that $B_j + kI$ are invertible matrices for all integers $k \geq 0$. Let C, D and $C + D$ be positive stable matrices such that $CD = DC$ and $m \in \mathbb{N}$. If $p \leq q + 1$, then the following integral representation is hold

$$\begin{aligned} & \int_0^1 {}_pF_q \left(\begin{matrix} A_1, A_2, \dots, A_p; \\ B_1, B_2, \dots, B_q; \end{matrix} \alpha x^m \right) x^{C-I} (t-x)^{D-I} dx \\ &= {}_{p+m}F_{q+m} \left(\begin{matrix} A_1, \dots, A_p, \frac{C}{m}, \dots, \frac{C+(m-1)I}{m}; \\ B_1, \dots, B_q, \frac{C+D}{m}, \dots, \frac{C+D+(m-1)I}{m}; \end{matrix} \alpha t^m \right) t^{C+D-I} B(C, D), \end{aligned} \quad (2.25)$$

where $B_j C = C B_j$ and

$$\begin{aligned} & \int_0^1 {}_pF_q \left(\begin{matrix} A_1, A_2, \dots, A_p; \\ B_1, B_2, \dots, B_q; \end{matrix} \alpha (t-x)^s \right) x^{C-I} (t-x)^{D-I} dx \\ &= {}_{p+s}F_{q+s} \left(\begin{matrix} A_1, \dots, A_p, \frac{D}{s}, \dots, \frac{D+(s-1)I}{s}; \\ B_1, \dots, B_q, \frac{C+D}{s}, \dots, \frac{C+D+(s-1)I}{s}; \end{matrix} \alpha t^s \right) t^{C+D-I} B(C, D), \end{aligned} \quad (2.26)$$

where $B_j D = D B_j$.

3. ON KONHAUSER MATRIX POLYNOMIALS

Let A be a matrix in $\mathbb{C}^{N \times N}$ satisfying the condition

$$\operatorname{Re}(\mu) > -1, \quad \text{for all eigenvalues } \mu \in \sigma(A) \quad (3.1)$$

and λ is a complex number with $\operatorname{Re}(\lambda) > 0$, $k \in \mathbb{N} = \{1, 2, 3, \dots\}$, and let us consider the Konhauser matrix polynomials $Z_n^{(A, \lambda)}(x; k)$ by means of the relation in the form

$$Z_n^{(A, \lambda)}(x; k) = \Gamma(knI + A + I) \sum_{m=0}^n \frac{(-1)^m (\lambda x)^{mk}}{(n-m)!m!} \Gamma^{-1}(kmI + A + I) \quad (3.2)$$

or

$$Z_n^{(A, \lambda)}(x; k) = \sum_{m=0}^n \frac{(-1)^m (\lambda x)^{mk}}{(n-m)!m!} (A + I)_{kn} [(A + I)_{km}]^{-1}$$

given in [34].

From (3.2) and (2.23), we obtain the hypergeometric matrix function representation

$$Z_n^{(A, \lambda)}(x; k) = \frac{(A + I)_{kn}}{n!} {}_1F_k \left(-nI; \frac{A + I}{k}, \dots, \frac{A + kI}{k}; \left(\frac{\lambda x}{k} \right)^k \right) \quad (3.3)$$

which can alternatively be used to derive the following properties of the Konhauser matrix polynomials $Z_n^{(A, \lambda)}(x; k)$ by simply specializing those of the generalized hypergeometric matrix function ${}_pF_q$ (see [34]).

The Konhauser matrix differential equation (3.4) below follows directly by using (2.10), we set $p = 1$, $q = k$, $z = (\frac{\lambda x}{k})^k$, $\theta = k^{-1}\delta$, where $\delta = x \frac{d}{dx}$, and apply the hypergeometric matrix function representation (3.3), and so we arrive at the following theorem:

Theorem 6. Suppose that A be a matrix in $\mathbb{C}^{N \times N}$ satisfying the condition (3.1). Then the Konhauser matrix polynomials is a solution of the matrix differential equation of order $k + 1$ in the form:

$$\left[\delta \prod_{i=1}^k (\delta I + A + (i - k)I) - (\lambda x)^k (\delta - kn)I \right] Z_n^{(A, \lambda)}(x; k) = \mathbf{0}. \quad (3.4)$$

3.1. Generating matrix functions for Konhauser matrix polynomials

The primary purpose of this section is to propose new generating matrix functions of a certain matrix functions in terms of the Konhauser matrix polynomials.

