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ON A NONLINEAR PARABOLIC DIFFERENTIAL EQUATION

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ABSTRACT. Our aim is to examine the nonlinear parabolic differential equation $u_{xx} - g(t,x)f(u_t,u_x) = 0$. We present three examples for the solution of the equation of some special forms. A maximum principle and some uniqueness results are given. Moreover, the approximate solution of the equation with $g(t,x) = 1$, obtained by the difference method is investigated.

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1. INTRODUCTION

We consider the differential equation

$$u_{xx} - g(t,x)\ f(u_t^{(k)}, u_x) = 0, \tag{1.1}$$

where $k = 1, 2$,

$$u = u(t,x), \quad u_t^{(k)} = \frac{\partial^k u}{\partial t^k}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

Throughout the paper we shall assume that the function $g(t,x) > 0$ and function f are homogeneous of the first degree, more precisely,

$$f(\lambda u, \lambda v) = \lambda\ f(u,v) \quad \text{for } \lambda > 0, \quad uv \neq 0,$$

and, moreover, the function f satisfies the condition

$$u\ f(u,v) > 0, \qquad uv \neq 0$$

and all the functions and derivatives involved here exist and are continuous in $\mathbb{R} \times \mathbb{R}$. When $k = 1$, equation (1.1) is a parabolic second order partial differential equation and if $k = 2$, it is a hyperbolic one.

For $g(t,x) = -q(x)$, equation (1.1) has solutions of the form $u(t,x) = e^t v(x)$ and $v = v(x)$ satisfies the second order differential equation

$$v'' + q(x)f(v, v') = 0. \tag{1.2}$$

A special case of equation (1.1) is the nonlinear parabolic differential equation

$$(\Phi_p(u_x))_x - g(t,x)\,\Phi_p(u_t) = 0, \tag{1.3}$$

where $\Phi_p(w) = |w|^{p-1}w$, $p > 0$, and the function Φ_p is increasing. The function v in the solutions of the form $u(t,x) = e^t v(x)$ with $g(t,x) = -q(x)$ satisfies the relation

$$(\Phi_p(v'))' + q(x)\,\Phi_p(v) = 0. \tag{1.4}$$

We shall consider the solvability of equation (1.1) for $k = 1$ and that of equation (1.3) with the conditions

$$\begin{aligned} u(0,x) &= \gamma(x), \\ u(t,0) &= \alpha(t), \\ u(t,l) &= \beta(t), \quad l > 0 \end{aligned} \tag{1.5}$$

and $\gamma(0) = \alpha(0)$, $\gamma(l) = \beta(0)$, $\gamma \in \mathbf{C}([0,l])$, $\alpha, \beta \in \mathbf{C}([0,T])$, $T > 0$. We suppose that $u(t,x)$ has continuous derivatives in the domain $\mathscr{D} = \{(t,x) : t \in [0,T],\ x \in [0,l]\}$ and $u(t,x)$ is continuous on the boundary of $\mathscr{D}$.

First we give three examples in which the solutions of (1.3) of some special forms are presented. A maximum principle and some uniqueness results are given for the solution of (1.1) and (1.3). In the last section, the approximate solution of (1.3) with $g(t,x) = 1$ obtained by the difference method is examined.

2. Solutions of a special form

We give the solution of the parabolic partial differential equation (1.1) or (1.3) provided the solution is of a special form.

Example 1. Let us consider the solution of (1.3) of the form $u(t,x) = \exp(at + bx)$, where a and b are constants. In this case, equation (1.3) gives

$$p\,|b|^{p+1} - g(t,x)\,\Phi_p(a) = 0.$$

If $g(t,x) > 0$, then it is obvious that $a > 0$. If $g(t,x) = 1$, then

$$a = p^{\frac{1}{p}}\,|b|^{\frac{p+1}{p}} \quad \text{or} \quad b = \pm\left(\frac{a^p}{p}\right)^{\frac{1}{p+1}}$$

and the solution has the form

$$u(t,x) = \exp\left(at \pm \left(\frac{a^p}{p}\right)^{\frac{1}{p+1}} x\right)$$

or

$$u(t,x) = \exp\left(p^{\frac{1}{p}}\,|b|^{\frac{p+1}{p}}\,t + bx\right).$$

In the plane (t,x), the solution $u(t,x) = e^C$ is constant on the straight lines $at \pm \left(\frac{a^p}{p}\right)^{\frac{1}{p+1}} x = C$, $C = \text{const}$.

