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NOTES ON GENERALIZED (α,β)-DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a prime ring with characteristics different from two, $\alpha,\beta \in \mathrm{Aut}(R)$, and I a nonzero ideal of R. Let $d: R \to R$ be a (α,β)-derivation and $f: R \to R$ be generalized (α,β)-derivation associated with d. In this case: (i) if $af(x) = 0$ ($f(x)a = 0$) for all $x \in I$, then $a = 0$; (ii) if $\beta d = d\beta$ and $[f(x), a] = 0$ for all $x \in I$, then $a \in Z$ or $d(a) = 0$; (iii) if f acts as an endomorphism or anti-homomorphism on I, then $d = 0$; (iv) if $\beta d = d\beta$ and $f(xy) = f(yx)$ for all $x, y \in I$ then R is a commutative ring; (v) if f satisfies $f(x^2) = f(x)\alpha(x) + \beta(x)d(x)$ for all $x \in I$, then $f(xy) = f(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in I$.

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1. INTRODUCTION

The notion of a generalized derivation of a prime ring R was introduced by M. Brešar [6] and B. Hvala [10]. An additive map f of an associative ring R is called a generalized derivation if there is a derivation d of R such that

$$f(xy) = f(x)y + xd(y) \text{ for all } x, y \in R.$$

A similar notion was used by A. Nakajima [12] and N. Argaç and E. Albaş [1], who established a series of categorical properties.

As a well-known result proved by I. N. Herstein in [9, Theorem] states that if R is a prime ring of characteristics different from two and if d is a nonzero derivation of R such that $[d(x), a] = 0$ for all $x \in R$, then either R is commutative or $a \in Z$. Later on, the authors extended this result to (σ,τ)-derivation in [3]. In [8], Jui-Chi Chang showed that several results hold for any generalized (α,β)-derivation of a prime ring. The purpose of this note is to generalize these theorems for a nonzero ideal of prime ring and generalized (α,β)-derivation of R.

In [4], H. E. Bell and L. C. Kappe proved that d is a derivation of R which is either an endomorphism or anti-endomorphism in semi-prime ring R or nonzero right ideal of R, then $d = 0$. In view of this result, it is natural to question whether this result is true for generalized (α,β)-derivations of a prime ring. Our second aim in this paper is to show that the above mentioned result holds for generalized (α,β)-derivation of a prime ring.

In [7], M. Brešar and J. Vukman proved that every (α,β)-Jordan derivation on a prime ring with characteristics different from two is a (α,β)-derivation. We showed that this result holds for generalized (α,β)-derivations on a nonzero ideal of prime ring R.

Throughout this paper, R will be a prime ring with char $R \neq 2$, I a nonzero ideal of R, Z the center of R. Let α and β be two automorphisms of R into itself. For any two elements x, y of R, we denote $[x, y] = xy - yx$ and make extensive use of the basic commutator identity $[xy,z] = x[y,z] + [x,z]y$.

2. Results

Definition 1 ([8, Definition 1]). Let R be a ring, α and β automorphisms of R and d a (α,β)-derivation of R. An additive mapping $f : R \to R$ is said to be right generalized (α,β)-derivation associated with d if

$$f(xy) = f(x)\alpha(y) + \beta(x)d(y) \text{ for all } x, y \in R. \tag{2.1}$$

and f is said to be left generalized (α,β)-derivation associated with d if

$$f(xy) = d(x)\alpha(y) + \beta(x)f(y) \text{ for all } x, y \in R. \tag{2.2}$$

f is said to be a generalized (α,β)-derivation associated with d if it is both a left and right generalized (α,β)-derivation associated with d.

Lemma 1 ([8, Lemma 4]). *Let $f \neq 0$ be a generalized (α,β)-derivation of prime ring R associated with d and $a \in R$.*

(i) *If $af(x) = 0$ for all $x \in R$, then $a = 0$.*
(ii) *If $f(x)a = 0$ for all $x \in R$, then $a = 0$.*

Lemma 2. *Let R be a prime ring with characteristics not equal to two, I a nonzero ideal of R and $f : R \to R$ be a generalized (α,β)-derivation of prime ring R associated with d. If $f(x) = 0$ for all $x \in I$, then $f = 0$.*