From (2.23) and (3.2), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} Z_n^{(A,\lambda)}(x;k)[(A+I)_{kn}]^{-1}t^n &= \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(-1)^m (\lambda x)^{mk}}{m!(n-m)!} [(A+I)_{km}]^{-1}t^n \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda x)^{mk}}{m!n!} [(A+I)_{km}]^{-1}t^{n+m} \\
 &= \sum_{n=0}^{\infty} \frac{1}{n!} t^n \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda x)^{mk}}{m!} [(A+I)_{km}]^{-1}t^m \\
 &= \sum_{n=0}^{\infty} \frac{1}{n!} t^n \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda x)^{mk}}{k^{km}m!} \left[\prod_{r=1}^k \left(\frac{A+rI}{k} \right)_m \right]^{-1} t^m \\
 &= e^t {}_0F_k \left(-; \frac{A+I}{k}, \frac{A+2I}{k}, \dots, \frac{A+kI}{k}; -\frac{(\lambda x)^k t}{k^k} \right).
 \end{aligned}$$

We can write the new generating matrix function which represents an explicit representation for the Konhauser matrix polynomials in the form

$$\begin{aligned}
 \sum_{n=0}^{\infty} Z_n^{(A,\lambda)}(x;k)[(A+I)_{kn}]^{-1}t^n \\
 = e^t {}_0F_k \left(-; \frac{A+I}{k}, \frac{A+2I}{k}, \dots, \frac{A+kI}{k}; -\frac{(\lambda x)^k t}{k^k} \right). \quad (3.5)
 \end{aligned}$$

Now, we consider the double series

$$\begin{aligned}
 \sum_{m=0}^{\infty} u^m \sum_{n=0}^{\infty} \binom{m+n}{n} [(A+I)_{k(m+n)}]^{-1} Z_{m+n}^{(A,\lambda)}(x;k)t^n \\
 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u^m \binom{m+n}{n} [(A+I)_{k(m+n)}]^{-1} Z_{m+n}^{(A,\lambda)}(x;k)t^n \\
 = \sum_{n=0}^{\infty} \sum_{m=0}^n u^m \binom{n}{m} [(A+I)_{kn}]^{-1} Z_n^{(A,\lambda)}(x;k)t^{n-m} \quad (3.6) \\
 = \sum_{n=0}^{\infty} [(A+I)_{kn}]^{-1} Z_n^{(A,\lambda)}(x;k)(u+t)^n \\
 = e^{u+t} {}_0F_k \left(-; \frac{A+I}{k}, \dots, \frac{A+kI}{k}; -\left(\frac{\lambda x}{k} \right)^k (u+t) \right).
 \end{aligned}$$

By (3.5), we have

$$\begin{aligned} & \sum_{n,v=0}^{\infty} \frac{(-(\lambda x)^k)^n}{n!v!} [(A+I)_{kn}]^{-1} \sum_{m=0}^{n+v} \binom{n+v}{m} t^{n+v-m} u^m \\ &= \sum_{m=0}^{\infty} u^m \sum_{n+v \geq m} \binom{n+v}{m} \frac{(-(\lambda x)^k)^n t^{n-m}}{n!} [(A+I)_{kn}]^{-1} \frac{t^v}{v!}. \end{aligned} \quad (3.7)$$

On equating the coefficients of u^m , the new generating matrix function which represents an explicit representation for the Konhauser matrix polynomials gets in the form

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{m+n}{n} [(A+I)_{k(m+n)}]^{-1} Z_{m+n}^{(A,\lambda)}(x;k) t^n \\ &= \sum_{n=m}^{\infty} \binom{n}{m} \frac{(-(\lambda x)^k)^n t^{n-m}}{n!} [(A+I)_{kn}]^{-1} {}_1F_1((n+1)I; (n-m+1)I; t) \end{aligned} \quad (3.8)$$

which holds true for every non-negative integer m .

Now, we prove a class of new generating matrix functions for the polynomials $Z_n^{(A,\lambda)}(x;k)$ in the following theorem:

Theorem 7. Let a and b be complex constants, not both zero. Then we have

$$\begin{aligned} & \sum_{n=0}^{\infty} Z_n^{(A,\lambda)}\left(\frac{x}{(a+bn)^{\frac{1}{k}}}; k\right) (a+bn)^n [(A+I)_{nk}]^{-1} t^n \\ &= \frac{e^{av}}{1-bu} {}_0F_k\left(-; \frac{A+I}{k}, \frac{A+2I}{k}, \dots, \frac{A+kI}{k}; -u \left(\frac{\lambda x}{k}\right)^k\right) \end{aligned} \quad (3.9)$$

where

$$u = t e^{bv}. \quad (3.10)$$

Proof. Starting with the left-hand side of (3.9), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} Z_n^{(A,\lambda)}\left(\frac{x}{\sqrt[k]{a+bn}}; k\right) (a+bn)^n [(A+I)_{nk}]^{-1} t^n \\ &= \sum_{r=0}^{\infty} \frac{(-tx^k)^r}{r!} [(A+I)_{kr}]^{-1} \sum_{n=0}^{\infty} \frac{(a+br+bn)^n}{n!} t^n. \end{aligned} \quad (3.11)$$

