Notes on generalized \((\alpha, \beta)\)-derivations in prime rings

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NOTES ON GENERALIZED \((\alpha, \beta)\)-DERIVATIONS IN PRIME RINGS

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Abstract. Let \( R \) be a prime ring with characteristics different from two, \( \alpha, \beta \in \text{Aut}(R) \), and \( I \) a nonzero ideal of \( R \). Let \( d : R \to R \) be a \((\alpha, \beta)\)-derivation and \( f : R \to R \) be generalized \((\alpha, \beta)\)-derivation associated with \( d \). In this case: (i) if \( af(x) = 0 \) \((f(x)a = 0) \) for all \( x \in I \), then \( a = 0 \); (ii) if \( \beta d = d \beta \) and \([f(x), a] = 0\) for all \( x \in I \), then \( a \in Z \) or \( d(a) = 0 \); (iii) if \( f \) acts as an endomorphism or anti-homomorphism on \( I \), then \( d = 0 \); (iv) if \( \beta d = d \beta \) and \( f(xy) = f(yx) \) for all \( x, y \in I \) then \( R \) is a commutative ring; (v) if \( f \) satisfies \( f(x^2) = f(x)a(x) + \beta(x)d(x) \) for all \( x \in I \), then \( f(xy) = f(x)a(y) + \beta(x)d(y) \) for all \( x, y \in I \).

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1. Introduction

The notion of a generalized derivation of a prime ring \( R \) was introduced by M. Brešar [6] and B. Hvala [10]. An additive map \( f \) of an associative ring \( R \) is called a generalized derivation if there is a derivation \( d \) of \( R \) such that

\[
    f(xy) = f(x)y + xd(y) \quad \text{for all } x, y \in R.
\]

A similar notion was used by A. Nakajima [12] and N. Argaç and E. Albaş [1], who established a series of categorical properties.

As a well-known result proved by I. N. Herstein in [9, Theorem] states that if \( R \) is a prime ring of characteristics different from two and if \( d \) is a nonzero derivation of \( R \) such that \([d(x), a] = 0\) for all \( x \in R \), then either \( R \) is commutative or \( a \in Z \). Later on, the authors extended this result to \((\sigma, \tau)\)-derivation in [3]. In [8], Jui-Chi Chang showed that several results hold for any generalized \((\alpha, \beta)\)-derivation of a prime ring. The purpose of this note is to generalize these theorems for a nonzero ideal of prime ring and generalized \((\alpha, \beta)\)-derivation of \( R \).

In [4], H. E. Bell and L. C. Kappe proved that \( d \) is a derivation of \( R \) which is either an endomorphism or anti-endomorphism in semi-prime ring \( R \) or nonzero right ideal of \( R \), then \( d \) = 0. In view of this result, it is natural to question whether this result is true for generalized \((\alpha, \beta)\)-derivations of a prime ring. Our second aim in this paper...
is to show that the above mentioned result holds for generalized \((\alpha, \beta)\)-derivation of a prime ring.

In [7], M. Brešar and J. Vukman proved that every \((\alpha, \beta)\)-Jordan derivation on a prime ring with characteristics different from two is a \((\alpha, \beta)\)-derivation. We showed that this result holds for generalized \((\alpha, \beta)\)-derivations on a nonzero ideal of prime ring \(R\).

Throughout this paper, \(R\) will be a prime ring with \(\text{char } R \neq 2\), \(I\) a nonzero ideal of \(R\), \(Z\) the center of \(R\). Let \(\alpha\) and \(\beta\) be two automorphisms of \(R\) into itself. For any two elements \(x, y\) of \(R\), we denote \([x, y] = xy - yx\) and make extensive use of the basic commutator identity \([xy, z] = x[y, z] + [x, z]y\).