Example 2. Let us consider the solution of (1.3) of the form $u(t,x) = v(x)\tau(t)$ with $g(t,x) = r(x)s(t)$.

Substituting u into equation (1.3) we have

$$p\,|v'|^{p-1}v''\,\Phi_p(\tau) = r(x)s(t)\Phi_p(v)\Phi_p(\tau').$$

We suppose that $v(x) \neq 0$ and $\tau(t) \neq 0$. Separating the variables, one can get for v

$$p\,|v'|^{p-1}v'' - \mu\, r(x)\,\Phi_p(v) = 0, \qquad \mu = \text{const}, \tag{2.1}$$

and for τ

$$\frac{\tau'}{\tau} = \left|\frac{\mu}{s(t)}\right|^{\frac{1}{p}-1}\frac{\mu}{s(t)}. \tag{2.2}$$

For the solution of (2.1) we refer to [**1–3**]. The most important property of these solutions is that for any given initial condition at $t_0 \in I$,

$$x\,(t_0) = x_0,$$
$$x'\,(t_0) = x'_0,$$

there exists a unique solution $x(t)$ defined for all $t \in I$. If $\mu\, r(x) < 0$ (this yields that $\mu\, s(t) < 0$), then v is oscillatory.

From this we can see that the solution of (2.2) has the form

$$\tau(t) = K\,\exp\left(\int_0^t \Phi_{\frac{1}{p}}\left(\frac{\mu}{s(w)}\right)dw\right), \quad K = \text{const}.$$

Let us consider the special case where $\mu > 0$, $r(x) > 0$, $s(t) > 0$, and

$$\alpha(t) = 1 + t, \qquad t \in [0, T]\,,$$
$$v(0) = 1,$$
$$\tau(0) = 1,$$

then from this it follows that

$$s(t) = \mu\,(1+t)^p\,,$$
$$v(x) = \gamma(x),$$
$$\tau(t) = 1 + t,$$
$$\beta(t) = (1+t)\,\gamma(l)$$

and function γ ($x \in [0, l]$) satisfies equation (2.1). For the solution of differential equation (1.3) of the form $u(t,x) = v(x)\,\tau(t)$, the relation

$$u(t,x) = (1+t)\,\gamma(x)$$

holds.

Example 3. Let us consider the solution of (1.3) of the form $u(t,x) = v(x) + \tau(t)$ with $g(t,x) = r(x)\,s(t)$.

In this case, equation (1.3) gives

$$p\,|v'|^{p-1}v'' = r(x)s(t)\,\Phi_p(\tau).$$

Separating the variables, we obtain

$$p\,|v'|^{p-1}v'' = \kappa\, r(x), \qquad \kappa = \text{const}, \tag{2.3}$$

and

$$s(t)\,\Phi_p(\tau) = \kappa. \tag{2.4}$$

From equation (2.3), we get

$$v(x) = \Phi_p(\kappa)\int_0^x \Phi_{\frac{1}{p}}\left(\int_0^\eta r(\xi)\,d\xi\right)d\eta$$

and from (2.4),

$$\tau(t) = \Phi_p(\kappa)\int_0^t \Phi_p^{-1}(s(\tau))\,d\tau.$$

In the special case where $r(x) = s(t) = 1$ ($g(t,x) = 1$), we have

$$v(x) = \Phi_p(\kappa)\int_0^x \Phi_{\frac{1}{p}}(\eta)\,d\eta = \frac{p}{p+1}\Phi_p(\kappa)\,x^{\frac{p+1}{p}},$$
$$\tau(t) = \Phi_p(\kappa)\,t,$$

and, therefore the solution of (1.3) has the form

$$u(t,x) = \Phi_p(\kappa)\left[\frac{p}{p+1}x^{\frac{p+1}{p}} + t\right].$$

3. Results

Theorem 1. *Let us suppose that there exists a solution of* (1.1) *for* $k = 1$ *or* (1.3) *in the domain* $\mathscr{D}_0 = \{(t,x) : t \in [0,T),\ x \in [0,l]\}$ *with the boundary conditions* (1.5). *Then the solution assumes its maximum on* $\partial\mathscr{D}_0$*, the boundary of* $\mathscr{D}_0$.

