A new result on the quasi power increasing sequences

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A NEW RESULT ON THE QUASI POWER INCREASING SEQUENCES

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Abstract. In this paper, we prove a general theorem dealing with generalized absolute convolution Cesàro mean summability factors under weaker conditions by using a general class of increasing sequences instead of an almost increasing sequence. Some new results have also been obtained.

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1. INTRODUCTION

A positive sequence \((b_n)\) is said to be an almost increasing sequence if there exists a positive increasing sequence \((c_n)\) and two positive constants \(M\) and \(N\) such that \(Mc_n \leq b_n \leq Nc_n\) (see [1]). A positive sequence \(X = (X_n)\) is said to be a quasi-\(\sigma\)-power increasing sequence if there exists a constant \(K = K(\sigma, X) \geq 1\) such that \(Kn^\sigma X_n \geq m^\sigma X_m\) for all \(n \geq m \geq 1\) and \(0 < \sigma < 1\). Every almost increasing sequence is quasi-\(\sigma\)-power increasing sequence for any nonnegative \(\sigma\), but the converse is not true for \(\sigma > 0\) (see [9]). Let \(\sum a_n\) be a given infinite series. We denote by \(t_n^{\alpha*\beta}\) the \(n\)th convolution Cesàro mean of order \((\alpha*\beta)\), with \(\alpha + \beta > -1\), of the sequence \((na_n)\), that is (see [5])

\[
t_n^{\alpha*\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^\beta v a_v,
\]

where

\[
A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.
\]

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Let \( \theta_n^{a*\beta} \) be a sequence defined by

\[
\theta_n^{a*\beta} = \begin{cases} 
|I_n^{a*\beta}|, & \alpha = 1, \beta > -1 \\
\max_{1 \leq v \leq n} |I_v^{a*\beta}|, & 0 < \alpha < 1, \beta > -1 
\end{cases}
\]  

(1.3)

The series \( \sum a_n \) is said to be summable \( \big| C, \alpha \ast \beta; \delta \big|_k \), \( k \geq 1 \) and \( \delta \geq 0 \), if (see [4])

\[
\sum_{n=1}^{\infty} n^{\delta k-1} |I_n^{a*\beta}|^k < \infty. 
\]  

(1.4)

If we take \( \delta = 0 \), then \( \big| C, \alpha \ast \beta; \delta \big|_k \) summability reduces to \( \big| C, \alpha \ast \beta \big|_k \) summability (see [6]). Also, if we take \( \beta = 0 \) and \( \delta = 0 \), then \( \big| C, \alpha \ast \beta; \delta \big|_k \) summability reduces to \( \big| C, \alpha \big|_k \) summability (see [7]). If we set \( \beta = 0 \), then we get \( \big| C, \alpha; \delta \big|_k \) summability (see [8]).

2. THE KNOWN RESULT

**Theorem 1** ([4]). Let \( (X_n) \) be an almost increasing sequence and let there be sequences \( (\eta_n) \) and \( (\lambda_n) \) such that

\[
|\Delta \lambda_n| \leq \eta_n, 
\]  

(2.1)

\[
\eta_n \to 0 \text{ as } n \to \infty, 
\]  

(2.2)

\[
\sum_{n=1}^{\infty} n |\Delta \eta_n| X_n < \infty, 
\]  

(2.3)

\[
|\lambda_n| X_n = O(1) \text{ as } n \to \infty. 
\]  

(2.4)

If the condition

\[
\sum_{n=1}^{m} n^{\delta k} \frac{|\theta_n^{a*\beta}|^k}{n} = O(X_m) \text{ as } m \to \infty 
\]  

(2.5)

satisfies, then the series \( \sum a_n \lambda_n \) is summable \( \big| C, \alpha \ast \beta; \delta \big|_k \), \( 0 < \alpha \leq 1, \beta > -1, k \geq 1, \delta \geq 0 \) and \( (\alpha + \beta - \delta) > 0 \).

It should be noted that if we take \( \beta = 0 \), then we get the result of Bor (see [2]).

