Some common fixed point theorems for a class of fuzzy contractive mappings

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SOME COMMON FIXED POINT THEOREMS FOR A CLASS OF FUZZY CONTRACTIVE MAPPINGS

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Abstract. The purpose of this paper is to state and prove a new lemma generalizing Lemma 3.1 of Arora and Sharma [1] and Proposition 3.2 of Lee and Cho [10]. Some common fixed point theorems for a type of fuzzy contractive mappings are also established. These theorems extend and generalize several previous results [3, 14, 21, 22].

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1. Introduction

Common fixed point theorems have been applied to diverse problems during the last few decades. These theorems provide techniques for solving a variety of applied problems in mathematical science and in dynamic programming (see, e.g., [4, 15, 16]). Extensions of the Banach contraction principle to multivalued mappings were initiated independently by Markin [11] and Nadler [13]. Therefore, results on fixed points of contractive type multivalued mappings have been carried out by many authors (see, for example, [2, 17, 21]).

The theory of fuzzy sets was investigated by Zadeh [24] in 1965. Some applications on results in this theory are discussed (see [9, 23]). In 1981, Heilpern [7] first introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set maps of Nadler [13]. Later, several fixed point theorems for types of fuzzy contractive mappings appeared (see, for instance, [1, 18–20]).

In this paper, we state and prove a new lemma generalizing Lemma 3.1 of Arora and Sharma [1] and Proposition 3.2 of Lee and Cho [10]. Two common fixed point theorems of a type of fuzzy contractive mappings are established. These theorems generalize and extend results in [3, 14, 21, 22]. Finally, we state a conclusion containing a brief of our results and future research.
2. Basic Preliminaries

The definitions and terminologies for further discussions are taken from Heilpern [7]. Let \( (X,d) \) be a metric linear space. A fuzzy set in \( X \) is a function with domain \( X \) and values in \([0,1]\). If \( A \) is a fuzzy set and \( x \in X \), then the function-value \( A(x) \) is called the grade of membership of \( x \) in \( A \). The collection of all fuzzy sets in \( X \) is denoted by \( \mathcal{F}(X) \).

Let \( A \in \mathcal{F}(X) \) and \( \alpha \in [0,1] \). The \( \alpha \)-level set of \( A \), denoted by \( A_\alpha \), is defined by the formula

\[
A_\alpha = \begin{cases} 
\{ x : A(x) \geq \alpha \} & \text{if } \alpha \in (0,1], \\
\{ x : A(x) > 0 \} & \text{if } \alpha = 0. 
\end{cases}
\]  

(2.1)

where \( \overline{B} \) is the closure of a (nonfuzzy) set \( B \).

**Definition 1.** A fuzzy set \( A \) in \( X \) is an approximate quantity if and only if its \( \alpha \)-level set is a nonempty compact convex (nonfuzzy) subset of \( X \) for each \( \alpha \in [0,1] \) and \( \sup_{x \in X} A(x) = 1 \).

The set of all approximate quantities, denoted \( W(X) \), is a subcollection of \( \mathcal{F}(X) \).

**Definition 2.** Let \( A,B \in W(X) \), \( \alpha \in [0,1] \) and \( CP(X) \) be the set of all nonempty compact subsets of \( X \). Then one puts \( p_\alpha(A,B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x,y) \), \( \delta_\alpha(A,B) = \sup_{x \in A_\alpha, y \in B_\alpha} d(x,y) \), and \( D_\alpha(A,B) = H(A_\alpha,B_\alpha) \), where \( H \) is the Hausdorff metric between two sets in the collection \( CP(X) \).

We define the functions \( p(A,B) = \sup_\alpha p_\alpha(A,B) \), \( \delta(A,B) = \sup_\alpha \delta_\alpha(A,B) \), and \( D(A,B) = \sup_\alpha D_\alpha(A,B) \).

Note that \( p_\alpha \) is nondecreasing function of \( \alpha \).

**Definition 3.** Let \( A,B \in W(X) \). Then \( A \) is said to be more accurate than \( B \) (or \( B \) includes \( A \)), denoted by \( A \subset B \), if and only if \( A(x) \leq B(x) \) for each \( x \in X \).

