On a nonlinear parabolic differential equation

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ON A NONLINEAR PARABOLIC DIFFERENTIAL EQUATION

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Abstract. Our aim is to examine the nonlinear parabolic differential equation \( u_{xx} - g(t, x)f(u_t, u_x) = 0 \). We present three examples for the solution of the equation of some special forms. A maximum principle and some uniqueness results are given. Moreover, the approximate solution of the equation with \( g(t, x) = 1 \), obtained by the difference method is investigated.

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1. Introduction

We consider the differential equation

\[
  u_{xx} - g(t, x)f(u_t^{(k)}, u_x) = 0, \tag{1.1}
\]

where \( k = 1, 2 \),

\[
  u = u(t, x), \quad u_t^{(k)} = \frac{\partial^k u}{\partial t^k}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.
\]

Throughout the paper we shall assume that the function \( g(t, x) > 0 \) and function \( f \) are homogeneous of the first degree, more precisely,

\[
  f(\lambda u, \lambda v) = \lambda f(u, v) \quad \text{for } \lambda > 0, \quad uv \neq 0,
\]

and, moreover, the function \( f \) satisfies the condition

\[
  uf(u, v) > 0, \quad uv \neq 0
\]

and all the functions and derivatives involved here exist and are continuous in \( \mathbb{R} \times \mathbb{R} \). When \( k = 1 \), equation (1.1) is a parabolic second order partial differential equation and if \( k = 2 \), it is a hyperbolic one.

For \( g(t, x) = -q(x) \), equation (1.1) has solutions of the form \( u(t, x) = e^t v(x) \) and \( v = v(x) \) satisfies the second order differential equation

\[
  v'' + q(x)f(v, v') = 0. \tag{1.2}
\]
A special case of equation (1.1) is the nonlinear parabolic differential equation
\[(\Phi_p(u_x))_x - g(t,x)\Phi_p(u_t) = 0,\]
where \(\Phi_p(w) = |w|^{p-1}w, p > 0,\) and the function \(\Phi_p\) is increasing. The function \(v\) in the solutions of the form \(u(t,x) = e^tv(x)\) with \(g(t,x) = -q(x)\) satisfies the relation
\[(\Phi_p(v'))' + q(x)\Phi_p(v) = 0.\]
(1.4)

We shall consider the solvability of equation (1.1) for \(k = 1\) and that of equation (1.3) with the conditions
\[u(0,x) = \gamma(x), \quad u(t,0) = \alpha(t), \quad u(t,l) = \beta(t), \quad l > 0, \quad \gamma(0) = \alpha(0), \quad \gamma(l) = \beta(0), \quad \alpha, \beta \in C([0,l]), \quad T > 0.\]
(1.5)

We suppose that \(u(t,x)\) has continuous derivatives in the domain \(D = \{(t,x) : t \in [0,T], x \in [0,l]\}\) and \(u(t,x)\) is continuous on the boundary of \(D\).

First we give three examples in which the solutions of (1.3) of some special forms are presented. A maximum principle and some uniqueness results are given for the solution of (1.1) and (1.3). In the last section, the approximate solution of (1.3) with \(g(t,x) = 1\) obtained by the difference method is examined.

2. Solutions of a special form

We give the solution of the parabolic partial differential equation (1.1) or (1.3) provided the solution is of a special form.

Example 1. Let us consider the solution of (1.3) of the form \(u(t,x) = \exp(at + bx)\), where \(a\) and \(b\) are constants. In this case, equation (1.3) gives
\[p |b|^{p+1} - g(t,x) \Phi_p(a) = 0.\]

If \(g(t,x) > 0\), then it is obvious that \(a > 0\). If \(g(t,x) = 1\), then
\[a = p^{\frac{1}{p}} |b|^{\frac{p+1}{p}} \quad \text{or} \quad b = \pm \left(\frac{a^p}{p}\right)^{\frac{1}{p+1}}\]
and the solution has the form
\[u(t,x) = \exp \left(at \pm \left(\frac{a^p}{p}\right)^{\frac{1}{p+1}} x\right)\]
or
\[u(t,x) = \exp \left(p^{\frac{1}{p}} |b|^{\frac{p+1}{p}} t + bx\right).\]

In the plane \((t,x)\), the solution \(u(t,x) = e^C\) is constant on the straight lines \(at \pm \left(\frac{a^p}{p}\right)^{\frac{1}{p+1}} x = C, C = \text{const.}\)
Example 2. Let us consider the solution of (1.3) of the form \( u(t, x) = v(x) \tau(t) \) with \( g(t, x) = r(x)s(t) \).