Proof. For all $x, y \in I$

$$0 = f(xy) = f(x)\alpha(y) + \beta(x)d(y)$$

and so,

$$I\beta^{-1}(d(y)) = 0 \text{ for all } y \in I.$$

Since R is a prime ring, we get $d(y) = 0$, for all $y \in I$. Hence, by the hypothesis, $0 = f(xy) = f(x)\alpha(y)$ for all $x \in I, r \in R$. This yields that $f = 0$. □

Lemma 3. *Let $f \neq 0$ be a generalized (α,β)-derivation of prime ring R associated with nonzero (α,β)-derivation d, I a nonzero ideal of R and $a \in R$.*

(i) *If $af(x) = 0$ for all $x \in I$, then $a = 0$.*
(ii) *If $f(x)a = 0$ for all $x \in I$, then $a = 0$.*

Proof. (i) For any $x \in I, r \in R$,

$$0 = af(xr) = af(x)\alpha(R) + a\beta(x)d(R)$$

and so,

$$\beta^{-1}(a)I\beta^{-1}(d(R)) = 0.$$

Since I is a nonzero ideal of prime ring R and $d \neq 0$, we get $a = 0$.

(ii) Similarly. □

Theorem 1. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R and $f : R \to R$ be a generalized (α,β)-derivation of prime ring R associated with nonzero (α,β)-derivation d. If $f(x) \in Z$ for all $x \in I$, then R is a commutative ring.*

Proof. By the hypothesis for all $x, y \in I$,

$$\begin{aligned}0 &= [\alpha(y), f(xy)] = [\alpha(y), f(x)\alpha(y) + \beta(x)d(y)]\\ &= \beta(x)[\alpha(y), d(y)] + [\alpha(y), \beta(x)]d(y)\\ &= \beta(x)\alpha(y)d(y) - \beta(x)d(y)\alpha(y) + \alpha(y)\beta(x)d(y) - \beta(x)\alpha(y)d(y)\end{aligned}$$

and so,

$$\alpha(y)\beta(x)d(y) - \beta(x)d(y)\alpha(y) = 0 \text{ for all } x, y \in I. \tag{2.3}$$

Writing $xz, z \in I$ by x in (2.3) and using this equation, we have

$$\begin{aligned}0 &= \alpha(y)\beta(x)\beta(z)d(y) - \beta(x)\beta(z)d(y)\alpha(y)\\ &= \alpha(y)\beta(x)\beta(z)d(y) - \beta(x)\alpha(y)\beta(z)d(y)\\ &= [\alpha(y), \beta(x)]\beta(z)d(y).\end{aligned}$$

Using I a nonzero ideal of prime ring R and $d \neq 0$, one can obtain that R is a commutative ring. □

Theorem 2. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R, d be a nonzero (α,β)-derivation such that $d\beta = \beta d$ and $f : R \to R$ be a generalized (α,β)-derivation of prime ring R associated with d and $a \in R$. If $[f(x), a] = 0$ for all $x \in I$, then $a \in Z$ or $d(a) = 0$.*

Proof. By the hypothesis for all $x \in R, y \in I$, we get

$$\begin{aligned}0 &= [f(xy), a] = [d(x)\alpha(y) + \beta(x)f(y), a]\\ &= d(x)[\alpha(y), a] + [d(x), a]\alpha(y) + [\beta(x), a]f(y)\\ &= d(x)\alpha(y)a - d(x)a\alpha(y) + d(x)a\alpha(y) - ad(x)\alpha(y)\\ &\quad + \beta(x)af(y) - a\beta(x)f(y)\end{aligned}$$

and so,

$$d(x)\alpha(y)a - ad(x)\alpha(y) + \beta(x)af(y) - a\beta(x)f(y) = 0 \tag{2.4}$$

for all $x \in R, y \in I$. Writing $yz, z \in R$ by y in (2.4) and using (2.4), we get

$$\begin{aligned} 0 &= d(x)\alpha(y)\alpha(z)a - ad(x)\alpha(y)\alpha(z) \\ &\quad + \beta(x)af(y)\alpha(z) + \beta(x)a\beta(y)d(z) - \\ &\quad - a\beta(x)f(y)\alpha(z) - a\beta(x)\beta(y)d(z) \\ &= d(x)\alpha(y)\alpha(z)a - ad(x)\alpha(y)\alpha(z) + (\beta(x)af(y) - \\ &\quad - a\beta(x)f(y))\alpha(z) + \beta(x)a\beta(y)d(z) - a\beta(x)\beta(y)d(z) \\ &= d(x)\alpha(y)\alpha(z)a - d(x)\alpha(y)a\alpha(z) + \beta(x)a\beta(y)d(z) \\ &\quad - a\beta(x)\beta(y)d(z) \end{aligned}$$

