Miskolc Mathematical Notes
Vol. 16 (2015), No 1, pp. 57-63

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1124

A new factor theorem on absolute Riesz summability

Hüseyin Bor

Miskolc Mathematical Notes HU e-ISSN 1787-2413
Vol. 16 (2015), No. 1, pp. 57–63

A NEW FACTOR THEOREM ON ABSOLUTE RIESZ SUMMABILITY

HÜSEYİN BOR

Received 06 February, 2014

Abstract. In this paper, a known theorem dealing with an application of quasi power increasing sequences has extended by using a new class of increasing sequences instead of a quasi-σ-power increasing sequence. This theorem also includes some new and known results.

2010 *Mathematics Subject Classification:* 26D15; 40D15; 40F05; 40G99; 46A45

Keywords: Riesz mean, absolute summability, infinite series, sequence spaces, power increasing sequences, Hölder inequality, Minkowski inequality

1. Introduction

A positive sequence $X = (X_n)$ is said to be a quasi-(σ,γ)-power increasing sequence if there exists a constant $K = K(X,(\sigma,\gamma)) \geq 1$ such that $Kn^\sigma(\log n)^\gamma Xn \geq m^\sigma(\log m)^\gamma Xm$ for all $n \geq m \geq 1$, where $\gamma \geq 0$ and $0 < \sigma < 1$ (see [14]). If we take γ=0, then we get a quasi-σ-power increasing sequence (see [13]). A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [1]). A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} \mid \lambda_n - \lambda_{n+1} \mid < \infty$. Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). We denote by t_n^α the nth Cesàro mean of order α, with $\alpha > -1$, of the sequences (na_n), that is (see[9])

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \qquad ({t_n}^1 = t_n) \tag{1.1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{1.2}$$

A series $\sum a_n$ is said to be summable $\mid C,\alpha;\delta\mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [11])

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid t_n^\alpha \mid^k < \infty. \tag{1.3}$$

If we set $\delta=0$, then we get $\mid C,\alpha\mid_k$ summablility (see [10]).
Let (p_n) be a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n} p_v \to \infty \quad as \quad n\to\infty, \quad (P_{-i}=p_{-i}=0, i\geq 1). \tag{1.4}$$

The sequence-to-sequence transformation

$$w_n=\frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.5}$$

defines the sequence (w_n) of the Riesz mean or simply the $(\bar{N},p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [12]). The series $\sum a_n$ is said to be summable $\mid \bar{N},p_n;\delta\mid_k$, $k\geq 1$ and $\delta\geq 0$, if (see [4])

$$\sum_{n=1}^{\infty}(P_n/p_n)^{\delta k+k-1}\mid w_n-w_{n-1}\mid^k<\infty. \tag{1.6}$$

If we set $\delta=0$, then we obtain $\mid \bar{N},p_n\mid_k$ summability (see [2]). In the special case $p_n=1$ for all values of n, the $\mid \bar{N},p_n;\delta\mid_k$ summability is the same as $\mid C,1;\delta\mid_k$ summability. Also if we take $\delta=0$ and $k=1$, then we get $\mid \bar{N},p_n\mid$ summability.

2. The known result

The following theorem is known dealing with the absolute Riesz summability factors of infinite series.

Theorem 1 ([5])**.** *Let (X_n) be a quasi-σ-power increasing sequence for some σ $(0<\sigma<1)$ and let there be sequences (β_n) and (λ_n) such that*

$$\mid \Delta\lambda_n\mid\leq\beta_n, \tag{2.1}$$

$$\beta_n\to 0 \quad as \quad n\to\infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n\mid\Delta\beta_n\mid X_n<\infty, \tag{2.3}$$

$$\mid\lambda_n\mid X_n=O(1). \tag{2.4}$$

If (p_n) is a sequence such that

$$P_n=O(np_n), \tag{2.5}$$

$$P_n\Delta p_n=O(p_n p_{n+1}), \tag{2.6}$$

$$\sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{\mid t_v\mid^k}{v}=O(X_n) \quad as \quad n\to\infty, \tag{2.7}$$

$$\sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} = O((\frac{P_v}{p_v})^{\delta k} \frac{1}{P_v}) \quad as \quad m \to \infty, \tag{2.8}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $| \bar{N}, p_n; \delta |_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.

Remark 1. In the hypothesis of Theorem 1, we have added the condition "$(\lambda_n) \in \mathcal{BV}$".