Now by setting $f(x) = e^{ax}$ and $\phi(x) = e^{bx}$ in the Lagrange expansion formula

$$\frac{f(v)}{1-t\phi'(v)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[D^n f(x) (\phi(x))^n \right]_{|x=0, u=t\phi(x)}, \quad (3.12)$$

given in [7], we obtain the following elegant result

$$\frac{e^{av}}{1-bu} = \sum_{n=0}^{\infty} \frac{(a+bn)^n}{n!} t^n \quad (3.13)$$

where u is given by (3.10).

From (3.11), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} Z_n^{(A,\lambda)} \left(\frac{x}{\sqrt[k]{a+bn}}; k \right) (a+bn)^n [(A+I)_{nk}]^{-1} t^n \\ &= \sum_{r=0}^{\infty} \frac{(-t(\lambda x)^k)^r}{r!} [(A+I)_{kr}]^{-1} \frac{e^{(a+br)v}}{1-bu} \\ &= \frac{e^{av}}{1-bu} {}_0F_k \left(-; \frac{A+I}{k}, \frac{A+2I}{k}, \dots, \frac{A+kI}{k}; -u \left(\frac{\lambda x}{k} \right)^k \right) \end{aligned} \quad (3.14)$$

and the proof is completed. \square

For instance, we mention the following particular case. For $a = 1$ and $b = 0$, our theorem yields the generating matrix function (3.5).

Theorem 8. Let A and B be matrices in $\mathbb{C}^{N \times N}$ such that A satisfies the condition (3.1) and λ is a complex number with $\operatorname{Re}(\lambda) > 0$. Then we have

$$\begin{aligned} & \sum_{n=0}^{\infty} (B)_{nk} Z_n^{(A,\lambda)}(x; k) [(A+I)_{nk}]^{-1} t^n = \sum_{r=0}^{\infty} \frac{(-t(\lambda x)^k)^r (B)_{rk}}{r!} [(A+I)_{kr}]^{-1} \\ & {}_kF_0 \left(\frac{B+rkI}{k}, \frac{B+(rk+1)I}{k}, \dots, \frac{B+(rk+k-1)I}{k}; -; t k^k \right), |t k^k| < 1. \end{aligned} \quad (3.15)$$

Proof. This can be proved by using equations (2.23) and (3.2). \square

Substituting $B = A + I$ in equation (3.15), we get the following corollary.

Corollary 2. Let A be a matrix in $\mathbb{C}^{N \times N}$ satisfying the condition (3.1), $|t k^k| < 1$ and λ is a complex number with $\operatorname{Re}(\lambda) > 0$. Then the Konhauser matrix polynomials satisfy the following generating matrix function:

$$\begin{aligned} & \sum_{n=0}^{\infty} Z_n^{(A,\lambda)}(x; k) t^n = \sum_{r=0}^{\infty} \frac{(-t(\lambda x)^k)^r}{r!} \\ & {}_kF_0 \left(\frac{A+(rk+1)I}{k}, \frac{A+(rk+2)I}{k}, \dots, \frac{A+(rk+k)I}{k}; -; t k^k \right). \end{aligned} \quad (3.16)$$

Theorem 9. Let A and B be matrices in $\mathbb{C}^{N \times N}$ such that $A - knI$ satisfies the condition (3.1), $|(\lambda x)^k t| < 1$, $|(-k)^k t| < 1$ and $\left| \frac{(-k)^k t}{1 + (\lambda x)^k t} \right| < 1$, then the Konhauser matrix polynomials have the linear generating matrix functions

$$\sum_{n=0}^{\infty} (B)_n Z_n^{(A-knI, \lambda)}(x; k) t^n = (1 + (\lambda x)^k t)^{-B} {}_k F_0 \left(B, -\frac{A}{k}, \frac{I-A}{k}, \dots, \frac{(k-1)I-A}{k}; -; \frac{(-k)^k t}{1 + (\lambda x)^k t} \right) \quad (3.17)$$

and

$$\sum_{n=0}^{\infty} Z_n^{(A-knI, \lambda)}(x; k) t^n = \exp(-(\lambda x)^k t) {}_k F_0 \left(-\frac{A}{k}, \frac{I-A}{k}, \dots, \frac{(k-1)I-A}{k}; -; (-k)^k t \right). \quad (3.18)$$