and so,

$$d(x)\alpha(y)[\alpha(z), a] + [\beta(x), a]\beta(y)d(z) = 0 \tag{2.5}$$

for all $x, z \in R, y \in I$. Replacing x by $\beta^{-1}(a)$ in (2.5), we have

$$d(\beta^{-1}(a))\alpha(y)[\alpha(z), a] = 0, \text{ for all } z \in R, y \in I.$$

Since I is a nonzero ideal of prime ring R and $d\beta = \beta d$, we get $d(a) = 0$ or $a \in Z$. □

Theorem 3. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R, d be a nonzero (α, β)-derivation such that $d\beta = \beta d$ and $f : R \to R$ be a generalized (α, β)-derivation of prime ring R associated with d. If $[f(x), f(y)] = 0$ for all $x, y \in I$, then R is a commutative ring or $df = 0$.*

Proof. By Theorem 2, we have $f(I) \subset Z$ or $df = 0$. The proof is completed by Theorem 1. □

Theorem 4. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R and $f : R \to R$ be a generalized (α, β)-derivation of prime ring R associated with d. If $f(xy) = f(x)f(y)$ for all $x, y \in I$, then $d = 0$.*

Proof. For any $x, y \in I$,

$$f(xy) = f(x)\alpha(y) + \beta(x)d(y) = f(x)f(y), \text{ for all } x, y \in I. \tag{2.6}$$

Writing $xw, w \in I$ for x in (2.6), we have

$$f(xw)\alpha(y) + \beta(xw)d(y) = f(xw)f(y)$$

Since d acts as a homomorphism on I, we get

$$\begin{aligned} f(x)f(w)\alpha(y) + \beta(xw)d(y) &= f(x)f(w)f(y) = f(x)f(wy) \\ &= f(x)f(w)\alpha(y) + f(x)\beta(w)d(y) \end{aligned}$$

and so, we have $\beta(x)\beta(w)d(y) = f(x)\beta(w)d(y)$, i. e.,

$$(\beta(x) - f(x))\beta(w)d(y) = 0, \text{ for all } x, y \in I.$$

Since I a nonzero ideal of prime ring R and β an automorphism of R, we conclude that

$$f(x)=\beta(x),\text{for all } x\in I \text{ or } d(y)=0, \text{ for all } y\in I.$$

Let us assume that $f(x)=\beta(x)$, for all $x\in I$. Replacing x by $xy, y\in I$, we have $f(xy)=\beta(xy)$ and $d(x)\alpha(y)+\beta(x)f(y)=\beta(x)\beta(y)=\beta(x)f(y)$.

Thus, we obtain that $d(x)=0$ for all $x\in I$ for any cases. That is $d=0$. This completes the proof. □

Theorem 5. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R and $f:R\to R$ be a generalized (α,β)-derivation of prime ring R associated with d. If $f(xy)=f(y)f(x)$ for all $x,y\in I$, then $d=0$.*

Proof. Since d acts as an anti-homorphism on I, we get

$$f(xy)=d(x)\alpha(y)+\beta(x)f(y)=f(y)f(x),\text{ for all } x,y\in I. \tag{2.7}$$

Replacing y by $xy, y\in I$ in (2.7), we obtain $d(x)\alpha(xy)+\beta(x)f(xy)=f(xy)f(x)$,

$$d(x)\alpha(xy)+\beta(x)f(y)f(x)=d(x)\alpha(y)f(x)+\beta(x)f(y)f(x)$$

and so,

$$d(x)\alpha(x)\alpha(y)=d(x)\alpha(y)f(x),\text{ for } x,y\in I. \tag{2.8}$$

Substituting $yw, w\in I$ for y in this equation and using (2.8), we get

$$\begin{aligned}0&=d(x)\alpha(x)\alpha(y)\alpha(w)-d(x)\alpha(y)\alpha(w)f(x)\\&=d(x)\alpha(y)f(x)\alpha(w)-d(x)\alpha(y)\alpha(w)f(x)\\&=d(x)\alpha(y)[f(x),\alpha(w)].\end{aligned}$$