2. Results

Definition 1 ([8, Definition 1]). Let \(R\) be a ring, \(\alpha\) and \(\beta\) automorphisms of \(R\) and \(d\) a \((\alpha, \beta)\)-derivation of \(R\). An additive mapping \(f : R \to R\) is said to be right generalized \((\alpha, \beta)\)-derivation associated with \(d\) if
\[
f(x, y) = f(x)\alpha(y) + \beta(y)d(x) \quad \text{for all } x, y \in R.
\]
and \(f\) is said to be left generalized \((\alpha, \beta)\)-derivation associated with \(d\) if
\[
f(x, y) = d(x)\alpha(y) + \beta(y)f(x) \quad \text{for all } x, y \in R.
\]
\(f\) is said to be a generalized \((\alpha, \beta)\)-derivation associated with \(d\) if it is both a left and right generalized \((\alpha, \beta)\)-derivation associated with \(d\).

Lemma 1 ([8, Lemma 4]). Let \(f \neq 0\) be a generalized \((\alpha, \beta)\)-derivation of prime ring \(R\) associated with \(d\) and \(a \in R\).

(i) If \(af(x) = 0\) for all \(x \in I\), then \(a = 0\).
(ii) If \(f(x)a = 0\) for all \(x \in R\), then \(a = 0\).

Lemma 2. Let \(R\) be a prime ring with characteristics not equal to two, \(I\) a nonzero ideal of \(R\) and \(f : R \to R\) be a generalized \((\alpha, \beta)\)-derivation of prime ring \(R\) associated with \(d\). If \(f(x) = 0\) for all \(x \in I\), then \(f = 0\).

Proof. For all \(x, y \in I\)
\[
0 = f(xy) = f(x)\alpha(y) + \beta(y)d(x)
\]
and so,
\[
I\beta^{-1}(d(y)) = 0 \quad \text{for all } y \in I.
\]
Since \(R\) is a prime ring, we get \(d(y) = 0\) for all \(y \in I\). Hence, by the hypothesis, \(0 = f(xy) = f(x)\alpha(y)\) for all \(x \in I, r \in R\). This yields that \(f = 0\). 

Lemma 3. Let \(f \neq 0\) be a generalized \((\alpha, \beta)\)-derivation of prime ring \(R\) associated with nonzero \((\alpha, \beta)\)-derivation \(d\), \(I\) a nonzero ideal of \(R\) and \(a \in R\).

(i) If \(af(x) = 0\) for all \(x \in I\), then \(a = 0\).
(ii) If \(f(x)a = 0\) for all \(x \in I\), then \(a = 0\).
Proof. (i) For any \( x \in I, r \in R \),
\[
0 = af(xr) = af(x)\alpha(R) + a\beta(x)d(R)
\]
and so,
\[
\beta^{-1}(a)I\beta^{-1}(d(R)) = 0.
\]
Since \( I \) is a nonzero ideal of prime ring \( R \) and \( d \neq 0 \), we get \( a = 0 \).
(ii) Similarly.

**Theorem 1.** Let \( R \) be a prime ring with characteristics not two, \( I \) a nonzero ideal of \( R \) and \( f : R \rightarrow R \) be a generalized \((\alpha, \beta)\)-derivation of prime ring \( R \) associated with nonzero \((\alpha, \beta)\)-derivation \( d \). If \( f(x) \in Z \) for all \( x \in I \), then \( R \) is a commutative ring.

**Proof.** By the hypothesis for all \( x, y \in I \),
\[
0 = [\alpha(y), f(xy)] = [\alpha(y), f(x)\alpha(y) + \beta(x)d(y)]
\]
\[
= \beta(x)[\alpha(y), d(y)] + [\alpha(y), \beta(x)]d(y)
\]
\[
= \beta(x)\alpha(y)d(y) - \beta(x)d(y)\alpha(y) + \alpha(y)\beta(x)d(y) - \beta(x)\alpha(y)d(y)
\]
and so,
\[
\alpha(y)\beta(x)d(y) - \beta(x)d(y)\alpha(y) = 0 \text{ for all } x, y \in I. \tag{2.3}
\]
Writing \( xz, z \in I \) by \( x \) in (2.3) and using this equation, we have
\[
0 = \alpha(y)\beta(x)\beta(z)d(y) - \beta(x)\beta(z)d(y)\alpha(y)
\]
\[
= \alpha(y)\beta(x)\beta(z)d(y) - \beta(x)\alpha(y)\beta(z)d(y)
\]
\[
= [\alpha(y), \beta(x)]\beta(z)d(y).
\]
Using \( I \) a nonzero ideal of prime ring \( R \) and \( d \neq 0 \), one can obtain that \( R \) is a commutative ring.