Proof. From (1.8), (2.23) and (3.2), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} (B)_n Z_n^{(A-knI, \lambda)}(x; k) t^n \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(-1)^m (\lambda x)^{mk}}{m!(n-m)!} (B)_n \Gamma(A+I) \Gamma^{-1}(A+I-k(n-m)I) t^n \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda x)^{mk}}{m!n!} (B)_{n+m} \Gamma(A+I) \Gamma^{-1}(A+I-knI) t^{n+m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda x)^{mk}}{m!n!} (B)_n (B+nI)_m \Gamma(A+I) \Gamma^{-1}(A+I-knI) t^{n+m} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (B)_n \Gamma(A+I) \Gamma^{-1}(A+I-knI) t^n \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda x)^{mk}}{m!} (B+nI)_m t^m \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (B)_n (-1)^{kn} (-A)_{kn} (1 + (\lambda x)^k t)^{-(B+nI)} t^n \\ &= (1 + (\lambda x)^k t)^{-B} \sum_{n=0}^{\infty} \frac{1}{n!} (B)_n (-1)^{kn} (-A)_{kn} \left(\frac{t}{1 + (\lambda x)^k t} \right)^n \\ &= (1 + (\lambda x)^k t)^{-B} \sum_{n=0}^{\infty} \frac{1}{n!} (B)_n (-1)^{kn} k^{kn} \prod_{r=1}^k \left(\frac{(r-1)I-A}{k} \right)_n \left(\frac{t}{1 + (\lambda x)^k t} \right)^n \end{aligned}$$

$$\begin{aligned}
&= (1 + (\lambda x)^k t)^{-B} \sum_{n=0}^{\infty} \frac{1}{n!} (B)_n (-k)^{kn} \prod_{r=1}^k \left(\frac{(r-1)I - A}{k} \right)_n \left(\frac{t}{1 + (\lambda x)^k t} \right)^n \\
&= (1 + (\lambda x)^k t)^{-B} {}_{k+1}F_0 \left(B, -\frac{A}{k}, \frac{I-A}{k}, \dots, \frac{(k-1)I-A}{k}; -; \frac{(-k)^k t}{1 + (\lambda x)^k t} \right).
\end{aligned}$$

which gives (3.17). Again, using the relations (1.8), (2.23) and (3.2), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} Z_n^{(A-knI, \lambda)}(x; k) t^n \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(-1)^m (\lambda x)^{mk}}{m!(n-m)!} \Gamma(A+I) \Gamma^{-1}(A+I-k(n-m)I) t^n \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda x)^{mk}}{m!n!} \Gamma(A+I) \Gamma^{-1}(A+I-knI) t^{n+m} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma(A+I) \Gamma^{-1}(A+I-knI) t^n \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda x)^{mk}}{m!} t^m \\
&= \exp(-(\lambda x)^k t) \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma(A+I) \Gamma^{-1}(A+I-knI) t^n \\
&= \exp(-(\lambda x)^k t) \sum_{n=0}^{\infty} \frac{(-1)^{kn}}{n!} (-A)_{kn} t^n \\
&= \exp(-(\lambda x)^k t) \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{kn} k^{kn} \prod_{r=1}^k \left(\frac{(r-1)I - A}{k} \right)_n t^n \\
&= \exp(-(\lambda x)^k t) {}_kF_0 \left(-\frac{A}{k}, \frac{I-A}{k}, \dots, \frac{(k-1)I-A}{k}; -; (-k)^k t \right)
\end{aligned}$$

which leads to (3.18). Thus, the proof is completed. \square

3.2. Matrix recurrence relations for Konhauser matrix polynomials

In this section, we give the matrix recurrence relations for Konhauser matrix polynomials $Z_n^{(A, \lambda)}(x; k)$ together with their proofs.

Theorem 10. Let A be a matrix in $\mathbb{C}^{N \times N}$ satisfying the condition (3.1) and λ is a complex number with $\operatorname{Re}(\lambda) > 0$. The Konhauser matrix polynomials satisfy

$$(x^{1-k} D_x)^r Z_n^{(A, \lambda)}(x; k) = (-k)^r \lambda^{kr} Z_{n-r}^{(A+krI, \lambda)}(x; k); n \geq r \geq 0. \quad (3.19)$$

Proof. From (2.19) and (2.20), whence, by setting $p = 1$, $q = k$, $z = (\frac{\lambda x}{k})^k$, $\frac{d}{dz} = \lambda^{-k} (\frac{k}{x})^{k-1} \frac{d}{dx}$, and applying (3.3), we obtain (3.19). \square