Since α is an automorphism and I a nonzero ideal of prime ring R, we obtain

$$d(x)=0 \text{ or } [f(x),\alpha(w)]=0,\text{ for all } x,w\in I.$$

Now let us define $A=\{x\in I\mid d(x)=0\}$ and $B=\{x\in I\mid[f(x),\alpha(w)]=0$, for all $w\in I\}$. Clearly each A and B is an additive subgroup of I. Moreover, I is the set theoretic union of A and B. But a group cannot be the set theoretic union of two proper subgroups, hence $A=I$ or $B=I$. In the former case, we have $d(R)=0$, which completes the proof. If $B=I$, then we get

$$0=[f(x),\alpha(wr)]=\alpha(w)[f(x),\alpha(R)]$$

for all $x,w\in I, r\in R$.

Thus, we obtain that $f(I)\subset Z$. Since f acts as an endomorphism of R, it follows that $d=0$, via Theorem 4. □

Theorem 6. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R and $f:R\to R$ be a generalized (α,β)-derivation of prime ring R associated with nonzero (α,β)-derivation d. If $\beta d=d\beta$ and $f(xy)=f(yx)$ for all $x,y\in I$, then R is a commutative ring.*

Proof. Let $c \in I$ be a constant element such that $f(c) = 0$ (i. e., $[x, y]$) and z be an arbitrary element of I. The condition that $f(cz) = f(zc)$ yields that $f(c)\alpha(z) + \beta(c)d(z) = d(z)\alpha(c) + \beta(z)f(c)$ and so,

$$\beta(c)d(z) = d(z)\alpha(c), \text{ for } z \in I.$$

Thus $[d(z), c]_{\alpha,\beta} = 0$, for all $z \in I$. Replacing z by $wz, w \in I$ and using this equation, we have

$$\begin{aligned} 0 &= [d(wz), c]_{\alpha,\beta} = [d(w)\alpha(z) + \beta(w)d(z), c]_{\alpha,\beta} \\ &= d(w)\alpha([z, c]) + [d(w), c]_{\alpha,\beta}\alpha(z) + \beta(w)[d(z), c]_{\alpha,\beta} + \beta([w, c])d(z) \end{aligned}$$

and so,

$$d(w)\alpha([z, c]) + \beta([w, c])d(z) = 0, \text{ for all } w, z \in I. \tag{2.9}$$

Writing c (since $c \in I$) by z in (2.9), we have $\beta([w, c])d(c) = 0$.This gives us $[w, c]\beta^{-1}(d(c)) = 0$, for all $w \in I$, all $c \in I$ such that $f(c) = 0$. If we take $rw, r \in R, w \in I$ instead of w in this equation, we have

$$[r, c]w\beta^{-1}(d(c)) = 0, \text{ for all } r \in R.$$

Since I is a nonzero ideal of prime ring R, we obtain

$$c \in Z \text{ or } d(c) = 0, \text{ for all } c \in I \text{ such that } f(c) = 0.$$

So, we obtain

$$[x, y] \in Z \text{ or } d([x, y]) = 0, \text{ for all } x, y \in I.$$

because of $f([x, y]) = 0$, for all $x, y \in I$. If $d([x, y]) = 0$, for all $y \in I\}$.

This shows that additive group I is the union of two of its additive subgroups $A = \{x \in I \mid d([x, y]) = 0, \text{for all } y \in I\}$ and $B = \{x \in I \mid [x, y] \in Z, \text{for all } y \in I\}$.

If $I = A$, then R is a commutative ring (i. e., $[x, y] \in Z$, for all $x, y \in I$) by [2, Theorem 3]. So, we can take $I = B$ and $[x, y] \in Z$, for all $x, y \in I$. Let us define an inner derivation $I_x : R \to R, I_x^2(y) = 0$, for all $y \in I$. This shows $I \subset Z$ and so R is a commutative ring by [5, Theorem 1]. □

Definition 2. Let R be a ring, β and α automorphisms of R and d a (α, β)-derivation of R. An additive mapping $f : R \to R$ is said to be right generalized (α, β)-Jordan derivation associated with d if

$$f(x^2) = f(x)\alpha(x) + \beta(x)d(x) \text{ for all } x \in R. \tag{2.10}$$

and f is said to be left generalized (α, β)-Jordan derivation associated with d if

$$f(x^2) = d(x)\alpha(x) + \beta(x)f(x) \text{for all } x \in R. \tag{2.11}$$

f is said to be a generalized (α, β)-Jordan derivation associated with d if it is both a left and right generalized (α, β)-Jordan derivation associated with d.