3. The main result

The aim of this paper is to extend Theorem 1 by using a quasi-(σ, γ)-power increasing sequence instead of quasi-σ-power increasing sequence. Now we shall prove the following theorem.

Theorem 2. *Let* $(\lambda_n) \in \mathcal{BV}$ *and let* (X_n) *be a quasi-*(σ, γ)*-power increasing sequence. If all conditions of Theorem 1 are satisfied, then the series* $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $| \bar{N}, p_n; \delta |_k$, $k \geq 1$, *and* $0 \leq \delta < 1/k$.

Remark 2. If we set γ=0, then we get Theorem 1. If we take $\delta = 0$, then we get a new result dealing with $| \bar{N}, p_n |_k$ summability factors. If we take $p_n = 1$ for all values of n, then we get a new result dealing with $| C, 1; \delta |_k$ summability factors. Also, if we take γ=0 and δ=0, the we get the known result (see [6]). Finally, if we take (X_n) as an almost increasing sequence, then we get the known result which was published in [7].

We need the following lemmas for the proof of the theorem.

Lemma 1 ([8])**.** *Except for the condition* $(\lambda_n) \in \mathcal{BV}$*, under the conditions on* (X_n)*,* (β_n) *and* (λ_n) *as expressed in the statement of the theorem, we have the following;*

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty, \tag{3.1}$$

$$n X_n \beta_n = O(1). \tag{3.2}$$

Lemma 2 ([3])**.** *If the conditions (2.5) and (2.6) are satisfied, then we have*

$$\Delta \left(\frac{P_n}{n^2 p_n} \right) = O \left(\frac{1}{n^2} \right). \tag{3.3}$$

4. Proof of Theorem 2

Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{r p_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{v p_v}. \tag{4.1}$$

Then, for $n \geq 1$ we obtain that

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v v \lambda_v}{v^2 p_v}.$$

Using Abel's transformation, we get that

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta \left(\frac{P_{v-1} P_v \lambda_v}{v^2 p_v} \right) \sum_{r=1}^{v} r a_r + \frac{\lambda_n}{n^2} \sum_{v=1}^{n} v a_v \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{p_v} (v+1) t_v p_v \frac{\lambda_v}{v^2} \\ &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v (v+1) \frac{t_v}{v^2 p_v} \\ &\quad - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} (v+1) t_v \Delta (P_v / v^2 p_v) \\ &\quad + \lambda_n t_n (n+1)/n^2 = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned}$$

To complete the proof of the theorem by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{4.2}$$

Now, applying Hölder's inequality, we have that

$$\begin{aligned} &\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \mid T_{n,1} \mid^k \\ &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v \mid t_v \mid\mid \lambda_v \mid \frac{1}{v} \right\}^k \\ &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k p_v \mid t_v \mid^k \mid \lambda_v \mid^k \frac{1}{v^k} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k p_v \mid t_v \mid^k \mid \lambda_v \mid^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid p_v \mid t_v \mid^k \frac{1}{v^k} \frac{1}{P_v} \left(\frac{P_v}{p_v}\right)^{\delta k} \end{aligned}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} \mid \lambda_v \mid\mid t_v \mid^k \frac{1}{v^k}(\frac{P_v}{p_v})^{\delta k}$$

$$= O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k} v^{k-1}\frac{1}{v^k} \mid \lambda_v \mid\mid t_v \mid^k$$

$$= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid (\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1} \Delta \mid \lambda_v \mid \sum_{r=1}^{v}(\frac{P_r}{p_r})^{\delta k}\frac{\mid t_r \mid^k}{r} + O(1) \mid \lambda_m \mid \sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1} \mid \Delta\lambda_v \mid X_v + O(1) \mid \lambda_m \mid X_m$$

$$= O(1)\sum_{v=1}^{m-1} \beta_v X_v + O(1) \mid \lambda_m \mid X_m = O(1)$$

as $m \to \infty$, by virtue of the hypotheses of the theorem and Lemma 1. Now, by using (2.5), we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,2} \mid^k$$

$$= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid\mid t_v \mid\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid\mid t_v \mid^k \times \left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m} P_v\beta_v \mid t_v \mid^k \sum_{n=v+1}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}\beta_v \mid t_v \mid^k = O(1)\sum_{v=1}^{m} v\beta_v(\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}(\frac{P_r}{p_r})^{\delta k}\frac{\mid t_r \mid^k}{r} + O(1)m\beta_m\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m$$