**Theorem 2.** Let \( R \) be a prime ring with characteristics not two, \( I \) a nonzero ideal of \( R \), \( d \) be a nonzero \((\alpha, \beta)\)-derivation such that \( d\beta = \beta d \) and \( f : R \rightarrow R \) be a generalized \((\alpha, \beta)\)-derivation of prime ring \( R \) associated with \( d \) and \( a \in R \). If \( [f(x), a] = 0 \) for all \( x \in I \), then \( a \in Z \) or \( d(a) = 0 \).

**Proof.** By the hypothesis for all \( x \in R, y \in I \), we get
\[
0 = [f(xy), a] = [d(x)\alpha(y) + \beta(x)f(y), a]
\]
\[
= d(x)[\alpha(y), a] + [d(x), a]\alpha(y) + [\beta(x), a]f(y)
\]
\[
= d(x)\alpha(y)a - d(x)a\alpha(y) + d(x)a\alpha(y) - ad(x)\alpha(y)
\]
\[
+ \beta(x)af(y) - a\beta(x)f(y)
\]
and so,
\[
d(x)\alpha(y)a - ad(x)a\alpha(y) + \beta(x)af(y) - a\beta(x)f(y) = 0 \tag{2.4}
\]
for all $x \in R, y \in I$. Writing $y, z \in R$ by $y$ in (2.4) and using (2.4), we get

\[
0 = d(x)\alpha(y)\alpha(z)a - ad(x)\alpha(y)\alpha(z) \\
+ \beta(x)af(y)\alpha(z) + \beta(x)a\beta(y)d(z) - \\
a\beta(x)f(y)\alpha(z) - a\beta(x)\beta(y)d(z)
\]

\[
= d(x)\alpha(y)\alpha(z)a - ad(x)\alpha(y)\alpha(z) + (\beta(x)af(y) - \\
- a\beta(x)f(y))\alpha(z) + \beta(x)a\beta(y)d(z) - a\beta(x)\beta(y)d(z)
\]

\[
= d(x)\alpha(y)\alpha(z)a - d(x)\alpha(y)a\alpha(z) + \beta(x)a\beta(y)d(z)
\]

and so,

\[
d(x)\alpha(y)[\alpha(z),a] + [\beta(x),a]\beta(y)d(z) = 0
\]

(2.5)

for all $x, z \in R, y \in I$. Replacing $x$ by $\beta^{-1}(x)$ in (2.5), we have

\[
d(\beta^{-1}(a))\alpha(y)[\alpha(z), a] = 0, \text{ for all } z \in R, y \in I.
\]

Since $I$ is a nonzero ideal of prime ring $R$ and $d\beta = \beta d$, we get $d(a) = 0$ or $a \in Z$.

**Theorem 3.** Let $R$ be a prime ring with characteristics not two, $I$ a nonzero ideal of $R, d$ be a nonzero $(\alpha, \beta)$-derivation such that $d\beta = \beta d$ and $f : R \to R$ be a generalized $(\alpha, \beta)$-derivation of prime ring $R$ associated with $d$. If $[f(x), f(y)] = 0$ for all $x, y \in I$, then $R$ is a commutative ring or $df = 0$.