Lemma 4. *Let $f : R \to R$ be a right generalized (α,β)-Jordan derivation on I. For all $x,y,z \in I$,*

$$f(xy+yx) = f(x)\alpha(y)+\beta(x)d(y)+f(y)\alpha(x)+\beta(y)d(x); \quad (1)$$

$$f(xyx) = f(x)\alpha(yx)+\beta(x)d(y)\alpha(x)+\beta(xy)d(x); \quad (2)$$

$$\begin{aligned} f(xyz+zyx) &= f(x)\alpha(yz)+\beta(x)d(y)\alpha(z)+\beta(xy)d(z) \\ &\quad + f(z)\alpha(yx)+\beta(z)d(y)\alpha(x)+\beta(zy)d(x) \\ &\quad + f(z)\alpha(yx)+\beta(z)d(y)\alpha(x)+\beta(zy)d(x). \end{aligned} \quad (3)$$

Proof. Assertion (1). Linearizing (2.10), we get

$$\begin{aligned} f((x+y)^2) &= f((x+y)(x+y) = f(x^2+xy+yx+y^2) \\ &= f(x^2)+f(xy+yx)+f(y^2) \\ &= f(x)\alpha(x)+\beta(x)d(x) \\ &\quad + f(xy+yx)+f(y)\alpha(y)+\beta(y)d(y) \end{aligned} \quad (2.12)$$

for all $x,y \in I$. On the other hand,

$$\begin{aligned} f((x+y)^2) &= f(x+y)\alpha(x+y)+\beta(x+y)d(x+y) \\ &= f(x)\alpha(x)+f(y)\alpha(x)+f(x)\alpha(y)+f(y)\alpha(y) \\ &+\beta(x)d(x)++\beta(x)d(y)+\beta(y)d(x)+\beta(y)d(y) \end{aligned} \quad (2.13)$$

for all $x,y \in I$. Comparing (2.12) and (2.13), we have

$$f(xy+yx) = f(x)\alpha(y)+\beta(x)d(y)+f(y)\alpha(x)+\beta(y)d(x)$$

for all $x,y \in I$.

Assertion (2). Replacing y by $xy+yx$ in (1), we get

$$\begin{aligned} f(x(xy+yx)+(xy+yx)x) &= f(x^2y+xyx+xyx+yx^2) \\ &= f(x^2y+yx^2)+2f(xyx) \\ &= f(x^2)\alpha(y)+\beta(x^2)d(y)+f(y)\alpha(x^2) \\ &\quad +\beta(y)d(x^2)+2f(xyx) \\ &= f(x)\alpha(xy)+\beta(x)d(x)\alpha(y)+\beta(x^2)d(y) \\ &\quad + f(y)\alpha(x^2)+\beta(y)d(x)\alpha(x) \\ &\quad +\beta(yx)d(x)+2f(xyx) \end{aligned} \quad (2.14)$$

for all $x, y \in I$. On the other hand, we have

$$\begin{aligned} f(x(xy+yx)+(xy+yx)x) &= f(x)\alpha(xy+yx)+\beta(x)d(xy+yx) \\ &\quad + f(xy+yx)\alpha(x)+\beta(xy+yx)d(x) \\ &= f(x)\alpha(xy)+f(x)\alpha(yx)+\beta(x)d(x)\alpha(y)+\beta(x^2)d(y) \\ &\quad +\beta(x)d(y)\alpha(x)+\beta(xy)d(x)+f(x)\alpha(yx) \\ &\quad +\beta(x)d(y)\alpha(x)+f(y)\alpha(x^2)+\beta(y)d(x)\alpha(x) \\ &\quad +\beta(xy)d(x)+\beta(yx)d(x) \end{aligned} \tag{2.15}$$

for all $x, y \in I$. Comparing (2.14) and (2.15) and using $\operatorname{char} R \neq 2$, we get the required result.