Substituting \( u \) into equation (1.3) we have

\[
 p |v'|^{p-1}v'' \Phi(p) = r(x)s(t)\Phi_p(v)\Phi_p(\tau').
\]

We suppose that \( v(x) \neq 0 \) and \( \tau(t) \neq 0 \). Separating the variables, one can get for \( v \)

\[
 p |v'|^{p-1}v'' - \mu r(x) \Phi_p(v) = 0, \quad \mu = \text{const}, \quad (2.1)
\]

and for \( \tau \)

\[
 \frac{\tau'}{\tau} = \left| \frac{\mu}{s(t)} \right|^{\frac{1}{p}-1} \frac{\mu}{s(t)}, \quad (2.2)
\]

For the solution of (2.1) we refer to [1–3]. The most important property of these solutions is that for any given initial condition at \( t_0 \in I \),

\[
 x(t_0) = x_0, \\
 x'(t_0) = x'_0,
\]

there exists a unique solution \( x(t) \) defined for all \( t \in I \). If \( \mu r(x) < 0 \) (this yields that \( \mu s(t) < 0 \)), then \( v \) is oscillatory.

From this we can see that the solution of (2.2) has the form

\[
 \tau(t) = K \exp \left( \int_0^t \Phi \left( \frac{\mu}{s(w)} \right) \, dw \right), \quad K = \text{const}.
\]

Let us consider the special case where \( \mu > 0, \ r(x) > 0, \ s(t) > 0, \) and

\[
 \alpha(t) = 1 + t, \quad t \in [0, T], \\
 v(0) = 1, \\
 \tau(0) = 1,
\]

then from this it follows that

\[
 s(t) = \mu (1 + t)^p, \\
 v(x) = \gamma(x), \\
 \tau(t) = 1 + t, \\
 \beta(t) = (1 + t) \gamma(t)
\]

and function \( \gamma \ (x \in [0, l]) \) satisfies equation (2.1). For the solution of differential equation (1.3) of the form \( u(t, x) = v(x) \tau(t) \), the relation

\[
 u(t, x) = (1 + t) \gamma(x)
\]

holds.

Example 3. Let us consider the solution of (1.3) of the form \( u(t, x) = v(x) + \tau(t) \) with \( g(t, x) = r(x) s(t) \).
In this case, equation (1.3) gives 
\[ p |v'|^{p-1} v'' = r(x) s(t) \Phi_p(\tau). \]
Separating the variables, we obtain 
\[ p |v'|^{p-1} v'' = \kappa r(x), \quad \kappa = \text{const}, \tag{2.3} \]
and 
\[ s(t) \Phi_p(\tau) = \kappa. \tag{2.4} \]
From equation (2.3), we get 
\[ v(x) = \Phi_p(\kappa) \int_0^x \Phi_{1/p} \left( \int_0^\eta r(\xi) d\xi \right) d\eta \]
and from (2.4), 
\[ \tau(t) = \Phi_p(\kappa) \int_0^t \Phi_{-1}^{-1}(s(\tau)) d\tau. \]
In the special case where \( r(x) = s(t) = 1 \) (\( g(t, x) = 1 \)), we have 
\[ v(x) = \Phi_p(\kappa) \int_0^x \Phi_{1/p} (\eta) d\eta = \frac{p}{p+1} \Phi_p(\kappa) x^{p+1}, \]
\[ \tau(t) = \Phi_p(\kappa) t, \]
and, therefore the solution of (1.3) has the form 
\[ u(t, x) = \Phi_p(\kappa) \left[ \frac{p}{p+1} x^{p+1} + t \right]. \]

3. RESULTS

Theorem 1. Let us suppose that there exists a solution of (1.1) for \( k = 1 \) or (1.3) in the domain \( D_0 = \{ (t, x) : t \in [0, T], \ x \in [0, l] \} \) with the boundary conditions (1.5). Then the solution assumes its maximum on \( \partial D_0 \), the boundary of \( D_0 \).

