ON THE PRIME SPECTRUM OF MODULES

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Abstract. Let \( R \) be a commutative ring and let \( M \) be an \( R \)-module. Let us denote the set of all prime submodules of \( M \) by \( \text{Spec}(M) \). In this article, we explore more properties of strongly top modules and investigate some conditions under which \( \text{Spec}(M) \) is a spectral space.

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1. INTRODUCTION, ETC

Throughout this article, all rings are commutative with identity elements, and all modules are unital left modules. \( \mathbb{N}, \mathbb{Z}, \) and \( \mathbb{Q} \) will denote respectively the natural numbers, the ring of integers and the field of quotients of \( \mathbb{Z} \). If \( N \) is a subset of an \( R \)-module \( M \), then \( N \subseteq M \) denotes \( N \) is a submodule of \( M \).

Let \( M \) be an \( R \)-module. For any submodule \( N \) of \( M \), we denote the annihilator of \( M/N \) by \( \text{Ann}(M/N) \), i.e. \( (N : M) = \{ r \in R | rM \subseteq N \} \). A submodule \( P \) of \( M \) is called prime if \( P \neq M \) and whenever \( r \in R \) and \( e \in M \) satisfy \( re \in P \), then \( r \in (P : M) \) or \( e \in P \).

The set of all prime submodule of \( M \) is denoted by \( \text{Spec}(M) \) (or \( X \)). For any ideal \( I \) of \( R \) containing \( \text{Ann}(M) \), \( \overline{I} \) and \( \overline{R} \) will denote \( I/\text{Ann}(M) \) and \( R/\text{Ann}(M) \), respectively. Also the map \( \psi : \text{Spec}(M) \rightarrow \text{Spec}(\overline{R}) \) given by \( P \mapsto (P : M) \) is called the natural map of \( X \). \( M \) is called primeful (resp. \( X \)-injective) if either \( M = 0 \) or \( M \neq 0 \) and the natural map \( \psi \) is surjective (resp. if either \( X = \emptyset \) or \( X \neq \emptyset \) and natural map \( \psi \) is injective). (See [3, 11] and [13].)

The Zariski topology on \( X \) is the topology \( \tau \) described by taking the set \( \Omega = \{ V(N) | N \text{ is a submodule of } M \} \) as the set of closed sets of \( X \), where \( V(N) = \{ P \in X | (P : M) \supseteq (N : M) \} \) [11].

The quasi-Zariski topology on \( X \) is described as follows: put \( V^*(N) = \{ P \in X | P \supseteq N \} \) and \( \Omega^* = \{ V^*(N) | N \text{ is a submodule of } M \} \). Then there exists a topology \( \tau^* \) on \( X \) having \( \Omega^* \) as the set of it’s closed subsets if and only if \( \Omega^* \) is closed under the finite union. When this is the case, \( \tau^* \) is called a quasi-Zariski topology on \( X \) and \( M \) is called a top \( R \)-module [15].
Let $Y$ be a topological space. $Y$ is irreducible if $Y \neq \emptyset$ and for every decomposition $Y = A_1 \cup A_2$ with closed subsets $A_i \subseteq Y, i = 1,2$, we have $A_1 = Y$ or $A_2 = Y$. A subset $T$ of $Y$ is irreducible if $T$ is irreducible as a space with the relative topology. For this to be so, it is necessary and sufficient that, for every pair of sets $F$, $G$ which are closed in $Y$ and satisfy $T \subseteq F \cup G$, $T \subseteq F$ or $T \subseteq G$. Let $F$ be a closed subset of $Y$. An element $y \in Y$ is called a generic point of $Y$ if $Y = \text{cl}(\{y\})$ (here for a subset $Z$ of $Y$, $\text{cl}(Z)$ denotes the topological closure of $Z$).

A topological space $X$ is a spectral space if $X$ is homeomorphic to $\text{Spec}(S)$ with the Zariski topology for some ring $S$. This concept plays an important role in studying of algebraic properties of an $R$-module $M$ when we have a related topology. For an example, when $\text{Spec}(M)$ is homeomorphic to $\text{Spec}(S)$, where $S$ is a commutative ring, we can transfer some of known topological properties of $\text{Spec}(S)$ to $\text{Spec}(M)$ and then by using these properties explore some of algebraic properties of $M$.