Assertion (3). Linearizing (2) on x, we get

$$\begin{aligned} f((x+z)y(x+y)) &= f(xyx+xyz+zyx+zyz) \\ &= f(xyx)+f(xyz+zyx)+f(zyz) \\ &= f(x)\alpha(yx)+\beta(x)d(y)\alpha(x)+\beta(xy)d(x) \\ &\quad + f(xyz+zyx)+f(z)\alpha(yz) \\ &\quad +\beta(z)d(y)\alpha(z)+\beta(zy)d(z) \end{aligned} \tag{2.16}$$

for all $x, y, z \in I$. Computing now $f((x+z)y(x+z))$ in another way, we get

$$\begin{aligned} f((x+z)y(x+z)) &= f(x+z)\alpha(yx+yz)+\beta(x+z)d(y)\alpha(x+z) \\ &\quad +\beta(xy+zy)d(x+z) \\ &= f(x)\alpha(yx)+f(x)\alpha(yz)+f(z)\alpha(yx)+f(z)\alpha(yz) \\ &\quad +\beta(x)d(y)\alpha(x)+\beta(x)d(y)\alpha(z) \\ &\quad +\beta(z)d(y)\alpha(x)+\beta(z)d(y)\alpha(z) \\ &\quad +\beta(xy)d(x)+\beta(xy)d(z)+\beta(zy)d(x) \\ &\quad +\beta(zy)d(z) \end{aligned} \tag{2.17}$$

for all $x, y, z \in I$. Comparing (2.16) and (2.17), we have

$$\begin{aligned} f(xyz+zyx) = f(x)\alpha(yz)+\beta(x)d(y)\alpha(z)+\beta(xy)d(z)+f(z)\alpha(yx) \\ +\beta(z)d(y)\alpha(x)+\beta(zy)d(x) \end{aligned}$$

for all $x, y, z \in I$. □

We introduce the abbreviation

$$x^y = f(xy) - f(x)\alpha(y) - \beta(x)d(y)$$

for all $x, y \in I$.

Observe that, by Lemma 4 (1), we have $f(xy+yx) = f(x)\alpha(y)+\beta(x)d(y)+f(y)\alpha(x)+\beta(y)d(x)$ and so, $f(xy) - f(x)\alpha(y) - \beta(x)d(y) = -(f(yx) - f(y)\alpha(x) - \beta(y)d(x))$. That is,

$$x^y = -y^x \text{ for all } x, y \in I. \tag{2.18}$$

Lemma 5. *Let $f : R \to R$ be a right generalized (α,β)-Jordan derivation on I. For all $x, y \in I$,*

$$x^y[\alpha(x), \alpha(y)] = 0.$$

Proof. Replacing z by xy in Lemma 4 (3), we get

$$\begin{aligned} f((xy)^2 + xy^2x) = f((xy)^2) + f(xy^2x) &= f(xy)\alpha(xy) + \beta(xy)d(xy) \\ &\quad + f(x)\alpha(x^2y) + \beta(x)d(y^2)\alpha(x) + \beta(xy^2)d(x) \\ &= f(xy)\alpha(xy) + \beta(xy)d(x)\alpha(y) + \beta(xyx)d(y) \\ &\quad + f(x)\alpha(x^2y) + \beta(x)d(y)\alpha(yx) \\ &\quad + \beta(xy)d(y)\alpha(x) + \beta(xy^2)d(x) \end{aligned} \tag{2.19}$$

for all $x, y \in I$. On the other hand, we get

$$\begin{aligned} f(xy(xy) + (xy)yx) &= f(x)\alpha(yxy) + \beta(x)d(y)\alpha(xy) + \beta(xy)d(xy) \\ &\quad + f(xy)\alpha(yx) + \beta(xy)d(y)\alpha(x) + \beta(xy^2)d(x) \\ &= f(x)\alpha(yxy) + \beta(x)d(y)\alpha(xy) + \beta(xy)d(x)\alpha(y) \\ &\quad + \beta(xyx)d(y) + f(xy)\alpha(yx) + \beta(xy)d(y)\alpha(x) \\ &\quad + \beta(xy^2)d(x) \end{aligned} \tag{2.20}$$

for all $x, y \in I$. Comparing these two equations, we have

$$\begin{aligned} &f(xy)\alpha(xy) + f(x)\alpha(y^2x) + \beta(x)d(y)\alpha(yx) \\ &\qquad = f(x)\alpha(yxy) + \beta(x)d(y)\alpha(xy) + f(xy)\alpha(yx). \end{aligned}$$