Theorem 11. Let A and $A - I$ be matrices in $\mathbb{C}^{N \times N}$ satisfying the condition (3.1) and λ is a complex number with $\operatorname{Re}(\lambda) > 0$. The Konhauser matrix polynomials verify the following properties

$$x D_x Z_n^{(A, \lambda)}(x; k) = (knI + A) Z_n^{(A-I, \lambda)}(x; k) - A Z_n^{(A, \lambda)}(x; k) \quad (3.20)$$

and

$$\begin{aligned} Z_n^{(A, \lambda)}(x; k) - Z_n^{(A-I, \lambda)}(x; k) \\ = k \Gamma(knI + A) \Gamma^{-1}(k(n-1)I + A + I) Z_{n-1}^{(A, \lambda)}(x; k). \end{aligned} \quad (3.21)$$

Proof. Similarly, from the known results (2.15), (2.16) and (2.18), involving the generalized hypergeometric matrix function (3.3), we readily obtain the following mixed recurrence relations (3.20). It is not difficult to verify that the recurrence relation (3.21) results from (3.20) by eliminating their common term $x D_x Z_n^{(A, \lambda)}(x; k)$. \square

It is worth mentioning that the Konhauser matrix polynomials satisfy the three terms recurrence relationship as given in the following theorem:

Theorem 12. Let A and $A - I$ be matrices in $\mathbb{C}^{N \times N}$ satisfying condition (3.1) and λ is a complex number with $\operatorname{Re}(\lambda) > 0$. The Konhauser matrix polynomials satisfying the following pure matrix recurrence relations:

$$k(\lambda x)^k Z_{n-1}^{(A+kI, \lambda)}(x; k) = A Z_n^{(A, \lambda)}(x; k) - (knI + A) Z_n^{(A-I, \lambda)}(x; k). \quad (3.22)$$

Proof. By derivation with respect to x in (3.20), we have desired relation. \square

3.3. Finite summation formulas for Konhauser matrix polynomials

As usual, we prefer to treat the problem by obtaining the finite summation formulas and then using generating matrix functions techniques.

Theorem 13. Let A be a matrix in $\mathbb{C}^{N \times N}$ satisfying condition (3.1) and λ is a complex number with $\operatorname{Re}(\lambda) > 0$, then the finite summation formula for Konhauser matrix polynomials are:

$$Z_n^{(A, \lambda)}(x; k) = \sum_{r=0}^n \frac{\lambda^{kr}}{r!} (y^k - x^k)^r Z_{n-r}^{(A+krI, \lambda)}(y; k), \quad (3.23)$$

where $A + krI$ is a matrix in $\mathbb{C}^{N \times N}$ satisfying condition (3.1).

Proof. Using (3.21) and (1.7), we have

$$\begin{aligned} \sum_{n=0}^{\infty} Z_n^{(A-knI, \lambda)}(x; k) t^n &= \exp(\lambda^k t (y^k - x^k)) \sum_{n=0}^{\infty} Z_n^{(A-knI, \lambda)}(y; k) t^n \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\lambda^{kr}}{r!} (y^k - x^k)^r Z_n^{(A-knI, \lambda)}(y; k) t^{n+r} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^n \frac{\lambda^{kr}}{r!} (y^k - x^k)^r Z_{n-r}^{(A-knI+krI, \lambda)}(y; k) t^n. \end{aligned}$$

Comparing the coefficients of t^n , we get

$$Z_n^{(A-knI, \lambda)}(x; k) = \sum_{r=0}^n \frac{\lambda^{kr}}{r!} (y^k - x^k)^r Z_{n-r}^{(A-knI+krI, \lambda)}(y; k).$$

Replacing A by $A + knI$, we obtain (3.23). \square

Theorem 14. Let A be a matrix in $\mathbb{C}^{N \times N}$ satisfying condition (3.1), then the Konhauser matrix polynomials have the following finite summation formulas:

$$\begin{aligned} Z_n^{(A, \lambda)}(x; k) &= \left(\frac{x}{y}\right)^{kn} \sum_{j=0}^n (A + I)_{kn} [(A + I)_{k(n-j)}]^{-1} \frac{1}{j!} \left(\frac{y^k - x^k}{x^k}\right)^j Z_{n-j}^{(A, \lambda)}(y; k) \quad (3.24) \end{aligned}$$

or equivalently,

$$\begin{aligned} Z_n^{(A, \lambda)}(x; k) &= \left(\frac{x}{y}\right)^{kn} \sum_{j=0}^n (A + I)_{kn} [(A + I)_{kj}]^{-1} \frac{1}{(n-j)!} \left(\frac{y^k - x^k}{x^k}\right)^{n-j} Z_j^{(A, \lambda)}(y; k). \quad (3.25) \end{aligned}$$