Spectral spaces have been characterized by M. Hochster as quasi-compact $T_0$-spaces $X$ having a quasi-compact open base closed under finite intersection and each irreducible closed subset of $X$ has a generic point [9, p. 52, Proposition 4].

The concept of strongly top modules was introduced in [2] and some of its properties have been studied. In this article, we get more information about this class of modules and explore some conditions under which $\text{Spec}(M)$ is a spectral space for its Zariski or quasi-Zariski topology.

In the rest of this article, $X$ will denote $\text{Spec}(M)$. Also the set of all maximal submodules of $M$ is denoted by $\text{Max}(M)$.

2. Main results

**Definition 1** (Definition 3.1 in [1]). Let $M$ be an $R$-module. $M$ is called a strongly top module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $V^*(N) = V^*(IM)$.

**Definition 2** (Definition 3.1 in [2]). Let $M$ be an $R$-module. $M$ is called a strongly top module if $M$ is a top module and $\tau^* = \tau$.

**Remark 1.** Definition 1 and Definition 2 are equivalent. This follows from the fact that if $N$ is a submodule of $M$, then by [11, Result 3], we have


**Remark 2** (Theorem 6.1 in [11]). Let $M$ be an $R$-module. Then the following are equivalent:

(a) $(X, \tau)$ is a $T_0$ space;

(b) The natural map of $X$ is injective;

(c) $V(P) = V(Q)$, that is, $(P : M) = (Q : M)$ implies that $P = Q$ for any $P, Q \in X$;
(d) $|Spec_p(M)| \leq 1$ for every $p \in Spec(R)$.

Remark 3. (a) Let $M$ be an $R$-module and $p \in Spec(R)$. The saturation of a submodule $N$ with respect to $p$ is the contraction of $N_p$ in $M$ and denoted by $S_p(N)$. It is known that

$$S_p(N) = N^{ec} = \{ x \in M | tx \in N \text{ for some } t \in R \setminus p \}.$$  

(b) Let $M$ be an $R$-module and $N \leq M$. The radical of $N$, denoted by $rad(N)$, is the intersection of all prime submodules of $M$ containing $N$; that is,

$$rad(N) = \bigcap_{P \in V^+(N)} P$$  

(c) A topological space $X$ is Noetherian provided that the open (respectively, closed) subsets of $X$ satisfy the ascending (respectively, descending) chain condition ([4, p. 79, Exercises 5-12]).

**Proposition 1.** Let $M$ be an strongly top module and $\psi$ be the natural map of $X$. Then

(a) $(X, \tau) = (X, \tau^*) \cong Im\psi$.

(b) If $X$ is Noetherian, then $X$ is a spectral space.

**Proof.** (a) By [15, Theorem 3.5] and Remark 2, $\psi|Im\psi$ is bijective. Also we have

$$\psi(V(N)) = \{(P : M) | P \in X, (P : M) \supseteq (N : M)\}.$$  

Now by [11, Proposition 3.1] and the above arguments, $\psi$ is continuous and a closed map. Consequently we have $(X, \tau) = (X, \tau^*) \cong Im\psi$.

(b) Let $Y = V^+(N)$ be an irreducible closed subset of $X$. Now by [6, Theorem 3.4], we have

$$V^+(N) = V^+(rad(N)) = cl(\{rad(N)\}).$$  

Hence $Y$ has a generic point. Also $X$ is Noetherian and it is a $T_0$-space by [6, Proposition 3.8 (i)]. Hence it is a spectral space by [9, Pages 57 and 58].

An $R$-module $M$ is said to be a *weak multiplication module* if either $X = Spec(M) = \emptyset$ or $X \neq \emptyset$ and for every prime submodule $P$ of $M$, we have $P = IM$ for some ideal $I$ of $R$ (see [5]).

The following theorem extends [1, Proposition 3.5], [1, Corollary 3.6], [1, Theorem 3.9 (1)], and [1, Theorem 3.9 (7)]. In fact, in part (a) of this theorem, we withdraw the restrictions of finiteness and Noetherian property from [1, Proposition 3.5] and [1, Corollary 3.6], respectively. In part (b), we remove the conditions “$M$ is primeful ” and “$R$ is a Noetherian ring ” in [1, Theorem 3.9 (1)] and instead of them, we put the weaker conditions “ $Im(\psi)$ is closed in $Spec(\overline{R})$ ” and “ $Spec(\overline{R})$ is a Noetherian space ”. In part (c), we withdraw the condition “ $R$ has Noetherian spectrum ” from [1, Theorem 3.9 (7)] and put the weaker condition “ the intersection of every infinite family of maximal ideals of $R$ is zero ”.

