A NEW FACTOR THEOREM ON ABSOLUTE RIESZ
SUMMABILITY

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Abstract. In this paper, a known theorem dealing with an application of quasi power increasing
sequences has extended by using a new class of increasing sequences instead of a quasi-σ-power
increasing sequence. This theorem also includes some new and known results.

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1. INTRODUCTION

A positive sequence $X = (X_n)$ is said to be a quasi-(σ, γ)-power increasing se-
quence if there exists a constant $K = K(X, (σ, γ)) \geq 1$ such that $Kn^\sigma (\log n)^\gamma Xn \geq
m^\sigma (\log m)^\gamma Xm$ for all $n \geq m \geq 1$, where $γ \geq 0$ and $0 < σ < 1$ (see [14]). If we
take $γ = 0$, then we get a quasi-σ-power increasing sequence (see [13]). A pos-
teive sequence $(b_n)$ is said to be an almost increasing sequence if there exists a
positive increasing sequence $(c_n)$ and two positive constants $M$ and $N$ such that
$Mc_n \leq b_n \leq Nc_n$ (see [1]). A sequence $(λ_n)$ is said to be of bounded variation,
denoted by $(λ_n) \in BV$, if $\sum_{n=1}^{∞} |\Delta λ_n| = \sum_{n=1}^{∞} |λ_n − λ_{n+1}| < ∞$. Let $\sum a_n$ be a
given infinite series with the sequence of partial sums $(s_n)$. We denote by $t^\alpha_n$ the n
th Cesàro mean of order $α$, with $α > −1$, of the sequences $(na_n)$, that is (see[9])

$$t^\alpha_n = \frac{1}{A^\alpha_n} \sum_{v=1}^{n} A^\alpha_{n−v} va_v, \quad (t_n^{−1} = t_n) \quad (1.1)$$

where

$$A^\alpha_n = \frac{(α + 1)(α + 2)\ldots(α + n)}{n!} = O(n^\alpha), \quad A^\alpha_{−n} = 0 \quad \text{for} \quad n > 0. \quad (1.2)$$

A series $\sum a_n$ is said to be summable $\mid C, α; δ \mid_k, k \geq 1$ and $δ \geq 0$, if (see [11])

$$\sum_{n=1}^{∞} n^{δk−1} |t^\alpha_n|^k < ∞. \quad (1.3)$$

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If we set $\delta = 0$, then we get $|C, \alpha|_k$ summability (see [10]).

Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty. \quad (P_{-i} = p_{-i} = 0, i \geq 1) \quad (1.4)$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (1.5)$$

defines the sequence $(w_n)$ of the Riesz mean or simply the $|N, p_n|$ mean of the sequence $(s_n)$, generated by the sequence of coefficients $(p_n)$ (see [12]). The series $\sum a_n$ is said to be summable $| \tilde{N}, p_n; \delta |_k$, $k \geq 1$ and $\delta \geq 0$, if (see [4])

$$\sum_{n=1}^{\infty} (p_n / p_{n+1})^{\delta k + k-1} |w_n - w_{n-1}|^k < \infty. \quad (1.6)$$

If we set $\delta = 0$, then we obtain $| \tilde{N}, p_n |_k$ summability (see [2]). In the special case $p_n = 1$ for all values of $n$, the $| \tilde{N}, p_n; \delta |_k$ summability is the same as $|C, 1; \delta |_k$ summability. Also if we take $\delta = 0$ and $k = 1$, then we get $| \tilde{N}, p_n |$ summability.

2. THE KNOWN RESULT

The following theorem is known dealing with the absolute Riesz summability factors of infinite series.

**Theorem 1** ([5]). Let $(X_n)$ be a quasi-$\sigma$-power increasing sequence for some $\sigma$ ($0 < \sigma < 1$) and let there be sequences $(\beta_n)$ and $(\lambda_n)$ such that

$$| \Delta \lambda_n | \leq \beta_n, \quad (2.1)$$

$$\beta_n \to 0 \text{ as } n \to \infty, \quad (2.2)$$

$$\sum_{n=1}^{\infty} n | \Delta \beta_n | X_n < \infty, \quad (2.3)$$

$$| \lambda_n | X_n = O(1). \quad (2.4)$$

If $(p_n)$ is a sequence such that

$$P_n = O(np_n), \quad (2.5)$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \quad (2.6)$$

$$\sum_{v=1}^{n} \left( \frac{P_v}{P_v} \right)^{\delta k} \frac{|l_v|^k}{v} = O(X_n) \text{ as } n \to \infty, \quad (2.7)$$
A NEW FACTOR THEOREM

\[ \sum_{n=\nu+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{P_n-1} = O \left( \frac{P_v}{p_v} \right)^{\delta k} \frac{1}{P_v} \quad \text{as} \quad m \to \infty, \quad (2.8) \]

then the series \( \sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{p_n} \) is summable \( | \tilde{N}, p_n; \delta | k, k \geq 1 \) and \( 0 \leq \delta < 1/k \).