**Proof.** By Theorem 2, we have $f(I) \subset Z$ or $df = 0$. The proof is completed by Theorem 1. □

**Theorem 4.** Let $R$ be a prime ring with characteristics not two, $I$ a nonzero ideal of $R$ and $f : R \to R$ be a generalized $(\alpha, \beta)$-derivation of prime ring $R$ associated with $d$. If $f(xy) = f(x)f(y)$ for all $x, y \in I$, then $d = 0$.

**Proof.** For any $x, y \in I$,

\[
f(xy) = f(x)\alpha(y) + \beta(x)d(y) = f(x)f(y), \text{ for all } x, y \in I.
\]

(2.6)

Writing $xw, w \in I$ for $x$ in (2.6), we have

\[
f(xw)\alpha(y) + \beta(xw)d(y) = f(xw)f(y)
\]

Since $d$ acts as a homomorphism on $I$, we get

\[
f(x)f(w)\alpha(y) + \beta(xw)d(y) = f(x)f(w)f(y) = f(x)f(w)f(y)
\]

\[
= f(x)f(w)\alpha(y) + f(x)\beta(w)d(y)
\]

and so, we have $\beta(x)\beta(w)d(y) = f(x)\beta(w)d(y), i.e.,$

\[
(\beta(x) - f(x))\beta(w)d(y) = 0, \text{ for all } x, y \in I.
\]
Since $I$ a nonzero ideal of prime ring $R$ and $\beta$ an automorphism of $R$, we conclude that
$$f(x) = \beta(x), \text{for all } x \in I \text{ or } d(y) = 0, \text{ for all } y \in I.$$ Let us assume that $f(x) = \beta(x)$, for all $x \in I$. Replacing $x$ by $xy, y \in I$, we have $f(xy) = \beta(xy)$ and
$$d(x) \alpha(y) + \beta(x) f(y) = \beta(x) \beta(y) = \beta(x)f(y).$$
Thus, we obtain that $d(x) = 0$ for all $x \in I$ for any cases. That is $d = 0$. This completes the proof. □

**Theorem 5.** Let $R$ be a prime ring with characteristics not two, $I$ a nonzero ideal of $R$ and $f : R \to R$ be a generalized $(\alpha, \beta)$-derivation of prime ring $R$ associated with $d$. If $f(xy) = f(y)f(x)$ for all $x, y \in I$, then $d = 0$.

**Proof.** Since $d$ acts as an anti-homorphism on $I$, we get
$$f(xy) = d(x)\alpha(y) + \beta(x)f(y) = f(y)f(x), \text{ for all } x, y \in I. \hspace{1cm} (2.7)$$ Replacing $y$ by $xy, y \in I$ in $(2.7)$, we obtain
$$d(x)\alpha(xy) + \beta(x)f(xy) = f(xy)f(x),$$
and so,
$$d(x)\alpha(xy)\alpha(y) = d(x)\alpha(y)f(x), \text{ for } x, y \in I. \hspace{1cm} (2.8)$$ Substituting $yw, w \in I$ for $y$ in this equation and using $(2.8)$, we get
$$0 = d(x)\alpha(x)\alpha(y)\alpha(w) - d(x)\alpha(y)\alpha(w)f(x)$$
$$= d(x)\alpha(y)f(x)\alpha(w) - d(x)\alpha(y)\alpha(w)f(x)$$
$$= d(x)\alpha(y)[f(x),\alpha(w)].$$
Since $\alpha$ is an automorphism and $I$ a nonzero ideal of prime ring $R$, we obtain
$$d(x) = 0 \text{ or } [f(x),\alpha(w)] = 0, \text{ for all } x, w \in I.$$ Now let us define $A = \{x \in I \mid d(x) = 0\}$ and $B = \{x \in I \mid [f(x),\alpha(w)] = 0\}$, for all $w \in I$. Clearly each $A$ and $B$ is an additive subgroup of $I$. Moreover, $I$ is the set theoretic union of $A$ and $B$. But a group cannot be the set theoretic union of two proper subgroups, hence $A = I$ or $B = I$. In the former case, we have $d(R) = 0$, which completes the proof. If $B = I$, then we get
$$0 = [f(x),\alpha(wr)] = \alpha(w)[f(x),\alpha(R)]$$
for all $x, w \in I, r \in R$.
Thus, we obtain that $f(I) \subset Z$. Since $f$ acts as an endomorphism of $R$, it follows that $d = 0$, via Theorem 4. □