**Theorem 1.** Let $M$ be an $R$-module. Then we have the following.
(a) Let \((M_i)_{i \in I}\) be a family of \(R\)-modules and let \(M = \bigoplus_{i \in I} M_i\). If \(M\) is an strongly top \(R\)-module, then each \(M_i\) is a strongly top \(R\)-module.

(b) If \(M\) be an strongly top \(R\)-module and \(\psi\) be the natural map of \(X\), then we have

(i) If \(\text{Im}(\psi)\) is closed in \(\text{Spec}(\overline{R})\), then \((X, \tau) = (X, \tau^*)\) is a spectral space.

(ii) If \(\text{Spec}(\overline{R})\) is Noetherian, then \((X, \tau) = (X, \tau^*)\) is a spectral space.

(c) Suppose \(R\) is a one dimensional integral domain such that the intersection of every infinite family of its maximal ideals is zero. If \(M\) is a weak multiplication \(R\)-module, then \(M\) is a top module.

Proof. (a) Each \(M_i\) is a homomorphic image of \(M\), hence it is strongly top by \([1, \text{Proposition 3.3}]\).

(b) (i) By Proposition 1, we have \((X, \tau) = (X, \tau^*) \cong \text{Im}(\psi)\). Now the claim follows by \([11, \text{Theorem 6.7}]\).

(ii) As \(\text{Spec}(\overline{R})\) is Noetherian, \(\text{Im}(\psi)\) is also Noetherian. Now the claim follows from Proposition 1.

(c) Use the technique of \([3, \text{Theorem 3.18}]\). \(\square\)

The following theorem extends \([1, \text{Theorem 3.9(3)}]\).

**Theorem 2.** Suppose \(R\) is a one dimensional integral domain such that the intersection of every infinite family of its maximal ideals is zero. If \(M\) is \(X\)-injective with \(S_0(0) \subseteq \text{rad}(0)\), then \(M\) is a top module.

Proof. If \(S_0(0) = M\), then \(X = \emptyset\) and there is nothing to prove. Otherwise, by \([12, \text{Corollary 3.7}]\), \(S_0(0)\) is a prime submodule so that \(S_0(0) = \text{rad}(0)\). Hence the natural map \(f : \text{Spec}(M/S_0(0)) \to \text{Spec}(M)\) is a homeomorphism by \([7, \text{Proposition 1.4}]\). But by \([3, \text{Theorem 3.7 (a)}]\) and \([3, \text{Theorem 3.15 (e)}]\), \(M/S_0(0)\) is a weak multiplication module. Now the result follows because by Theorem 1 (c), \(M/S_0(0)\) is a top module. \(\square\)

Let \(M\) be an \(R\)-module. Then \(M\) is called a content module if for every \(x \in M\), \(x \in c(x)M\), where \(c(x) = \bigcap\{I \mid I\text{ is an ideal of } R\text{ such that } x \in IM\}\) (see \([13, p. 140])\).

In below we generalize \([1, \text{Theorem 3.9(4)}]\).

**Theorem 3.** Suppose \(R\) is a one dimensional integral domain and let \(M\) be a content \(R\)-module. Then we have the following.

(a) If \(M\) is \(X\)-injective, then \(M\) is a top module.