Proof. We suppose the opposite that solution \( u \) assumes its maximum at an inner point \( (t^*, x^*) \) of \( D_0 \) or on the line \( t = T \). By this assumption, 
\[ u(t^*, x^*) - \max_{(t, x) \in \partial D_0} u(t, x) = \delta > 0. \]
For the auxiliary function 
\[ w(t, x) = u(t, x) + \frac{\delta}{2} \frac{T - t}{T}, \]
we have 
\[ u(t, x) < w(t, x) < u(t, x) + \frac{\delta}{2}. \]
Introducing the new variable $\vartheta$

By the Lagrange Mean Value Theorem, there exists some $\xi$, which gives that
\[
\begin{align*}
  w(t^*, x^*) &\leq \max_{(t, x) \in \partial \Omega} u(t, x) + \delta \\
  &\leq \frac{\delta}{2} T + \frac{\delta}{2} > w(\tilde{r}, \tilde{x}).
\end{align*}
\]

Therefore, $w(t, x)$ assumes its maximum at $(\tilde{r}, \tilde{x}) \in D_0 \setminus \bar{D}$. At this point,
\[
w_x = 0, \quad w_{xx} \leq 0, \quad w_t \geq 0,
\]
which implies that
\[
  u_x = 0, \quad u_{xx} \leq 0, \quad u_t = w_t + \frac{\delta}{2T} > 0. \tag{3.1}
\]

From this observation, it follows that
\[u_{xx} - g(t, x) f_u(t, u_x) \leq 0 \quad \text{or} \quad (\Phi_p(u_x))_x - g(t, x) \Phi_p(u_t) < 0 \quad \text{at} \quad (\tilde{r}, \tilde{x}),\]
which is a contradiction.

We remark that $w_t > 0$ and also $u_t > 0$ at $\tilde{t} = T$. Theorem 2.

There are no any two solutions $u, v$ of (1.3) with $g(t, x) = 1$ such that $u = v$ on $\mathbb{R}$ and $u \neq v, u_x > v_x$ and $u_t > v_t$ in $D_0$.

Proof. We suppose that $u$ and $v$ are different solutions of the differential equation with $g(t, x) = 1$, then
\[
(\Phi_p(u_x))_x - \Phi_p(u_t) = 0
\]
and
\[
(\Phi_p(v_x))_x - \Phi_p(v_t) = 0,
\]
which gives that
\[
(\Phi_p(u_x) - \Phi_p(v_x))_x - (\Phi_p(u_t) - \Phi_p(v_t)) = 0. \tag{3.2}
\]

By the Lagrange Mean Value Theorem, there exists some $\xi \in (a, b)$ such that
\[
\Phi_p(b) - \Phi_p(a) = (b - a) p |\xi|^{p-1}. \tag{3.3}
\]

Let us introduce the notation $w = u - v$; for equation (3.2) we can write
\[
(A(t, x) w_x)_x - B(t, x) w_t = 0, \tag{3.4}
\]
where $A(t, x) = p |\xi_1|^{p-1}$ for $v_x < \xi_1 < u_x$ and $B(t, x) = p |\xi_2|^{p-1}$ for $v_t < \xi_2 < u_t$. Introducing the new variable $\theta$ instead of $x$, by
\[
\theta = \int^t A(t, \zeta) \frac{d\zeta}{\zeta}.
\]
we get for $w(t, x) = \tilde{w}(t, \theta)$ that $A(t, x) w_x = \tilde{w}_\theta$ and equation (3.4) is transformed to
\[
\tilde{w}_{\theta \theta} - Q(t, \theta) \tilde{w}_\theta = 0 \tag{3.5}
\]
where \( Q(t, \vartheta) = \bar{A}(t, \vartheta) \bar{B}(t, \vartheta), A(t, x) = \bar{A}(t, \vartheta) \) and \( B(t, x) = \bar{B}(t, \vartheta). \)