**Theorem 6.** Let $R$ be a prime ring with characteristics not two, $I$ a nonzero ideal of $R$ and $f : R \to R$ be a generalized $(\alpha, \beta)$-derivation of prime ring $R$ associated with nonzero $(\alpha, \beta)$-derivation $d$. If $\beta d = d\beta$ and $f(xy) = f(yx)$ for all $x, y \in I$, then $R$ is a commutative ring.
Proof. Let $c \in I$ be a constant element such that $f(c) = 0$ (i.e., $[x, y]$) and $z$ be an arbitrary element of $I$. The condition that $f(cz) = f(za) + \beta(c) d(z)$ yields that $f(c) \alpha(z) + \beta(c) d(z) = d(z) \alpha(c) + \beta(z) f(c)$ and so,

$\beta(c) d(z) = d(z) \alpha(c), \text{ for } z \in I.$

Thus $[d(z), c]_{\alpha, \beta} = 0$, for all $z \in I$. Replacing $z$ by $wz$, $w \in I$ and using this equation, we have

$0 = [d(wz), c]_{\alpha, \beta} = [d(w) \alpha(z) + \beta(w) d(z), c]_{\alpha, \beta}$

$= d(w) \alpha([z, c]) + [d(w), c]_{\alpha, \beta} \alpha(z) + \beta(w) [d(z), c]_{\alpha, \beta} + \beta([w, c]) d(z)$

and so,

$d(w) \alpha([z, c]) + \beta([w, c]) d(z) = 0, \text{ for all } w, z \in I. \quad (2.9)$

Writing $c$ (since $c \in I$) by $z$ in (2.9), we have $\beta([w, c]) d(c) = 0$. This gives us

$[w, c] \beta^{-1}(d(c)) = 0, \text{ for all } w \in I, \text{ all } c \in I \text{ such that } f(c) = 0.$

If we take $rw, r \in R, w \in I$ instead of $w$ in this equation, we have

$[r, c] w^\beta^{-1}(d(c)) = 0, \text{ for all } r \in R.$

Since $I$ is a nonzero ideal of prime ring $R$, we obtain

$c \in Z \text{ or } d(c) = 0, \text{ for all } c \in I \text{ such that } f(c) = 0.$

So, we obtain

$[x, y] \in Z \text{ or } d([x, y]) = 0, \text{ for all } x, y \in I.$

because of $f([x, y]) = 0, \text{ for all } x, y \in I.$

This shows that additive group $I$ is the union of two of its additive subgroups $A = \{x \in I \mid d([x, y]) = 0, \text{ for all } y \in I\}$ and $B = \{x \in I \mid [x, y] \in Z, \text{ for all } y \in I\}$.

If $I = A$, then $R$ is a commutative ring (i.e., $[x, y] \in Z, \text{ for all } x, y \in I$) by [2, Theorem 3]. So, we can take $I = B$ and $[x, y] \in Z, \text{ for all } x, y \in I.$ Let us define an inner derivation $I_x : R \rightarrow R, I_x^2 (y) = 0, \text{ for all } y \in I.$ This shows $I \subset Z$ and so $R$ is a commutative ring by [5, Theorem 1].

**Definition 2.** Let $R$ be a ring, $\beta$ and $\alpha$ automorphisms of $R$ and $d$ a $(\alpha, \beta)$-derivation of $R$. An additive mapping $f : R \rightarrow R$ is said to be right generalized $(\alpha, \beta)$-Jordan derivation associated with $d$ if

$f(x^2) = f(x) \alpha(x) + \beta(x) d(x) \text{ for all } x \in R. \quad (2.10)$

and $f$ is said to be left generalized $(\alpha, \beta)$-Jordan derivation associated with $d$ if

$f(x^2) = d(x) \alpha(x) + \beta(x) f(x) \text{ for all } x \in R. \quad (2.11)$

$f$ is said to be a generalized $(\alpha, \beta)$-Jordan derivation associated with $d$ if it is both a left and right generalized $(\alpha, \beta)$-Jordan derivation associated with $d$. 