Remark 1. In the hypothesis of Theorem 1, we have added the condition "\( (\lambda_n) \in B^V \)."

3. THE MAIN RESULT

The aim of this paper is to extend Theorem 1 by using a quasi-\((\sigma, \gamma)\)-power increasing sequence instead of quasi-\(\sigma\)-power increasing sequence. Now we shall prove the following theorem.

**Theorem 2.** Let \( (\lambda_n) \in B^V \) and \( (X_n) \) be a quasi-\((\sigma, \gamma)\)-power increasing sequence. If all conditions of Theorem 1 are satisfied, then the series \( \sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{p_n} \) is summable \( | \tilde{N}, p_n; \delta | k, k \geq 1 \) and \( 0 \leq \delta < 1/k \).

Remark 2. If we set \( \gamma = 0 \), then we get Theorem 1. If we take \( \delta = 0 \), then we get a new result dealing with \( | \tilde{N}, p_n | k \) summability factors. If we take \( p_n = 1 \) for all values of \( n \), then we get a new result dealing with \( | C, 1: \delta | k \) summability factors. Also, if we take \( \gamma = 0 \) and \( \delta = 0 \), then we get the known result (see [6]). Finally, if we take \( (X_n) \) as an almost increasing sequence, then we get the known result which was published in [7].

We need the following lemmas for the proof of the theorem.

**Lemma 1** ([8]). Except for the condition \( (\lambda_n) \in B^V \), under the conditions on \( (X_n), (\beta_n) \) and \( (\lambda_n) \) as expressed in the statement of the theorem, we have the following:

\[ \sum_{n=1}^{\infty} \beta_n X_n < \infty, \quad (3.1) \]

\[ n X_n \beta_n = O(1). \quad (3.2) \]

**Lemma 2** ([3]). If the conditions (2.5) and (2.6) are satisfied, then we have

\[ \Delta \left( \frac{P_n}{n^2 p_n} \right) = O \left( \frac{1}{n^2} \right), \quad (3.3) \]

4. PROOF OF THEOREM 2

Let \( (T_n) \) be the sequence of \( (\tilde{N}, p_n) \) mean of the series \( \sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{p_n} \). Then, by definition, we have

\[ T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{r p_r} = \frac{1}{P_n} \sum_{v=1}^{n} \left( P_{n} - P_{v-1} \right) \frac{\alpha_v P_v \lambda_v}{v p_v}. \quad (4.1) \]
Then, for \( n \geq 1 \) we obtain that
\[
T_n - T_{n-1} = \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v \lambda_v}{v^2 p_v}.
\]

Using Abel’s transformation, we get that
\[
T_n - T_{n-1} = \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} \Delta \left( \frac{P_{v-1} P_v \lambda_v}{v^2 p_v} \right) \sum_{r=1}^{v} r a_r + \lambda_n \sum_{v=1}^{n} v a_v
\]
\[
= \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} P_v (v+1) t_v P_v \frac{\lambda_v}{v^2}
\]
\[
+ \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v (v+1) t_v \frac{P_v}{v^2}
\]
\[
- \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_v (v+1) t_v \frac{P_v}{v^2 p_v}
\]
\[
+ \lambda_n t_n (n+1) / n^2 = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\]

To complete the proof of the theorem by Minkowski’s inequality, it is sufficient to show that
\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\delta_k + k-1} | T_{n,r} |^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \quad (4.2)
\]