From the condition \( u = v \) on \( \partial \), it follows that \( \bar{w}(t, \vartheta) = 0 \) on \( \partial \), which, by Theorem 1, implies that \( \bar{w}(t, \vartheta) = 0 \) and \( u = v \) in \( D_0 \). This contradicts the assumption on \( u \) and \( v \). \( \square \)

4. Application of the difference method

We shall use the difference method for the determination of the approximate solution of the parabolic differential equation

\[
(\Phi_p(u_x))_x - \Phi_p(u_t) = 0 \quad (4.1)
\]

with conditions \((1.5)\).

Let \( m \) be a positive integer and

\[
\frac{l}{m} = h, \quad x_i = ih, \quad i = 0, 1, 2, \ldots, m.
\]

Obviously, \( x_0 = 0 \) and \( x_m = l \).

Let us denote by \( u_i(t) \) the solution of the first order system of ordinary differential equations

\[
\Phi_p\left( \frac{du_i}{dt} \right) = \frac{1}{h} \left[ \Phi_p\left( \frac{u_{i+1} - u_i}{h} \right) - \Phi_p\left( \frac{u_i - u_{i-1}}{h} \right) \right]
\]

or equivalently

\[
\Phi_p\left( \frac{du_i}{dt} \right) = \frac{1}{h^{p+1}} \left[ \Phi_p\left( u_{i+1} - u_i \right) - \Phi_p\left( u_i - u_{i-1} \right) \right] \quad (4.2)
\]

with the initial conditions

\[
u_i(0) = \gamma(x_i), \quad i = 1, 2, \ldots, m - 1, \quad u_0(t) = \alpha(t), \quad u_m(t) = \beta(t). \quad (4.3)
\]

Thus, system \((4.2)\) involves \( m - 1 \) equations with \( m - 1 \) unknowns.

We intend to show that for an arbitrary \( T \) problem \((4.2)\) \((4.3)\) has a uniquely determined solution. We also show that \( u(t, x_i) \) can be approximated by \( u_i(t) \) with arbitrary accuracy, i. e., for every \( \varepsilon > 0 \), there exists an \( h(\varepsilon) > 0 \) such that \( |u(t, x_i) - u_i(t)| < \varepsilon \) when \( h < h(\varepsilon) \) in \( 0 \leq t \leq T, \ i = 1, 2, \ldots, m - 1. \)

**Lemma 1.** For \( p > 0 \), the function \( \bar{u}_i(t) = u(t, x_i) \) satisfies the system

\[
Q_i(t) = \Phi_p\left( \frac{d\bar{u}_i}{dt} \right) - \frac{1}{h^{p+1}} \left[ \Phi_p\left( \bar{u}_{i+1} - \bar{u}_i \right) - \Phi_p\left( \bar{u}_i - \bar{u}_{i-1} \right) \right], \ i = 1, 2, \ldots, m - 1
\]

where \( |Q_i(t)| \leq Q_0(h) \), with \( Q_0(h) \to 0 \) as \( h \to 0. \)
where $0 \leq \theta_1, \theta_2 < 1$ we have

$$\bar{u}_{i+1}(t) - \bar{u}_i(t) = hu_i(t, x_i) + \frac{h^2}{2} u_{xx}(t, x_i + \theta_1 h),$$

$$\bar{u}_{i-1}(t) - \bar{u}_{i-1}(t) = hu_i(t, x_i) - \frac{h^2}{2} u_{xx}(t, x_i - \theta_2 h).$$

We rewrite problem (4.1) by

$$\Phi_p((\bar{u}_i)_i) = \left( \Phi_p(u_i(t, x_i)) \right)_i = p |u_i(t, x_i)|^{p-1} u_{xx}(t, x_i).$$

From this, together with (4.4) and (4.5), it follows that

$$Q_i(t) = p |u_i(t, x_i)|^{p-1} u_{xx}(t, x_i) - \frac{1}{h^{p+1}} \left[ \Phi_p \left( hu_i(t, x_i) + \frac{h^2}{2} u_{xx}(t, x_i + \theta_1 h) \right) \right.\nonumber$$

$$\left. - \Phi_p \left( hu_i(t, x_i) - \frac{h^2}{2} u_{xx}(t, x_i - \theta_2 h) \right) \right].$$