(b) If \(M\) is \(X\)-injective and \(S_0(0) \subseteq \text{rad}(0)\), then \(M\) is an strongly top module. Furthermore, if \(\text{Spec}(\overline{R})\) is Noetherian, then \((X, \tau^*)\) is spectral.
Proof. (a) By [3, Theorem 3.21], we have
\[ \text{Spec}(M) = \{ S_p(pM) \mid p \in V(\text{Ann}(M)), S_p(pM) \neq M \} = \{ S_0(0) \} \cup \text{Max}(M), \]
where
\[ \text{Max}(M) = \{ pM \mid p \in \text{Max}(R), pM \neq M \}. \]
Let \( N \subseteq M \) and let \( N \not\subseteq S_0(0) \). Then
\[ \text{rad}(N) = \bigcap_{N \subseteq P \in \text{Spec}(M)} P = \bigcap_{N \subseteq P \in \text{Max}(M)} P. \]
So by the above arguments, there is an index set \( I \) such that \( \text{rad}(N) = \bigcap_{i \in I} (p_i M) \).
Since \( M \) is content module,
\[ V^*(N) = V^*(\text{rad}(N)) = V^*(\bigcap_{i \in I} (p_i M)) = V((\bigcap_{i \in I} p_i M)). \]
Now if \( N \subseteq S_0(0) \), then by [10, Lemma 2],
\[ V^*(N) = V^*(\text{rad}(N)) = V^*(S_0(0) \cap (\bigcap_{i \in I} (p_i M))) \]
\[ = V^*(S_0(0)) \cap (\bigcap_{i \in I} p_i M) \]
\[ = V^*(S_0(0)) \cup V^*((\bigcap_{i \in I} p_i M)) \]
\[ = V^*(S_0(0)) \cup V((\bigcap_{i \in I} p_i M)). \]
By the above arguments, it follows that \( M \) is a top module.
(b) By [3, Theorem 3.21],
\[ \text{Spec}(M) = \{ S_0(0) \} \cup \text{Max}(M) \text{ and } \text{Max}(M) = \{ pM \mid p \in \text{Max}(R), pM \neq M \}. \]
Let \( N \subseteq M \). If \( N \subseteq S_0(0) \), then \( V^*(N) = V^*(0) = X \). Otherwise, we have \( \text{rad}(N) = \bigcap_{i \in I} (p_i M) \) by [3, Theorem 3.21]. Since \( M \) is content, by [11, Result 3] we have
\[ V^*(N) = V^*(\text{rad}(N)) = V^*(\bigcap_{i \in I} (p_i M)) = V((\bigcap_{i \in I} p_i M)). \]
Hence \( M \) is an strongly top module. The second assertion follows from Theorem 1 (b).

Theorem 4. If \( M \) is content weak multiplication, then \( M \) is an strongly top module. Moreover, if \( \text{Spec}(R) \) is Noetherian, then \((X, \tau^*)\) is a spectral space.
Proof. Let \( N \leq M \). Then we have

\[
V^*(N) = V^*(\text{rad}(N)) = V^*(\bigcap_{P} P).
\]

Since \( M \) is a weak multiplication module, for each prime submodule \( P \) of \( M \) containing \( N \), there exists an ideal \( I_P \) of \( R \) such that \( P = I_PM \). Hence since \( M \) is a content module,

\[
V^*(N) = V^*(\bigcap_{N \leq P} (I_PM)) = V^*(\bigcap_{N \leq P} I_PM).
\]

This implies that \( M \) is an strongly top module. Since \( \text{Spec}(R) \) is Noetherian, so is \( \text{Spec}(\overline{R}) \). Hence by Theorem 1 (b), \((X, \tau^*)\) is a spectral space. \( \square \)

**Theorem 5.** Let \( R \) be a one-dimensional integral domain and let \( M \) be an \( X \)-injective \( R \)-module such that \( S_0(0) \subseteq \text{rad}(0) \). If the intersection of every infinite number of maximal submodules of \( M \) is zero, then \( M \) is strongly top and \((X, \tau^*)\) is a spectral space.

Proof. If \( S_0(0) = M \), then \( X = \emptyset \) and there is nothing to prove. Otherwise, by [3, Theorem 3.21], we have \( \text{Spec}(M) = \{S_0(0)\} \cup \text{Max}(M) \) and \( \text{Max}(M) = \{pM \mid p \in \text{Max}(R), pM \neq M\} \). Now let \( N \leq M \). If \( N = 0 \), then claim clear because \( V^*(N) = V^*(0) = V^*(0M) = \text{Spec}(M) \). So we assume that \( N \neq 0 \). We consider two cases.

1. \( N \subseteq S_0(0) \). In this case, we have \( V^*(N) = V^*(0) = V^*(0M) = \text{Spec}(M) \).
2. \( N \not\subseteq S_0(0) \). Then since \( N \neq 0 \) and the intersection of every infinite number of maximal submodules of \( M \) is zero, \( \text{rad}(N) = \bigcap_{i=1}^{n} (p_iM) \), where \( p_iM \in \text{Max}(M) \) for each \( i (1 \leq i \leq n) \). Hence we have

\[
V^*(N) = V^*(\text{rad}(N)) = V^*(\bigcap_{i=1}^{n} (p_iM)).
\]