Lemma 4. Let $f : R \to R$ be a right generalized $(\alpha, \beta)$-Jordan derivation on $I$. For all $x, y, z \in I$,

$$f(xy + yx) = f(x)\alpha(y) + \beta(x)d(y) + f(y)\alpha(x) + \beta(y)d(x);$$

(1)

$$f(xy) = f(x)\alpha(yx) + \beta(x)d(y)\alpha(x) + \beta(xy)d(x);$$

(2)

$$f(yz + zx) = f(x)\alpha(yz) + \beta(x)d(y)\alpha(z) + \beta(xy)d(z)$$

$$+ f(z)\alpha(y) + \beta(z)d(y)\alpha(x) + \beta(z)d(x)$$

$$+ f(z)\alpha(yx) + \beta(z)d(y)\alpha(x) + \beta(z)d(x).$$

(3)

Proof. Assertion (1). Linearizing (2.10), we get

$$f((x + y)^2) = f((x + y)(x + y) = f(x^2 + xy + yx + y^2)$$

$$= f(x^2) + f(xy + yx) + f(y^2)$$

$$= f(x)\alpha(x) + \beta(x)d(x)$$

$$+ f(xy + yx) + f(y)\alpha(y) + \beta(y)d(y)$$

(2.12)

for all $x, y \in I$. On the other hand,

$$f((x + y)^2) = f(x + y)\alpha(x + y) + \beta(x + y)d(x + y)$$

$$= f(x)\alpha(x) + f(y)\alpha(x) + f(x)\alpha(y) + f(y)\alpha(y)$$

$$+ \beta(x)d(x) + \beta(x)d(y) + \beta(y)d(x) + \beta(y)d(y)$$

(2.13)

for all $x, y \in I$. Comparing (2.12) and (2.13), we have

$$f(xy + yx) = f(x)\alpha(y) + \beta(x)d(y) + f(y)\alpha(x) + \beta(y)d(x)$$

for all $x, y \in I$.

Assertion (2). Replacing $y$ by $xy + yx$ in (1), we get

$$f(x(xy + yx)) = f(x^2y + xyx + xyx + yx^2)$$

$$= f(x^2y + yx^2) + 2f(xy)$$

$$= f(x)\alpha(y) + \beta(x)d(y) + f(y)\alpha(x)$$

$$+ \beta(y)d(x^2) + 2f(xy)$$

$$= f(x)\alpha(xy) + \beta(x)d(x)\alpha(y) + \beta(x^2)d(y)$$

$$+ f(y)\alpha(x^2) + \beta(y)d(x)\alpha(x)$$

$$+ \beta(xy)d(x) + 2f(xy)$$

(2.14)
for all $x, y \in I$. On the other hand, we have

$$f(x(xy + yx) + (xy + yx)x) = f(x)\alpha(xy + yx) + \beta(x) d(xy + yx)$$

$$+ f(xy + yx)\alpha(x) + \beta(xy + yx)d(x)$$

$$= f(x)\alpha(xy) + f(x)\alpha(yx) + \beta(x) d(x)\alpha(y) + \beta(x^2)d(y)$$

$$+ \beta(x) d(y)\alpha(x) + f(y)\alpha(x^2) + \beta(y) d(x)\alpha(x)$$

$$+ \beta(xy) d(x) + \beta(yx)d(x)$$

(2.15)

for all $x, y \in I$. Comparing (2.14) and (2.15) and using char $R \neq 2$, we get the required result.