Now, applying Hölder’s inequality, we have that
\[
\sum_{n=2}^{m+1} \left( \frac{p_n}{p_n} \right)^{\delta_k + k-1} | T_{n,1} |^k
\]
\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{p_n}{p_n} \right)^{\delta_k - k-1} \left( \sum_{v=1}^{n-1} \frac{P_v}{p_v} \cdot t_v \cdot | \lambda_v | \cdot 1 \right) \left( \sum_{v=1}^{n-1} \frac{P_v}{p_v} \cdot t_v \cdot | \lambda_v | \cdot 1 \right)^{-k-1}
\]
\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{p_n}{p_n} \right)^{\delta_k - k-1} \left( \sum_{v=1}^{n-1} \frac{P_v}{p_v} \cdot t_v \cdot | \lambda_v | \cdot 1 \right) \left( \sum_{v=1}^{n-1} \frac{P_v}{p_v} \cdot t_v \cdot | \lambda_v | \cdot 1 \right)^{-k-1}
\]
\[
= O(1) \sum_{v=1}^{m} \frac{p_v}{p_v} \cdot t_v \cdot | \lambda_v | \cdot 1 \left( \sum_{v=1}^{n-1} \frac{P_v}{p_v} \cdot t_v \cdot | \lambda_v | \cdot 1 \right) \left( \sum_{v=1}^{n-1} \frac{P_v}{p_v} \cdot t_v \cdot | \lambda_v | \cdot 1 \right)^{-k-1}
\]
\[
= O(1) \sum_{v=1}^{m} \frac{p_v}{p_v} \cdot t_v \cdot | \lambda_v | \cdot 1 \left( \sum_{v=1}^{n-1} \frac{P_v}{p_v} \cdot t_v \cdot | \lambda_v | \cdot 1 \right) \left( \sum_{v=1}^{n-1} \frac{P_v}{p_v} \cdot t_v \cdot | \lambda_v | \cdot 1 \right)^{-k-1}
\]
\[ \begin{align*}
&= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{k-1} | \lambda_v | | t_v |^k \frac{1}{v^k} \left( \frac{P_v}{p_v} \right)^{\delta_k} \\
&= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta_k} v^{k-1} \frac{1}{v^k} | \lambda_v | | t_v |^k \\
&= O(1) \sum_{v=1}^{m} | \lambda_v | (\frac{P_v}{p_v})^{\delta_k} \frac{| t_v |^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta | \lambda_v | \sum_{r=1}^{v} \left( \frac{P_r}{p_r} \right)^{\delta_k} \frac{| t_r |^k}{r} + O(1) | \lambda_m | \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta_k} \frac{| t_v |^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} | \Delta \lambda_v | X_v + O(1) | \lambda_m | X_m \\
&= O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) | \lambda_m | X_m = O(1)
\end{align*} \]

as \( m \to \infty \), by virtue of the hypotheses of the theorem and Lemma 1. Now, by using (2.5), we have that

\[ \begin{align*}
& \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} | T_{n,2} |^k \\
&= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_n} \sum_{v=1}^{n-1} \left| P_v \right| \Delta \lambda_v | | t_v |^k \left\{ \sum_{v=1}^{n-1} \left| P_v \right| \Delta \lambda_v | | t_v | \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_n} \sum_{v=1}^{n-1} \left| P_v \right| \Delta \lambda_v | | t_v |^k \left( \frac{1}{P_n} \right) \sum_{v=1}^{n-1} \left| P_v \right| \Delta \lambda_v | \left\{ \sum_{v=1}^{n-1} \left| P_v \right| \Delta \lambda_v | \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k} \beta_v | t_v |^k \sum_{n=1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_n} \\
&= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k} \beta_v | t_v |^k \left( \frac{1}{p_v} \right) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k} | t_v |^k \\
&= O(1) \sum_{v=1}^{m} \Delta (\beta v) \sum_{r=1}^{v} \left( \frac{P_r}{p_r} \right)^{\delta k} \frac{| t_r |^k}{r} + O(1)m \beta_m \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k} \frac{| t_v |^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1)m \beta_m X_m
\end{align*} \]
\[ = O(1) \quad \text{as} \quad m \to \infty, \]

in view of the hypotheses of the theorem and Lemma 1. By using Lemma 2, as in \( T_{n,1} \), we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,3}|^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{p_{n-1}} \left( \sum_{v=1}^{n-1} P_v | \lambda_{v+1} | \left| t_v \right| \frac{1}{v} \right)^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{p_{n-1}} \left( \sum_{v=1}^{n-1} P_v | \lambda_{v+1} | \left| t_v \right| \right)^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \right)^k \frac{1}{v^k} \left| \lambda_{v+1} \right| \left| t_v \right|^k \times \left( \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} P_v \right)^{-1}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k \frac{1}{v^k} \left| \lambda_{v+1} \right| \left| t_v \right|^k \times \frac{1}{p_{n-1}} \sum_{v=1}^{n-1} P_v \left( \frac{p_v}{p_n} \right)^{\delta k}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k \frac{1}{v^k} \left| \lambda_{v+1} \right| \left| t_v \right|^k
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k \frac{1}{v^k} \left| \lambda_{v+1} \right| \left| t_v \right|^k
\]

\[
= O(1) \quad \text{as} \quad m \to \infty.
\]

Finally, as in \( T_{n,1} \), we have that

\[
\sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,4}|^k
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k} \left( \frac{P_n}{p_n} \right)^{k-1} \frac{n+1}{n} \frac{1}{n^k} \left| \lambda_n \right|^k \left| t_n \right|^k
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k} n^{k-1} \frac{1}{n^k} \left| \lambda_n \right|^k \left| t_n \right|^k
\]

\[
= O(1) \sum_{n=1}^{m} \left| \lambda_n \right| \left( \frac{P_n}{p_n} \right)^{\delta k} \frac{t_n}{n} \frac{1}{n^k} = O(1) \quad \text{as} \quad m \to \infty.
\]

This completes the proof of the theorem.
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