The relation \( \subset \) induces a partial ordering on \( W(X) \).

**Definition 4.** Let \( X \) be an arbitrary set and \( Y \) be a metric linear space. \( F \) is said to be a fuzzy mapping if and only if \( F \) is a mapping from the set \( X \) into \( W(Y) \), i.e., \( F(x) \in W(Y) \) for each \( x \in X \).

The following lemma and proposition are used in the sequel.

**Lemma 1** ([12]). Suppose that \( \gamma : [0,\infty) \rightarrow [0,\infty] \) is a right continuous function such that \( \gamma(t) < t \) for all \( t > 0 \). Then for every \( t > 0 \), \( \lim_{n \to \infty} \gamma^n(t) = 0 \), where \( \gamma^n \) is the \( n \)-th iterate of \( \gamma \), \( n \in \mathbb{N} \cup \{0\} \).

**Proposition 1** ([13]). If \( A,B \in CP(X) \) and \( a \in A \), then there exists \( b \in B \) such that \( d(a,b) \leq H(A,B) \).

*\( \mathbb{N} \) is the set of all positive integers.*
We consider the set $\Phi$ of all functions $\phi : [0, \infty)^5 \to [0, \infty)$ with the following properties:

(i) $\phi$ is nondecreasing with respect to each variable,
(ii) $\phi$ is right continuous with respect to each variable,
(iii) for each $t > 0$, $\Psi(t) = \max\{\phi(t,t,t,t,t), \phi(t,t,t,2t,0), \phi(t,t,t,0,2t)\} < t$.

3. Main Results

Throughout this paper, let $(X, d)$ be a metric space. We consider a subcollection of $\mathcal{F}(X)$ denoted by $W^*(X)$. Each fuzzy set $A \in W^*(x)$, its $\alpha$-level set is a nonempty compact (nonfuzzy) subset of $X$ for each $\alpha \in [0, 1]$. It is obvious that each element $A \in W(X)$ leads one to $A \in W^*(X)$ but the converse is not true. Now, we introduce the improvements of the lemmas in Heilpern [7] as follows.

**Lemma 2.** If $\{x_0\} \subset A$ for each $A \in W^*(X)$ and $x_0 \in X$, then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $B \in W^*(X)$.

**Lemma 3.** $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for all $x, y \in X$ and $A \in W^*(X)$.

**Lemma 4.** Let $x \in X$, $A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.

**Proof.** If $\{x\} \subset A$, then $x \in A_\alpha$ for each $\alpha \in [0, 1]$. This implies that $p_\alpha(x, A) = \inf_{y \in A_\alpha} d(x, y) = 0$ for any $\alpha \in [0, 1]$. Conversely, if $p_\alpha(x, A) = 0$, then we have $\inf_{y \in A_\alpha} d(x, y) = 0$. It follows that $x \in A_\alpha = A_\alpha$ for an arbitrary $\alpha \in [0, 1]$. Then $\{x\} \subset A$. \qed

Also, we state and prove a new lemma in the following way.

**Lemma 5.** Let $(X, d)$ be a complete metric space, $F : X \to W^*(X)$ be a fuzzy map and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

**Proof.** For $n \in \mathbb{N}$, $((F(x_0))_{n/(n+1)})$ is a decreasing sequence of nonempty compact subsets of $X$. Thus we have from Proposition 11.4 and Remark 11.5 of [25, pp. 495–496] that $\bigcap_{n=1}^{\infty} (F(x_0))_{n/(n+1)}$ is nonempty and compact.

Let $x_1 \in \bigcap_{n=1}^{\infty} (F(x_0))_{n/(n+1)}$. Then $\frac{n}{n+1} \leq (F(x_0))(x_1) \leq 1$. As $n \to \infty$, we get that $(F(x_0))(x_1) = 1$. This implies that $\{x_1\} \subset F(x_0)$. \qed

**Remark 1.** It is clear that Lemma 5 is a generalization of Lemma 3.1 of Arora and Sharma [1] and Proposition 3.2 of Lee and Cho [10].