Proof. From (3.5), if we set $t = \left(\frac{y}{k}\right)^k z$, we get

$$\begin{aligned} \sum_{n=0}^{\infty} [(A + I)_{kn}]^{-1} Z_n^{(A, \lambda)}(x; k) \left(\frac{y}{k}\right)^{nk} z^n &= \exp\left(\left(\frac{y}{k}\right)^k z\right) {}_0F_k\left(-; \frac{A+I}{k}, \dots, \frac{A+kI}{k}; -\left(\frac{\lambda xy}{k^2}\right)^k z\right) \quad (3.26) \end{aligned}$$

which, on interchanging x and y , gives us

$$\begin{aligned} & \sum_{n=0}^{\infty} [(A+I)_{kn}]^{-1} Z_n^{(A,\lambda)}(y;k) \left(\frac{x}{k}\right)^{nk} z^n \\ &= \exp\left(\left(\frac{x}{k}\right)^k z\right) {}_0F_k\left(-; \frac{A+I}{k}, \dots, \frac{A+kI}{k}; -\left(\frac{\lambda xy}{k^2}\right)^k z\right). \end{aligned} \quad (3.27)$$

Using (3.26) and (3.27), it follows that

$$\begin{aligned} & \sum_{n=0}^{\infty} [(A+I)_{kn}]^{-1} Z_n^{(A,\lambda)}(x;k) \left(\frac{y}{k}\right)^{nk} z^n \\ &= \exp\left(z\left[\left(\frac{y}{k}\right)^k - \left(\frac{x}{k}\right)^k\right]\right) \sum_{n=0}^{\infty} [(A+I)_{kn}]^{-1} Z_n^{(A,\lambda)}(y;k) \left(\frac{x}{k}\right)^{nk} z^n. \end{aligned}$$

Comparing the coefficients of z^n , we obtain (3.24). Thus the proof is completed. \square

Corollary 3. *The Konhauser matrix polynomials satisfy the multiplication formula follows:*

$$Z_n^{(A,\lambda)}(\mu x; k) = \sum_{r=0}^n (A+I)_{kn} [(A+I)_{k(n-r)}]^{-1} \frac{1}{r!} \mu^{k(n-r)} (1-\mu^k)^r Z_{n-r}^{(A,\lambda)}(x; k). \quad (3.28)$$

Proof. The summation formula (3.24) can indeed be derived directly. \square

3.4. Expansions of Konhauser matrix polynomials

Now, we state and prove the following theorem which will be instrumental in the sequel.

Theorem 15. *Let A be a matrix in $\mathbb{C}^{N \times N}$ satisfy the condition (3.1). Then we have*

$$(\lambda x)^{kn} I = \sum_{m=0}^n \frac{(-1)^m n!}{(n-m)!} Z_m^{(A,\lambda)}(x; k) (A+I)_{kn} [(A+I)_{km}]^{-1}. \quad (3.29)$$

Proof. From (3.5), we get

$$\begin{aligned} & {}_0F_k\left(-; \frac{A+I}{k}, \frac{A+2I}{k}, \dots, \frac{A+kI}{k}; -\frac{(\lambda x)^k t}{k^k}\right) \\ &= e^{-t} \sum_{m=0}^{\infty} Z_m^{(A,\lambda)}(x; k) [(A+I)_{km}]^{-1} t^m. \end{aligned}$$

Therefore, we can write the following

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-(\lambda x)^k)^n t^n}{n! k^{kn}} \left[\left(\frac{A+I}{k} \right)_n \right]^{-1} \left[\left(\frac{A+2I}{k} \right)_n \right]^{-1} \cdots \left[\left(\frac{A+kI}{k} \right)_n \right]^{-1} \\ = \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(-1)^{n-m}}{(n-m)!} Z_m^{(A,\lambda)}(x; k) t^n [(A+I)_{km}]^{-1}. \end{aligned} \quad (3.30)$$

Comparing the coefficients of t^n in (3.30), we have the proof. \square

Expansions of Hermite and Legendre matrix polynomials in a series of Konhauser matrix polynomials relevant to the present investigation are summarized following theorem.