**Assertion (3).** Linearizing (2) on $x$, we get

$$f((x + z)y(x + y)) = f(xy)\alpha(xy) + yz + \beta* x + z y z$$

$$= f(xy) + f(xy + z y z)$$

$$= f(x)\alpha(xy) + \beta(x) d(y)\alpha(x) + \beta(xy) d(x)$$

$$+ f(xy + z y z) + f(z)\alpha(yz)$$

$$+ \beta(z) d(y)\alpha(z) + \beta(z y d(z))$$

(2.16)

for all $x, y, z \in I$. Computing now $f((x + z)y(x + z))$ in another way, we get

$$f((x + z)y(x + z)) = f(x + z)\alpha(xy + yz) + \beta(x + z)d(y)\alpha(x + z)$$

$$+ \beta(xy + yz)d(x + z)$$

$$= f(x)\alpha(xy) + f(x)\alpha(yz) + f(z)\alpha(yx) + f(z)\alpha(yz)$$

$$+ \beta(x) d(y)\alpha(x) + \beta(x) d(y)\alpha(z)$$

$$+ \beta(z) d(y)\alpha(x) + \beta(z) d(y)\alpha(z)$$

$$+ \beta(xy) d(x) + \beta(xy) d(z) + \beta(yz)d(x)$$

$$+ \beta(z y d(z))$$

(2.17)

for all $x, y, z \in I$. Comparing (2.16) and (2.17), we have

$$f(xyz + z y x) = f(x)\alpha(yz) + \beta(x) d(y)\alpha(z) + \beta(xy)d(z) + f(z)\alpha(yx)$$

$$+ \beta(z) d(y)\alpha(x) + \beta(z y d(x))$$

for all $x, y, z \in I$.

We introduce the abbreviation

$$x^y = f(xy) - f(x)\alpha(y) - \beta(x)d(y)$$

for all $x, y \in I$. 

□
Observe that, by Lemma 4 (1), we have
\[ f(xy) = f(x)\alpha(y) + \beta(x)d(y) + f(y)\alpha(x) + \beta(y)d(x) \]
and so, \( f(xy) - f(x)\alpha(y) - \beta(x)d(y) = -(f(yx) - f(y)\alpha(x) - \beta(y)d(x)) \). That is,
\[ x^y = -y^x \text{ for all } x, y \in I. \] (2.18)

**Lemma 5.** Let \( f : R \to R \) be a right generalized \((\alpha, \beta)\)-Jordan derivation on \( I \).
For all \( x, y \in I \),
\[ x^y[\alpha(x), \alpha(y)] = 0. \]

**Proof.** Replacing \( z \) by \( xy \) in Lemma 4 (3), we get
\[
f((xy)^2 + xy^2x) = f((xy)^2) + f(xy^2x) = f(xy)\alpha(xy) + \beta(xy)d(xy)
+ f(x)\alpha(x^2y) + \beta(x)d(y^2)\alpha(x) + \beta(xy^2)d(x)
= f(xy)\alpha(xy) + \beta(xy)d(x)\alpha(y) + \beta(xy)d(x)\alpha(y)
+ f(x)\alpha(x^2y) + \beta(x)d(y)\alpha(xy)
+ \beta(xy)d(y)d(x)\alpha(x) + \beta(xy^2)d(x) \tag{2.19}
\]
for all \( x, y \in I \). On the other hand, we get
\[
f(xy(xy) + (xy)yx) = f(x)\alpha(yxy) + \beta(x)d(y)\alpha(xy) + \beta(xy)d(xy)
+ f(xy)\alpha(xy) + \beta(xy)d(y)\alpha(x) + \beta(xy^2)d(x)
= f(xy)\alpha(yxy) + \beta(xy)d(y)\alpha(xy) + \beta(xy)d(x)\alpha(y)
+ \beta(xy)d(y)d(y)\alpha(y)
+ \beta(xy^2)d(x) \tag{2.20}
\]
for all \( x, y \in I \). Comparing these two equations, we have
\[
f(xy)\alpha(xy) + f(x)\alpha(y^2x) + \beta(x)d(y)\alpha(yx)
= f(x)\alpha(yxy) + \beta(xy)d(y)\alpha(xy) + f(xy)\alpha(yx).
\]
That is, \( x^y[\alpha(x), \alpha(y)] = 0 \) for all \( x, y \in I \). \qed