Now we show that \( V^*(\bigcap_{i=1}^{n} (p_iM)) = V^*(\bigcap_{i=1}^{n} p_iM) \). Clearly, \( V^*(\bigcap_{i=1}^{n} (p_iM)) \subseteq V^*(\bigcap_{i=1}^{n} p_iM) \). Too see this reverse inclusion, let \( P \in V^*(\bigcap_{i=1}^{n} p_iM) \). If \( P = S_0(0) \), then \( \bigcap_{i=1}^{n} p_iM \subseteq S_0(0) \) implies that \( \bigcap_{i=1}^{n} p_i \subseteq (\bigcap_{i=1}^{n} p_iM : M) \subseteq (S_0(0) : M) = 0 \). Thus, there exists \( j \) \( (1 \leq j \leq n) \) such that \( p_j = 0 \), a contradiction. Hence we must have \( P = qM \), where \( q \in \text{Max}(R) \). Then, similar the above arguments, there exists \( j \) \( (1 \leq j \leq n) \) such that \( q = p_j \). Therefore, \( P = qM = p_j M \in V^*(\bigcap_{i=1}^{n} p_iM) \). So we have

\[
V^*(N) = V^*(\bigcap_{i=1}^{n} (p_iM)) = V^*(\bigcap_{i=1}^{n} p_iM).
\]

Hence \( M \) is strongly top so that \( \tau = \tau^* \). On the other hand, \( \tau = \tau^* \) is a subset of a finite complement topology. This implies that \((X, \tau^*)\) is Noetherian. Now by Proposition 1, \((X, \tau^*) = (X, \tau)\) is spectral. \( \square \)
**Theorem 6.** If for each submodule \( N \) of \( M \), \( \text{rad}(N) = \sqrt{(N : M)M} \), then \( M \) is an strongly top module. Moreover, if \( \text{Spec}(R) \) is Noetherian, then \( (X, \tau^*) \) is spectral.

**Proof.** Let \( N \leq M \). Then we have

\[
\]

Hence \( M \) is an strongly top module. Now the result follows by using similar arguments as in the proof of Theorem 4. □

**Remark 4.** Theorems 4, 5, and 6 improve respectively [1, Theorem 3.9(5)], [1, Theorem 3.9(8)], and [1, Theorem 3.9(6)]. They show that the notion of “top modules” can be replaced by “strongly top modules” and the proofs can be shortened considerably.

In below we generalize [1, Theorem 3.36].

**Theorem 7.** Let \( M \) be a primeful \( R \)-module. Then we have the following.

(a) If \((X, \tau)\) is discrete, then \( \text{Spec}(M) = \text{Max}(M) \).

(b) If \( R \) is Noetherian and \( \text{Spec}(M) = \text{Max}(M) \), then \((X, \tau)\) is a finite discrete space.

**Proof.** (a) Since \((X, \tau)\) is discrete, it is a \( T_1 \)-space. Now by [3, Theorem 4.3], we have \( \text{Spec}(M) = \text{Max}(M) \).

(b) By [3, Theorem 4.3], \( \text{Spec}(R) = \text{Max}(R) \). Hence \( R \) is Artinian. Now by [3, Theorem 4.3], \((X, \tau)\) is a \( T_0 \)-space. Thus by Remark 2, \( M \) is \( X \)-injective. But \( M \) is a cyclic \( R \)-module and hence a cyclic \( R \)-module by [3, Remark 3.13] and [3, Theorem 3.15]. Also \((\text{Spec}(M), \tau)\) is homeomorphic to \( \text{Spec}(R) \) by [11, Theorem 6.5(5)]. Hence \( X \) is a finite discrete space by [4, Chapter 8, Exe 2]. □

It is well known that if \( R \) is a PID and \( \text{Max}(R) \) is not finite, then the intersection every infinite number of maximal ideals of \( R \) is zero. Now it is natural to ask the following question: Is the same true when \( R \) is a one dimensional integral domain with infinite maximal ideals? In below, we show that this true when \( \text{Spec}(R) \) is a Noetherian space. Although this is not a simple fact, it used by some authors without giving any proof.

**Theorem 8.**

(a) Let \( I \) be an ideal of \( R \) and let \( k, n \in \mathbb{N} \). Then \( (\sqrt{I} : a^k) = (\sqrt{I} : a^n). \)

(b) Let \( I \) be an ideal of \( R \) and let \( a \in R, n \in \mathbb{N} \). Then \( \sqrt{I} = \sqrt{(\sqrt{I} : a^n) \cap \sqrt{(\sqrt{I}, a^n)}}. \)
(c) Suppose $\text{Spec}(R)$ is a Noetherian topological space. Then for every ideal $I$ of $R$, $\sqrt{I}$ has a primary decomposition.