Theorem 16. Let A be a matrix in $\mathbb{C}^{N \times N}$ satisfy the condition (3.1) and λ is a complex number with $\text{Re}(\lambda) > 0$. Hermite, Legendre and Konhauser matrix polynomials satisfy respectively the relation:

$$\begin{aligned} H_n((\lambda x)^k, A) = n! \sum_{m=0}^n \sum_{s=0}^{\lfloor \frac{n-m}{2} \rfloor} \frac{(-1)^{m+s} (\sqrt{2A})^{n-2s}}{(n-m-2s)! s!} \\ Z_m^{(A,\lambda)}(x; k) (A+I)_{k(n-2s)} [(A+I)_{km}]^{-1}. \end{aligned} \quad (3.31)$$

where $H_n(x, A)$ stands for Hermite matrix polynomials [8, 11].

$$\begin{aligned} P_n((\lambda x)^k, A) = n! \sum_{m=0}^n \sum_{s=0}^{\lfloor \frac{1}{2}(n-m) \rfloor} \frac{(-1)^{s+m} \left(\frac{1}{2}\right)_{n-s} (\sqrt{2A})^{n-2s}}{s! (n-m-2s)!} \\ Z_m^{(A,\lambda)}(x; k) (A+I)_{k(n-2s)} [(A+I)_{km}]^{-1} t^n. \end{aligned} \quad (3.32)$$

where $P_n(x, A)$ is Legendre matrix polynomials [32].

Proof. We consider the series involving Hermite matrix polynomials

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n((\lambda x)^k, A) &= \exp \left((\lambda x)^k t \sqrt{2A} - t^2 I \right) = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (\lambda x)^{kn} (\sqrt{2A})^n}{n! s!} t^{n+2s} \\ &= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{m=0}^n \frac{(-1)^s (\sqrt{2A})^n}{n! s!} \frac{(-1)^m n!}{(n-m)!} Z_m^{(A,\lambda)}(x; k) (A+I)_{kn} [(A+I)_{km}]^{-1} t^{n+2s} \\ &= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{s+m} (\sqrt{2A})^{n+m}}{s! n!} Z_m^{(A,\lambda)}(x; k) (A+I)_{k(n+m)} [(A+I)_{km}]^{-1} t^{n+m+2s} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{m+s} (\sqrt{2A})^{n+m-2s}}{s! (n-2s)!} Z_m^{(A,\lambda)}(x; k) (A+I)_{k(n+m-2s)} [(A+I)_{km}]^{-1} t^{n+m}. \end{aligned}$$

Replacing n by $n - m$, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n((\lambda x)^k, A) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{s=0}^{\lfloor \frac{n-m}{2} \rfloor} \frac{(-1)^{m+s} (\sqrt{2A})^{n-2s}}{(n-m-2s)! s!} Z_m^{(A, \lambda)}(x; k) (A + I)_{k(n-2s)} [(A + I)_{km}]^{-1} t^n. \end{aligned}$$

Comparing the coefficients of t^n , we obtain (3.31). In a similar fashion, using the result (3.29) in the generating relation for Legendre matrix polynomials, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n((\lambda x)^k, A) \\ &= \sum_{n=0}^{\infty} \sum_{s=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{(-1)^s \left(\frac{1}{2}\right)_{n-s} ((\lambda x)^k \sqrt{2A})^{n-2s}}{s! (n-2s)!} t^n \\ &= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{1}{2}\right)_{n+s} ((\lambda x)^k \sqrt{2A})^n}{s! n!} t^{n+2s} \\ &= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{1}{2}\right)_{n+s} (\lambda x)^{nk} (\sqrt{2A})^n}{s! n!} t^{n+2s} \\ &= \sum_{n,s=0}^{\infty} \sum_{m=0}^n \frac{(-1)^{s+m} \left(\frac{1}{2}\right)_{n+s} (\sqrt{2A})^n}{s! (n-m)!} Z_m^{(A, \lambda)}(x; k) (A + I)_{kn} [(A + I)_{km}]^{-1} t^{n+2s} \\ &= \sum_{n,s,m=0}^{\infty} \frac{(-1)^{s+m} \left(\frac{1}{2}\right)_{n+m+s} (\sqrt{2A})^{n+m}}{s! n!} Z_m^{(A, \lambda)}(x; k) (A + I)_{k(n+m)} \\ & \quad [(A + I)_{km}]^{-1} t^{n+m+2s} \\ &= \sum_{n,m=0}^{\infty} \sum_{s=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{(-1)^{s+m} \left(\frac{1}{2}\right)_{n+m-s} (\sqrt{2A})^{n+m-2s}}{s! (n-2s)!} Z_m^{(A, \lambda)}(x; k) (A + I)_{k(n+m-2s)} \\ & \quad [(A + I)_{km}]^{-1} t^{n+m} \end{aligned}$$

Replacing n by $n - m$, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n((\lambda x)^k, A) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{s=0}^{\lfloor \frac{1}{2}(n-m) \rfloor} \frac{(-1)^{s+m} \left(\frac{1}{2}\right)_{n-s} (\sqrt{2A})^{n-2s}}{s!(n-m-2s)!} Z_m^{(A,\lambda)}(x; k) (A+I)_{k(n-2s)} \\ & \quad [(A+I)_{km}]^{-1} t^n. \end{aligned}$$

Comparing the coefficients of t^n , we obtain (3.32). Thus the proof is completed. \square

In conclusion, we shall now show that there exist generating matrix functions between Konhauser matrix polynomials and the various type of Mittag-Leffler and Bessel-Maitland matrix functions, more results could be obtained, but the details are omitted for reasons of brevity.