By the Lagrange Mean Value Theorem (3.3), we can write

$$\Phi_p \left( hu_i(t, x_i) + \frac{h^2}{2} u_{xx}(t, x_i + \theta_1 h) \right) - \Phi_p \left( hu_i(t, x_i) - \frac{h^2}{2} u_{xx}(t, x_i - \theta_2 h) \right) \nonumber$$

$$= \frac{h^2}{2} \left[ u_{xx}(t, x_i + \theta_1 h) + u_{xx}(t, x_i - \theta_2 h) \right] p |\xi|^{p-1},$$

where

$$\xi = hu_i(t, x_i) + \frac{h^2}{2} u_{xx}(t, x_i + \theta_1 h), \quad -\theta_2 < \xi < \theta_1.$$

Then we have

$$Q_i(t) = p |u_i(t, x_i)|^{p-1} u_{xx}(t, x_i) - \frac{p}{2} \left[ u_{xx}(t, x_i + \theta_1 h) + u_{xx}(t, x_i - \theta_2 h) \right] |\eta|^{p-1},$$

where

$$\eta = u_i(t, x_i) + \frac{h}{2} u_{xx}(t, x_i + \theta_1 h) = u_i(t, x_i) + O(h), \quad h \to 0,$$
and

$$Q(t) = p |u_x(t, x_i)|^{p-1} u_{xx}(t, x_i)$$

$$- \frac{p}{2} [u_{xx}(t, x_i + \theta_1 h) + u_{xx}(t, x_i - \theta_2 h)] |u_x(t, x_i) + O(h)|^{p-1}$$

$$= \frac{p}{2} |u_x(t, x_i)|^{p-1} \left[ u_{xx}(t, x_i) - u_{xx}(t, x_i + \theta_1 h) + u_{xx}(t, x_i - \theta_2 h) \right] + O(h^{p-1}).$$

We note that $u_x(t, x_i)$ is bounded, $u_{xx}(t, x_i)$ is uniformly continuous in $\mathcal{D}_0$ and $p > 1$, then for any $\varepsilon > 0$ there exists a function $h(\varepsilon)$ such that $|Q(t)| < Q_0(h)$ when $h < h(\varepsilon)$, $0 \leq t \leq T$.

Now our goal is to state a maximum principle for problem (4.2), (4.3).

**Theorem 3.** Consider a solution $u_i(t)$, $i = 1, 2, \ldots, m - 1$, of (4.2), (4.3), where the function $\gamma$ is positive, increasing, and convex. Then the maxima of $u_i(t)$, $i = 1, 2, \ldots, m - 1$, cannot be greater than the maxima of $\alpha(t)$, $\beta(t)$ and $h(x)$; moreover, the minima of $u_i(t)$, $i = 1, 2, \ldots, m - 1$, cannot be less than the minima of $\alpha(t)$, $\beta(t)$, and $\gamma(x)$.

**Proof.** Let us suppose that there exists a $u_k(t^*)$ where $t^* > 0$ and $k \neq m$ such that $\max_i u_i(t) = u_k(t^*)$. Then the following two cases are possible:

(i) At least one of the inequalities $u_{k+1} - u_k < 0$ and $u_k - u_{k-1} > 0$ holds, moreover $u'_k(t_0) \geq 0$ if $t^* \neq T$ and $u'_k(t^*) > 0$ if $t^* = T$. In (4.2) we have different signs on the left side and on the right side, which is a contradiction. By the convexity of function $\gamma$ we see that

$$\Phi_p \left( u'_k(O) \right) = \frac{1}{h^{p+1}} \left[ \Phi_p \left( \gamma_{i+1} - \gamma_i \right) - \Phi_p \left( \gamma_i - \gamma_{i-1} \right) \right] > 0,$$

then the maxima of $u_i(t)$ cannot be taken at $t^* = 0$. This means that $k = 0$ or $k = m$ can be taken for any $t^* \in (0, T]$.