(d) Suppose $R$ is a one dimensional integral domain and $\text{Spec}(R)$ is a Noetherian topological space. Then the intersection of every infinite number of maximal ideals is zero.

**Proof.** (a) It is clear.

(b) Let $f \in \sqrt{(\sqrt{T} : a^n) \cap (\sqrt{T}, a^n)}$. Then there is $m \in \mathbb{N}$ such that $f^m \in (\sqrt{T} : a^n) \cap (\sqrt{T}, a^n)$. It follows that $f^m = g + xa^n$ for some $g \in \sqrt{T}$ and $x \in R$ and we also get $a^n f^m \in \sqrt{T}$. Hence $a^n f^m = a^n g + xa^{2n}$. This implies that $xa^{2n} \in \sqrt{T}$ and so $x \in (\sqrt{T} : a^n)$ by part (a). Thus $xa^n \in \sqrt{T}$. It follows that $f \in \sqrt{T}$. The reverse inclusion is clear.

(c) Set $\dot{\Sigma} = \{\sqrt{I} \mid I \text{ is a proper ideal of } R \text{ and } \sqrt{I} \text{ doesn't have any primary decomposition}\}$. Since $\text{Spec}(R)$ is Noetherian, the radicals of ideals satisfy the a.c.c. condition. So $\dot{\Sigma}$ has a maximal member, $\sqrt{T}_0$ say. Thus $\sqrt{T}_0 \subseteq \sqrt{T}$ and $\hat{T}_0 \subseteq \sqrt{T}_0$. Since $\sqrt{\sqrt{T}_0 : b}$ and $\sqrt{\sqrt{T}_0, b}$ have primary decompositions by hypothesis, $\sqrt{T}_0$ has a primary decomposition, a contradiction.

(d) Since $R$ is one dimensional integral domain, $\text{Spec}(R) = \{0\} \cup \text{Max}(R)$. Suppose $\{m_i\}_{i \in I}$ is an infinite family of maximal ideals of $R$ such that $\bigcap_{i \in I} m_i \neq 0$. By part (c), $\bigcap_{i \in I} m_i$ has a primary decomposition. Hence

$$\sqrt{\bigcap_{i \in I} m_i} = \bigcap_{j=1}^n m_j', \quad m_j' \in \text{Max}(R).$$

This implies that $\{m_i\}_{i \in I}$ is a finite family, a contradiction. So the proof is completed. \qed

**Example 1.** We show that $\mathbb{Z}[i \sqrt{5}]$ is a one dimensional Noetherian integral domain which has infinite number of maximal ideals and it is not a PID. To see this, let $\phi : \mathbb{Z}[X] \rightarrow \mathbb{Z}[i \sqrt{5}]$ be the natural epimorphism given by $p(x) \mapsto p(i \sqrt{5})$. By using [8] or [16], one can see that

$$\text{Spec}(\mathbb{Z}[X]) = \{(p), (f), (q, g) \mid p \text{ and } q \text{ are prime numbers, } f \text{ is a primary irreducible polynomial in } \mathbb{Q}[X], \text{ and } g \text{ is an irreducible polynomial in } \mathbb{Z}_q[X]\}.$$  

Now we have $\ker \phi = \langle X^2 + 5 \rangle$. A simple verification shows that $\text{Spec}(\mathbb{Z}[i \sqrt{5}]) = \{0\} \cup \text{Max}(\mathbb{Z}[i \sqrt{5}])$.
$= \{0\} \cup \{(q, g(\sqrt{-5})) | (q, g) \in \text{Spec}(\mathbb{Z}[X]) \text{ and } X^2 + 5 \in \langle q, g \rangle\}.$

Further $\mathbb{Z}[i \sqrt{5}]$ contains a finite number elements which are invertible by [17, p. 38]. So $\mathbb{Z}[i \sqrt{5}]$ is a Noetherian one dimensional integral domain with infinite number of maximal ideals. Hence the intersection of every infinite number of maximal ideals of $\mathbb{Z}[i \sqrt{5}]$ is zero by Theorem 8 (c). Note that $\mathbb{Z}[i \sqrt{5}]$ is not a PID by [17, p. 38].

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