3. THE MAIN RESULT

The aim of this paper is to generalize Theorem 1 under weaker conditions to the \( \big| C, \alpha \ast \beta; \delta \big|_k \) summability by using a quasi-\( \sigma \)-power increasing sequence, which is a wider class of sequences, instead of an almost increasing sequence. We shall prove the following main theorem.
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Theorem 2. Let \((X_n)\) be a quasi-\(\sigma\)-power increasing sequence for some \(\sigma \in (0, 1)\) and let there be sequences \((\eta_n)\) and \((\lambda_n)\) such that conditions (2.1)-(2.4) of Theorem A are satisfied. If the condition

\[
\sum_{n=1}^{m} n^{\delta k} \left( \frac{\theta_{n}^{\alpha \beta}}{n X_n^{k-1}} \right)^k = O(X_m) \quad \text{as} \quad m \to \infty
\]

is satisfies, then the series \(\sum a_n \lambda_n\) is summable \(\mathcal{C}; \alpha, \beta; \delta \, \middle| \, k \) for \(0 < \alpha \leq 1, \delta \geq 0, \beta > -1, k \geq 1\) and \((\alpha + \beta - \delta - 1) > 0\).

Remark 1. It should also be noted that condition (3.1) is the same as condition (2.5) when \(k=1\). When \(k > 1\), condition (3.1) is weaker than condition (2.5) but the converse is not true. As in [10] we can show that if (2.5) is satisfied, then we get that

\[
\sum_{n=1}^{m} n^{\delta k} \left( \frac{\theta_{n}^{\alpha \beta}}{n X_n^{k-1}} \right)^k = O\left( \frac{1}{X_1^{k-1}} \right) \sum_{n=1}^{m} n^{\delta k} \left( \frac{\theta_{n}^{\alpha \beta}}{n X_n^{k-1}} \right)^k = O(X_m).
\]

Also if (3.1) is satisfied, then for \(k > 1\) we obtain that

\[
\sum_{n=1}^{m} n^{\delta k} \left( \frac{\theta_{n}^{\alpha \beta}}{n X_n^{k-1}} \right)^k = \sum_{n=1}^{m} \left( \frac{\theta_{n}^{\alpha \beta}}{n X_n^{k-1}} \right)^k X_n^{k-1} = O(X_m^{k-1}) \sum_{n=1}^{m} n^{\delta k} \left( \frac{\theta_{n}^{\alpha \beta}}{n X_n^{k-1}} \right)^k = O(X_m^{k}) \neq O(X_m).
\]

We need the following lemmas for the proof of our theorem.

Lemma 1 ([3]). If \(0 < \alpha \leq 1, \beta > -1\) and \(1 \leq v \leq n\), then

\[
| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} B_p A_p | \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} B_p a_p \right|.
\]

Lemma 2 ([9]). Under the conditions on \((X_n), (\eta_n)\) and \((\lambda_n)\) as expressed in the statement of the theorem, we have the following :

\[
nX_n \eta_n = O(1); \quad \sum_{n=1}^{\infty} \eta_n X_n < \infty.
\]

4. PROOF OF THE THEOREM

Let \(T_n^{\alpha \beta}\) be the \(n\)th \((C, \alpha \ast \beta)\) mean of the sequence \((n\alpha_n \lambda_n)\). Then, by (1.1), we have

\[
T_n^{\alpha \beta} = \frac{1}{A_n^{\alpha + \beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} B_v a_v \lambda_v.
\]
Applying Abel’s transformation first and then using Lemma 1, we have that

\[ T_{\alpha^* \beta} = \frac{1}{A_n^{\alpha + \beta}} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha - 1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha + \beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha - 1} A_v^{\beta} v a_v, \]

thus,

\[ |T_{\alpha^* \beta}| \leq \frac{1}{A_n^{\alpha + \beta}} \sum_{v=1}^{n-1} |\Delta \lambda_v| \sum_{p=1}^{v} A_{n-p}^{\alpha - 1} A_p^{\beta} p a_p | + \frac{\lambda_n}{A_n^{\alpha + \beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha - 1} A_v^{\beta} v a_v | \]

\[ \leq \frac{1}{A_n^{\alpha + \beta}} \sum_{v=1}^{n-1} A_v^{(\alpha + \beta)} (\partial_v^{\alpha^* \beta}) \Delta \lambda_v | + |\lambda_n| (\partial_n^{\alpha^* \beta}) \]

\[ = T_{\alpha^* \beta}^{(1)} + T_{\alpha^* \beta}^{(2)}. \]