Now, we are ready to prove our main theorems.
Theorem 1. Let \((X, d)\) be a complete metric space and \(F_1, F_2\) be fuzzy mappings from \(X\) into \(W^*(X)\). If there is a \(\phi \in \Phi\) such that for all \(x, y \in X\),

\[
D(F_1(x), F_2(y)) \leq \phi(d(x, y), p(x, F_1(x)), p(y, F_2(y)), p(x, F_1(x)), p(y, F_1(x))), \quad (3.1)
\]
then there exists \(z \in X\) such that \(\{z\} \subset F_1(z)\) and \(\{z\} \subset F_2(z)\).

Proof. Let \(x_0 \in X\). Then by Lemma 5, there exists \(x_1 \in X\) such that \(\{x_1\} \subset F_1(x_0)\). For \(x_1 \in X\), the set \((F_2(x_1))_1\) is nonempty compact subset of \(X\). Since \((F_1(x_0))_1\) and \((F_2(x_1))_1\) belong to \(CP(X)\) and \(x_1 \in (F_1(x_0))_1\), Proposition 1 asserts that there exists \(x_2 \in (F_2(x_1))_1\) such that \(d(x_1, x_2) \leq D_1(F_1(x_0), F_2(x_1))\). So, we have from Lemma 4 and the property (i) of \(\phi\) that

\[
d(x_1, x_2) \leq D_1(F_1(x_0), F_2(x_1)) \leq D(F_1(x_0), F_2(x_1)) \leq \phi(d(x_0, x_1), p(x_0, F_1(x_0)), p(x_1, F_2(x_1)), p(x_0, F_2(x_1)), p(x_1, F_1(x_0))) \\
\leq \phi(d(x_0, x_1), d(x_0, x_2), d(x_1, x_1), d(x_0, x_1) + d(x_1, x_2), 0).
\]

If \(d(x_1, x_2) > d(x_0, x_1)\), then

\[
d(x_1, x_2) \leq \phi(d(x_1, x_2), d(x_1, x_2), d(x_1, x_2), 2d(x_1, x_2), 0) < d(x_1, x_2).
\]
This contradiction demands that

\[
d(x_1, x_2) \leq \phi(d(x_0, x_1), d(x_0, x_1), d(x_0, x_1), 2d(x_0, x_1), 0).
\]

Similarly, one can deduce that

\[
d(x_2, x_3) \leq \phi(d(x_1, x_2), d(x_1, x_2), d(x_1, x_2), 0, 2d(x_1, x_2)).
\]

By induction, we have a sequence \((x_n)\) of points in \(X\) such that, for all \(n \in \mathbb{N} \cup \{0\}\),

\[
\{x_{2n+1}\} \subset F_1(x_{2n}), \quad \{x_{2n+2}\} \subset F_2(x_{2n+1}).
\]

It follows by induction that \(d(x_{n}, x_{n+1}) \leq \Psi^n(d(x_0, x_1))\), where \(\Psi\) is defined in the property (iii) of \(\phi\). Then, Lemma 1 gives that \(\lim_{n \to \infty} d(x_{n}, x_{n+1}) = 0\). Since

\[
d(x_{n}, x_{m}) \leq d(x_{n}, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_{m}),
\]

then \(\lim_{n, m \to \infty} d(x_{n}, x_{m}) = 0\). Therefore, \((x_n)\) is a Cauchy sequence. Since \(X\) is a complete metric space, then there exists \(z \in X\) such that \(\lim_{n \to \infty} x_n = z\). Next, we show that \(\{z\} \subset F_i(z), i = 1, 2\). Now, we get from Lemma 2 and Lemma 3 that