That is, $x^y[\alpha(x), \alpha(y)] = 0$ for all $x, y \in I$. □

Theorem 7. *Let R be a non-commutative prime ring with characteristics not two, I a nonzero ideal of R. If f be a right generalized (α,β)-Jordan derivation on I, then f is generalized (α,β)-derivation on I.*

Proof. From Lemma 4 (3), we have

$$\begin{aligned} f(xwy + ywx) &= f(x)\alpha(wy) + \beta(x)d(w)\alpha(y) + \beta(xw)d(y) \\ &\quad + f(y)\alpha(wx) + \beta(y)d(w)\alpha(x) + \beta(yw)d(x) \end{aligned} \tag{2.21}$$

for all $x,y,w \in I$. Replacing x by xy and y by yx in (2.21), we get

$$\begin{aligned} f((xy)w(yx)+(yx)w(xy)) &= f(xy)\alpha(wyx)+\beta(xy)d(w)\alpha(yx) \\ &\quad +\beta(xyw)d(yx)+f(yx)\alpha(wxy) \\ &\quad +\beta(yx)d(w)\alpha(xy)+\beta(yxw)d(xy) \\ &= f(xy)\alpha(wyx)+\beta(xy)d(w)\alpha(yx)+\beta(xyw)d(y)\alpha(x) \\ &\quad +\beta(xywy)d(x)+f(yx)\alpha(wxy)+\beta(yx)d(w)\alpha(xy) \\ &\quad +\beta(yxw)d(x)\alpha(y)+\beta(yxwx)d(y) \end{aligned} \tag{2.22}$$

for all $x,y,w \in I$. On the other hand, we have

$$\begin{aligned} f((xy)w(yx)+(yx)w(xy)) &= f(x(ywy)x)+f(y(xwx)y) \\ &= f(x)\alpha(ywyx)+\beta(x)d(ywy)\alpha(x)+\beta(xywy)d(x) \\ &\quad +f(y)\alpha(xwxy)+\beta(y)d(xwx)\alpha(y)+\beta(yxwx)d(y) \\ &= f(x)\alpha(ywyx)+\beta(x)d(y)\alpha(wyx)+\beta(xy)d(w)\alpha(yx) \\ &\quad +\beta(xyw)d(y)\alpha(x)+\beta(xywy)d(x)+f(y)\alpha(xwxy) \\ &\quad +\beta(y)d(x)\alpha(wxy)+\beta(yx)d(w)\alpha(xy) \\ &\quad +\beta(yxw)d(x)\alpha(y)+\beta(yxwx)d(y) \end{aligned} \tag{2.23}$$

for all $x,y,w \in I$. Comparing equations (2.22) and (2.23), we obtain

$$\begin{aligned} &\{f(yx)-f(y)\alpha(x)-\beta(y)d(x)\}\alpha(w)\alpha(xy) \\ &\qquad +\{f(xy)-f(x)\alpha(y)-\beta(x)d(y)\}\alpha(w)\alpha(yx)=0 \end{aligned}$$

and hence

$$y^x\alpha(w)\alpha(xy)+x^y\alpha(w)\alpha(yx)=0$$

for all $x,y,w \in I$. Using (2.18), we get $x^y\alpha(I)\alpha([y,x])=0$ for all $x,y \in I$. Since I is a nonzero ideal of prime ring R, we obtain for each pair $x,y \in I$ either $x^y=0$ or $[x,y]=0$. Notice that the mappings $(x,y)\to x^y$ and $(x,y)\to [x,y]$ satisfy the requirements of [5, Lemma 4]. Hence $x^y=0$ for all $x,y \in I$ or $[x,y]^2=0$ for all $x,y \in I$. If $[x,y]^2=0$ for all $x,y \in I$, then for each $x \in I$, $I_x(y)^2=0$, for all $y \in I$, where I_x is the inner derivation. This yields that R is commutative, a contradiction by [11, Theorem 3]. Thus, we have $x^y=0$ for all $x,y \in I$. This completes the proof. □

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