(ii) At least one of equalities $u_{k+1} - u_k = 0$ and $u_k - u_{k-1} = 0$ holds. We assume that $u_{k+1} - u_k = 0$. Then stepping from index $k$ to $k + 1$, either we obtain a contradiction or we get

$$u_k = u_{k+1} = \cdots = u_m = 0 = \beta(t^*),$$

which we had to prove.

The proof concerning the minima is similar except that $t^* = 0$ is also allowed as $u'_0(0) > 0$.

Now we consider the existence and uniqueness of a solution of problem (4.2), (4.3).
Theorem 4. Let us suppose that function $\gamma$ is continuous, increasing, convex and 
\[
\min_{x,h} \Phi_1 \left\{ \frac{1}{h^{p+1}} \left[ \Phi_p (\gamma(x + h) - \gamma(x)) - \Phi_p (\gamma(x) - \gamma(x - h)) \right] \right\} > L,
\]
where $L$ is independent of $h$, moreover, $u_i(0) = \gamma_i$ with $\gamma_i = \gamma(ih)$ for $i = 1, 2, \ldots, m - 1$, and $u_0(t) \equiv \alpha(t)$, $u_m(t) \equiv \beta(t)$, where $\alpha < \beta$, $\alpha' > 0$, $\beta' > 0$ for all $t \geq 0$. Then problem (4.2), (4.3) has a uniquely determined solution for $0 \leq t \leq T$, where $T$ is positive.

Proof. By the conditions on $\gamma$, we get that
\[
\frac{du_i}{dt} = \Phi_1 \left\{ \frac{1}{h^{p+1}} \left[ \Phi_p (u_{i+1} - u_i) - \Phi_p (u_i - u_{i-1}) \right] \right\}
\]  
(4.6)
are continuous and satisfy the Lipschitz condition for any $t < \tau$, with a small $\tau > 0$. This implies that the solution exists and is unique for $t < \tau$. The conditions on $\gamma$ also gives that
\[
\frac{du_i}{dt}(0) = \Phi_1 \left\{ \frac{1}{h^{p+1}} \left[ \Phi_p (\gamma_{i+1} - \gamma_i) - \Phi_p (\gamma_i - \gamma_{i-1}) \right] \right\} > L,
\]
then
\[
u_i'(t) > 0 \text{ for } t < \tau.
\]
Since $u_i(0) = \gamma(x_i) > 0$, $i = 1, 2, \ldots, m - 1$, then either $u_i'(t)$ remains positive for $t > 0$ or there is a smallest value $\tau_k$ for which $u_i'(\tau_k) = 0$ for some $k = i$.

Taking the derivative of (4.6) we obtain
\[
|u_i'|^{p-1}u_i'' = \frac{1}{h^{p+1}} \left[ |\Delta u_{k+1}|^{p-1} \left( u_{k+1}' - u_k' \right) - |\Delta u_k|^{p-1} \left( u_k' - u_{k-1}' \right) \right],
\]  
(4.7)
where $\Delta u_k = u_k - u_{k-1}$. At $t = \tau_k$ we have that $\Delta u_k = \Delta u_{k+1}$. For small $\varepsilon > 0$, in interval $(\tau_k - \varepsilon, \tau_k)$ we obtain that $|\Delta u_k| > 0$, $|\Delta u_{k+1}| > 0$, $u_k' = o(1)$, $u_{k+1}' - u_k' > 0$, $u_k' - u_{k-1}' < 0$. From these it follows that in (4.7) the right side is positive while the left side is negative as $u_k'' < 0$ for $t \in (\tau_k - \varepsilon, \tau_k)$.

In the case $u_k''(\tau_k) = 0$, passing from $k$ to $k + 1$ and carrying on, we obtain
\[
u_{m-1}' = u_m' = \beta' (\tau_h) = 0,
\]
which is a contradiction. From the argument above, it follows that such a finite $\tau_h$ does not exist.

Consequently, we have that
\[
\Delta u_1 < \Delta u_2 < \cdots < \Delta u_{m-1} < \Delta u_m.
\]
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