\[
p_a(z, F_2(z)) \leq d(z, x_{2n+1}) + D_1(F_1(x_{2n+1}), F_2(z)) \leq d(z, x_{2n+1}) + D_a(F_1(x_{2n+1}), F_2(z)),
\]
for each $\alpha \in [0, 1]$. Taking the supremum on $\alpha$ in the last inequality, we obtain from the property (i) of $\phi$ that
\[
p(z, F_2(z)) \leq d(z, x_{2n+1}) + D(F_1(x_{2n}), F_2(z))
\leq d(z, x_{2n+1}) + \phi(d(x_{2n}, z), p(x_{2n}, F_1(x_{2n})), p(z, F_2(z)), p(x_{2n}, F_2(z)), p(z, F_1(x_{2n})))
\leq d(z, x_{2n+1}) + \phi(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), p(z, F_2(z)), p(x_{2n}, F_2(z)), d(z, x_{2n+1})).
\]
As $n \to \infty$, we have from the properties (i), (ii) and (iii) of $\phi$ with $p(z, F_2(z)) \neq 0$ that
\[
p(z, F_2(z)) \leq \phi(0,0, p(z, F_2(z)), p(z, F_2(z)), 0)
\leq \phi(p(z, F_2(z)), p(z, F_2(z)), p(z, F_2(z)), p(z, F_2(z)), p(z, F_2(z)))
< p(z, F_2(z)).
\]
This contradiction yields $p(z, F_2(z)) = 0$. We then get from Lemma 4 that $\{z\} \subset F_2(z)$. Similarly, one can show that $\{z\} \subset F_1(z)$.

Example 1. Let $X = [0, 1]$ endowed with the metric $d$ defined by $d(x, y) = |x - y|$. It is clear that $(X, d)$ is a complete metric space. Assume that $\phi(t_1, t_2, t_3, t_4, t_5) = \frac{3}{4} t_1$ for arbitrary $t_i \in [0, \infty)$, $i = 1, 5$. It is obvious that $\Psi(t) < t$ for all $t > 0$. Let $F_1 = F_2 = F$. Define a fuzzy mapping $F$ on $X$ such that for all $x \in X$, $F(x)$ is the characteristic function for $\{\frac{3}{4} x\}$. For each $x, y \in X$,
\[
D(F(x), F(y)) = \frac{3}{4} d(x, y)
= \phi(d(x, y), p(x, F(x)), p(y, F(y)), p(x, F(y)), p(y, F(x))).
\]
The characteristic function for $\{0\}$ is the fixed point of $F$.

As corollaries of Theorem 1, we get the following statements.

Corollary 1. Let $(X, d)$ be a complete metric space and $F_1, F_2$ be fuzzy mappings from $X$ into $W^+(X)$ satisfying the following conditions: for any $x, y \in X$,
\[
D(F_1(x), F_2(y)) \leq a_1 p(x, F_1(x)) + a_2 p(y, F_2(y)) + a_3 p(y, F_1(x)) + a_4 p(y, F_1(x)) + a_5 d(x, y),
\]
where $a_1, a_2, a_3, a_4$, and $a_5$ are non-negative real numbers, $\sum_{j=1}^{5} a_i < 1$ and $a_1 = a_2$ or $a_3 = a_5$. Then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. We consider the function $\phi : [0, \infty)^5 \to [0, \infty)$ defined by the formula
\[
\phi(x_1, x_2, x_3, x_4, x_5) = a_1 x_2 + a_2 x_3 + a_3 x_5 + a_4 x_4 + a_5 x_1,
\]
where $\sum_{i=1}^{5} a_i < 1$ such that $a_1 = a_2$ or $a_3 = a_4$. Since $\phi \in \Phi$, we have from Theorem 1 that there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.
The following corollary is a fuzzy version of the fixed point theorem of Singh and Whitfield [21] for multivalued mappings.

**Corollary 2.** Let \((X, d)\) be a complete metric space and \(F_1, F_2\) be fuzzy mappings from \(X\) into \(W^*(X)\). If there is a constant \(\alpha, 0 \leq \alpha < 1\), such that, for each \(x, y \in X\),

\[
D(F_1(x), F_2(y)) \leq \alpha \max \left\{ d(x, y), \frac{1}{2}[p(x, F_1(x)) + p(y, F_2(y))], \frac{1}{2}[p(x, F_2(y)) + p(y, F_1(x))] \right\},
\]

then there exists \(z \in X\) such that \(\{z\} \subset F_1(z)\) and \(\{z\} \subset F_2(z)\).