$$= O(1) \quad as \quad m \to \infty,$$

in view of the hypotheses of the theorem and Lemma 1. By using Lemma 2, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,3} \mid^k$$
$$= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v \mid \lambda_{v+1} \mid\mid t_v \mid \frac{1}{v} \frac{v+1}{v} \right\}^k$$
$$= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v \mid \lambda_{v+1} \mid \frac{1}{v} \mid t_v \mid \right\}^k$$
$$= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k p_v \frac{1}{v^k} \mid \lambda_{v+1} \mid^k \mid t_v \mid^k \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$
$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v \frac{1}{v^k} \mid \lambda_{v+1} \mid^{k-1} \mid \lambda_{v+1} \mid\mid t_v \mid^k \times \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} \frac{1}{v^k} \mid \lambda_{v+1} \mid\mid t_v \mid^k (\frac{P_v}{p_v})^{\delta k}$$
$$= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} v^{k-1} \frac{1}{v^k} \mid \lambda_{v+1} \mid\mid t_v \mid^k$$
$$= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} \mid \lambda_{v+1} \mid \frac{\mid t_v \mid^k}{v} = O(1) \quad as \quad m \to \infty.$$

Finally, as in $T_{n,1}$, we have that

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,4} \mid^k = O(1) \sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{n+1}{n}\right)^k \frac{1}{n^k} \mid \lambda_n \mid^k \mid t_n \mid^k$$
$$= O(1) \sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k} n^{k-1} \frac{1}{n^k} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid\mid t_n \mid^k$$
$$= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid (\frac{P_n}{p_n})^{\delta k} \frac{\mid t_n \mid^k}{n} = O(1) \quad as \quad m \to \infty.$$

This completes the proof of the theorem.

ACKNOWLEDGEMENT

The author expresses his thanks to the referee for his/her valuable comments and suggestions for the improvement of this paper.

REFERENCES

[1] N. K. Bari and S. B. Stečkin, "Best approximation and differential properties of two conjugate functions," *Trudy. Moskov. Mat. Obšč.*, vol. 5, no. 1, pp. 483–522, 1956.

[2] H. Bor, "On two summability methods," *Math. Proc. Camb. Philos Soc*, vol. 97, no. 1, pp. 147–149, 1985.

[3] H. Bor, "Absolute summability factors for infinite series," *Indian J. Pure Appl. Math.*, vol. 19, no. 7, pp. 664–671, 1988.

[4] H. Bor, "On local property of $|\bar{N}, p_n; \delta|_k$ summability of factored Fourier series," *J. Math. Anal. Appl.*, vol. 179, no. 2, pp. 646–649, 1993.

[5] H. Bor, "On some new applications of increasing sequences," *Advan. Stud. Contemp. Math.*, vol. 17, no. 2, pp. 231–239, 2008.

[6] H. Bor, "A theorem on the absolute summability factors," *Math. Notes*, vol. 86, no. 3-4, pp. 463–468, 2009.

[7] H. Bor, "On an application of almost increasing sequences," *Math. Commun.*, vol. 15, no. 1, pp. 57–64, 2010.

[8] H. Bor, "A new application of generalized power increasing sequences," *Filomat*, vol. 26, no. 3, pp. 631–635, 2012.

[9] E. Cesàro, "Sur la multiplication des séries," *Bull. Sci. Math.*, vol. 14, no. 1, pp. 114–120, 1890.

[10] T. M. Flett, "On an extension of absolute summability and some theorems of Littlewood and Paley," *Proc. London Math. Soc.*, vol. 7, no. 1, pp. 113–141, 1957.

[11] T. M. Flett, "Some more theorems concerning the absolute summability of fourier series," *Proc. London Math. Soc.*, vol. 8, no. 3, pp. 357–387, 1958.

[12] G. H. Hardy, *Divergent Series*. Oxford: Oxford Univ. Press, 1949.

[13] L. Leindler, "A new application of quasi power increasing sequences," *Publ. Math. Debrecen*, vol. 58, no. 4, pp. 791–796, 2001.

[14] W. T. Sulaiman, "Extension on absolute summability factors of infinite series," *J. Math. Anal. Appl.*, vol. 322, no. 2, pp. 1224–1230, 2006.

Author's address

Hüseyin Bor
P. O. Box 121, TR-06502 Bahçelievler, Ankara, Turkey
E-mail address: hbor33@gmail.com