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

\[ \sum_{n=1}^{\infty} n^{\delta k-1} |T_{\alpha^* \beta}^{(r)}|^k < \infty, \quad \text{for} \quad r = 1, 2. \]

Whenever \( k > 1 \), we can apply Hölder’s inequality with indices \( k \) and \( k' \), where \( \frac{1}{k} + \frac{1}{k'} = 1 \), we get that

\[ \sum_{n=2}^{m+1} n^{\delta k-1} |T_{\alpha^* \beta}^{(1)}|^k \]

\[ \leq \sum_{n=2}^{m+1} n^{\delta k-1} (A_n^{\alpha + \beta})^{-k} \left\{ \sum_{v=1}^{n-1} (A_v^{\alpha + \beta})^k (\partial_v^{\alpha^* \beta})^k \Delta \lambda_v | k \times \sum_{v=1}^{n-1} 1 \right\}^{k-1} \]

\[ = O(1) \sum_{n=2}^{m+1} n^{\delta k-2+k-(\alpha + \beta)k} \left\{ \sum_{v=1}^{n-1} (A_v^{\alpha + \beta})^k (\partial_v^{\alpha^* \beta})^k \right\} \]

\[ = O(1) \frac{\sum_{v=1}^{m+1} (A_v^{\alpha + \beta})^k (\partial_v^{\alpha^* \beta})^k}{\eta_v^{k-1}} \int_{n^{\delta k+k-1}}^{\infty} \frac{dx}{x^{2+(\alpha + \beta - \delta - 1)k}} \]

\[ = O(1) \sum_{v=1}^{m} (A_v^{\alpha + \beta})^k \eta_v^{k-1} \int_{n^{\delta k+k-1}}^{\infty} \frac{dx}{x^{2+(\alpha + \beta - \delta - 1)k}} \]

\[ = O(1) \sum_{v=1}^{m} (A_v^{\alpha + \beta})^k \eta_v^{k-1} \int_{n^{\delta k+k-1}}^{\infty} \frac{dx}{x^{2+(\alpha + \beta - \delta - 1)k}} \]

\[ = O(1) \sum_{v=1}^{m} (A_v^{\alpha + \beta})^k \eta_v^{k-1} n^{\delta k+k-1} \]

\[ = O(1) \sum_{v=1}^{m} (A_v^{\alpha + \beta})^k \eta_v^{k-1} \left( \frac{1}{n^{\delta k+k-1}} \right)^{k-1} \]

\[ = O(1) \sum_{v=1}^{m} (A_v^{\alpha + \beta})^k \eta_v^{k-1} \left( \frac{1}{n^{\delta k+k-1}} \right)^{k-1} \]
by virtue of the hypotheses of the theorem and Lemma 2. Finally, we have that
\[ \sum_{n=1}^{m} n^k \left| \frac{\lambda_n}{\eta_{n,2}} \right|^k = \sum_{n=1}^{m} \left| \lambda_n \right|^{k-1} \lambda_n \left| n^k \left( \frac{\eta_{n}^{\alpha*\beta}}{n} \right)^k \right| \]
\[ = O(1) \sum_{n=1}^{m-1} \Delta \left| \lambda_n \right| \sum_{v=1}^{n} v^k \left( \frac{\eta_{v}^{\alpha*\beta}}{vX_v^{k-1}} \right)^k \]
\[ + O(1) \lambda_m \sum_{n=1}^{m-1} n^k \left( \frac{\eta_{n}^{\alpha*\beta}}{nX_n^{k-1}} \right)^k \]
\[ = O(1) \sum_{n=1}^{m-1} \eta_n X_n + O(1) \lambda_m \left| X_m \right| = O(1) \text{ as } m \to \infty, \]
by virtue of the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

Remark 2. If we take \((X_n)\) as an almost increasing sequence, \(\beta = 0\) and \(\delta = 0\), then we obtain a theorem dealing with the \(|C, \alpha|_k\) summability factors. Also, if we take \(\delta = 0\), then we get a new result concerning the \(|C, \alpha* \beta|_k\) summability factors of infinite series.

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