**Proof.** We consider the function \(\phi : [0, \infty)^5 \to [0, \infty)\) defined by

\[
\phi(x_1, x_2, x_3, x_4, x_5) = \alpha \max \left\{ x_1, \frac{1}{2}[x_2 + x_3], \frac{1}{2}[x_4 + x_5] \right\}.
\]

Since \(\phi \in \Phi\), we get from Theorem 1 that there exists \(z \in X\) such that \(\{z\} \subset F_1(z)\) and \(\{z\} \subset F_2(z)\).

**Remark 2.** (1) If there is a \(\phi \in \Phi\) such that, for each \(x, y \in X\),

\[
\delta(F_1(x), F_2(y)) \leq \phi(d(x, y), p(x, F_1(x)), p(y, F_2(y)), p(x, F_2(y)), p(y, F_1(x))),
\]

then the conclusion of Theorem 1 remains valid. This result is considered as a special case of Theorem 1 because \(\delta(F_1(x), F_2(y)) \leq \delta(F_1(x), F_2(y)) \leq \delta(F_1(x), F_2(y))\) [8, p. 414]. Moreover, this result generalizes Theorem 3.3 of Park and Jeong [14].

(2) Corollary 1 is [22, Theorem 3.1] without condition (a), where condition (a) reads as follows: “for each \(x \in X\), there exists \(\alpha(x) \in (0, 1]\) such that \((F_1(x))_{\alpha(x)}\) and \((F_2(x))_{\alpha(x)}\) are nonempty closed bounded subsets of \(\mathcal{F}(X)\).” Also, Corollary 1 generalizes [3, Theorem 3.1].

(3) Theorems 3.1 and 3.4 of Park and Jeong [14] are special cases of Theorem 1.

The following theorem generalizes Theorem 1 to a sequence of fuzzy contractive mappings.

**Theorem 2.** Let \((F_n : n \in \mathbb{N} \cup \{0\})\) be a sequence of fuzzy mappings from a complete metric space \((X, d)\) into \(W^*(X)\). If there is a \(\phi \in \Phi\) such that, for all \(x, y \in X\),

\[
D(F_0(x), F_n(y)) \leq \phi(d(x, y), p(x, F_0(x)), p(y, F_n(y)), p(x, F_n(y)), p(y, F_0(x))) \quad \forall (n \in \mathbb{N}),
\]

then there exists a common fixed point of the family \((F_n : n \in \mathbb{N} \cup \{0\})\).

**Proof.** Putting \(F_1 = F_0\) and \(F_2 = F_n\) for all \(n \in \mathbb{N}\) in Theorem 1. Then there exists a common fixed point of the family \((F_n : n \in \mathbb{N} \cup \{0\})\).
Remark 3. If there is a $\phi \in \Phi$ such that, for all $x, y \in X$,

$$
\delta(F_0(x), F_n(y)) \leq \phi(d(x, y), p(x, F_0(x)), p(y, F_n(y)), p(x, F_n(y)), p(y, F_0(x))) \quad (\forall n \in \mathbb{N}), \quad (3.9)
$$

then the conclusion of Theorem 2 remains valid. This result is considered as a special case of Theorem 2 for the same reason as in Remark 2 (1).

4. Conclusion

This paper presents an improvement of some results in [1, 7, 10]. Also, it presents two common fixed point theorems for a type of fuzzy contractive mappings. These theorems generalize and extend results in [3, 14, 22] and [21], respectively. A fixed point theorem for fuzzy contractive mappings is stated generalizing [1, Theorem 3.5]. Many applications of our main theorems are possible, e. g., for differential and integral equations. In view of the references [5,6], some future research can be done, for example:

1. I believe that our results can be hold for $FC(X)$, where $FC(X) = \{ A \in \mathcal{F}(X) : A_\alpha$ is a nonempty closed (nonfuzzy) subset of $X$ for each $\alpha \in [0, 1]\}$,

2. it is also possible to generalize our results to quasi-metric spaces.

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References


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