**Theorem 7.** Let \( R \) be a non-commutative prime ring with characteristics not two, \( I \) a nonzero ideal of \( R \). If \( f \) be a right generalized \((\alpha, \beta)\)-Jordan derivation on \( I \), then \( f \) is generalized \((\alpha, \beta)\)-derivation on \( I \).

**Proof.** From Lemma 4 (3), we have
\[
f(xwy + ywx) = f(x)\alpha(wy) + \beta(x)d(w)\alpha(y) + \beta(wx)d(y)
+ f(y)\alpha(wx) + \beta(y)d(w)\alpha(x) + \beta(yw)d(x) \tag{2.21}
\]
for all \( x, y, w \in I \). Replacing \( x \) by \( xy \) and \( y \) by \( yx \) in (2.21), we get

\[
f((xy)w(yx) + (yx)w(xy)) = f(xy)\alpha(wyx) + \beta(xy)d(w)\alpha(yx)
+ \beta(xyw)d(yx) + f(y)x\alpha(wxy)
+ \beta(yxw)d(w)\alpha(xy) = f(xy)\alpha(wyx) + \beta(xy)d(w)\alpha(yx) + \beta(xyw)d(yx)
+ f(y)x\alpha(wxy) + \beta(yx)d(w)\alpha(xy)
+ \beta(yxw)d(w)\alpha(yx) + \beta(yxw)d(yx)\delta_{xy} (2.22)
\]
for all \( x, y, w \in I \). On the other hand, we have

\[
f((xy)w(yx) + (yx)w(xy)) = f(x(ywy)x) + f(y(xwy)x)
= f(x)\alpha(ywx) + \beta(x)d(ywy)\alpha(x) + \beta(xwy)d(x)
+ f(y)\alpha(xwy) + \beta(y)d(xwy)\alpha(y) + \beta(xy)\alpha(yx)
+ \beta(xy)d(x)\alpha(wyx) + \beta(xy)d(w)\alpha(xy)
+ \beta(xy)d(xy)\alpha(wyx) + \beta(xy)d(yx)\delta_{xy} (2.23)
\]
for all \( x, y, w \in I \). Comparing equations (2.22) and (2.23), we obtain

\[
\{ f(xy) - f(y)\alpha(x) - \beta(y)d(x) \} \alpha(w)\alpha(xy)
+ \{ f(xy) - f(x)\alpha(y) - \beta(x)d(y) \} \alpha(w)\alpha(yx) = 0
\]
and hence

\[
y^x\alpha(w)\alpha(xy) + x^y\alpha(w)\alpha(yx) = 0
\]
for all \( x, y, w \in I \). Using (2.18), we get \( x^y\alpha(I)\alpha([y,x]) = 0 \) for all \( x, y \in I \). Since \( I \) is a nonzero ideal of prime ring \( R \), we obtain for each pair \( x, y \in I \) either \( x^y = 0 \) or \( [x,y] = 0 \). Notice that the mappings \( (x,y) \rightarrow x^y \) and \( (x,y) \rightarrow [x,y] \) satisfy the requirements of [5, Lemma 4]. Hence \( x^y = 0 \) for all \( x, y \in I \) or \( [x,y] = 0 \) for all \( x, y \in I \). If \( [x,y]^2 = 0 \) for all \( x, y \in I \), then for each \( x \in I \), \( I_x (y)^2 = 0 \), for all \( y \in I \), where \( I_x \) is the inner derivation. This yields that \( R \) is commutative, a contradiction by [11, Theorem 3]. Thus, we have \( x^y = 0 \) for all \( x, y \in I \). This completes the proof. \( \square \)

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