4. APPLICATIONS

Notice that the so-called Mittag-Leffler matrix function $E_{kI, A+I}^B$ and the Bessel-Maitland matrix functions $\phi(kI, A+I; z)$, are indeed the familiar hypergeometric matrix functions ${}_1F_k$ and ${}_0F_k$ respectively, k is a positive integer. More precisely, we define, for $k = 1, 2, 3, \dots$,

$$\begin{aligned} E_{kI, A+I}^B(z) &= \sum_{m=0}^{\infty} \frac{(B)_m z^m}{m!} \Gamma^{-1}(kmI + A + I) \\ &= \Gamma^{-1}(A + I) {}_1F_k\left(B; \frac{A+I}{k}, \dots, \frac{A+kI}{k}; \frac{z}{k^k}\right) \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} \phi(kI, A+I; z) &= \sum_{m=0}^{\infty} \frac{z^m}{m!} \Gamma^{-1}(kmI + A + I) \\ &= \Gamma^{-1}(A + I) {}_0F_k\left(-; \frac{A+I}{k}, \dots, \frac{A+kI}{k}; \frac{z}{k^k}\right) \end{aligned} \quad (4.2)$$

where A satisfies the condition (3.1).

As a consequence of the definitions (4.1) and (4.2) the following results hold:

Theorem 17. Let A and B be matrices in $\mathbb{C}^{N \times N}$ such that A satisfies the condition (3.1) and $|t| < 1$. There hold the following relations:

$$\sum_{n=0}^{\infty} (B)_n \Gamma^{-1}(knI + A + I) Z_n^{(A,\lambda)}(x; k) t^n = (1-t)^{-B} E_{kI, A+I}^B\left(\frac{-(\lambda x)^k t}{1-t}\right) \quad (4.3)$$

and

$$\sum_{n=0}^{\infty} \Gamma^{-1}(knI + A + I) Z_n^{(A,\lambda)}(x; k) t^n = e^t \phi(kI, A + I; -(\lambda x)^k t). \quad (4.4)$$

Proof. Using (1.8), (4.1) and (3.2), we get another representation for the Konhauser matrix polynomials

$$\begin{aligned} & \sum_{n=0}^{\infty} (B)_n \Gamma^{-1}(knI + A + I) Z_n^{(A,\lambda)}(x; k) t^n \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(-1)^m (B)_n \Gamma^{-1}(kmI + A + I) (\lambda x)^{km}}{m!(n-m)!} t^n \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (B)_{n+m} \Gamma^{-1}(kmI + A + I) (\lambda x)^{km}}{m!n!} t^{n+m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (B+mI)_n (B)_m \Gamma^{-1}(kmI + A + I) (\lambda x)^{km}}{m!n!} t^{n+m} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(B+mI)_n}{n!} t^n \frac{(-1)^m (B)_m \Gamma^{-1}(kmI + A + I) (\lambda x)^{km}}{m!} t^m \\ &= \sum_{m=0}^{\infty} (1-t)^{-(B+mI)} \frac{(-1)^m (B)_m \Gamma^{-1}(kmI + A + I) (\lambda x)^{km}}{m!} t^m \\ &= (1-t)^{-B} \sum_{m=0}^{\infty} \frac{(B)_m \Gamma^{-1}(kmI + A + I)}{m!} \left(\frac{-(\lambda x)^k t}{1-t} \right)^m \\ &= (1-t)^{-B} E_{kI, A+I}^B \left(\frac{-(\lambda x)^k t}{1-t} \right). \end{aligned}$$

which gives (4.3). From (1.7), (3.2) and (4.2), we have

$$\begin{aligned} e^t \phi(kI, A + I; -(\lambda x)^k t) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda x)^{mk}}{n!m!} t^{n+m} \Gamma^{-1}(kmI + A + I) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(-1)^m (\lambda x)^{mk}}{(n-m)!m!} t^n \Gamma^{-1}(kmI + A + I) \\ &= \sum_{n=0}^{\infty} \Gamma^{-1}(knI + A + I) Z_n^{(A,\lambda)}(x; k) t^n \end{aligned}$$

Thus the proof is completed. \square

In the forthcoming work, we will discuss the properties of these new families of Mittag-Leffler and Bessel-Maitland matrix functions and we will analyze possible developments and applications of the theory. Hence, new results and further applications can be obtained.

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