Proof. We suppose the opposite that solution u assumes its maximum at an inner point (t^*, x^*) of $\mathscr{D}_0$ or on the line $t = T$. By this assumption,

$$u(t^*, x^*) - \max_{(t,x)\in\partial\mathscr{D}_0} u(t,x) = \delta > 0.$$

For the auxiliary function

$$w(t,x) = u(t,x) + \frac{\delta}{2}\frac{T-t}{T},$$

we have

$$u(t,x) < w(t,x) < u(t,x) + \frac{\delta}{2}.$$

For any point $(\bar{t},\bar{x})$ on $\mathscr{B} = \{(t,x) : t = 0,\ x = 0,\ x = l\}$, we see that

$$w(t^*,x^*) \ge u(t^*,x^*) = \max_{(t,x)\in\partial\mathscr{D}_0} u(t,x) + \delta \ge u(\bar{t},\bar{x}) + \delta$$
$$\ge w(\bar{t},\bar{x}) - \frac{\delta}{2}\frac{T-t}{T} + \delta \ge w(\bar{t},\bar{x}) + \frac{\delta}{2} > w(\bar{t},\bar{x}).$$

Therefore, $w(t,x)$ assumes its maximum at $(\tilde{t},\tilde{x}) \in \mathscr{D}_0 \setminus \mathscr{B}$. At this point,

$$w_x = 0, \quad w_{xx} \le 0, \quad w_t \ge 0,$$

which implies that

$$u_x = 0, \quad u_{xx} \le 0, \quad u_t = w_t + \frac{\delta}{2T} > 0. \tag{3.1}$$

From this observation, it follows that

$$u_{xx} - g(t,x)\ f(u_t,u_x) < 0 \quad \text{or} \quad (\Phi_p\,(u_x))_x - g(t,x)\,\Phi_p\,(u_t) < 0 \ \text{ at } \ (\tilde{t},\tilde{x}),$$

which is a contradiction.

We remark that $w_t > 0$ and also $u_t > 0$ at $\tilde{t} = T$. □

For the minimum of the solution of (1.1) or (1.3) in $\mathscr{D}_0$ we can formulate a similar statement, namely that $u(t,x)$ assumes its minimum on $\mathscr{B}$ in both cases.

Theorem 2. *There are no any two solutions u, v of* (1.3) *with $g(t,x) = 1$ such that $u = v$ on $\mathscr{B}$ and $u \ne v$, $u_x > v_x$ and $u_t > v_t$ in $\mathscr{D}_0$.*

Proof. We suppose that u and v are different solutions of the differential equation with $g(t,x) = 1$, then

$$(\Phi_p\,(u_x))_x - \Phi_p\,(u_t) = 0$$

and

$$(\Phi_p\,(v_x))_x - \Phi_p\,(v_t) = 0,$$

which gives that

$$(\Phi_p\,(u_x) - \Phi_p\,(v_x))_x - (\Phi_p\,(u_t) - \Phi_p\,(v_t)) = 0. \tag{3.2}$$

By the Lagrange Mean Value Theorem, there exists some $\xi \in (a,b)$ such that

$$\Phi_p(b) - \Phi_p(a) = (b-a)\,p\,|\xi|^{p-1}\,. \tag{3.3}$$

Let us introduce the notation $w = u - v$; for equation (3.2) we can write

$$(A(t,x)\,w_x)_x - B(t,x)\,w_t = 0, \tag{3.4}$$

where $A(t,x) = p\,|\xi_1|^{p-1}$ for $v_x < \xi_1 < u_x$ and $B(t,x) = p\,|\xi_2|^{p-1}$ for $v_t < \xi_2 < u_t$. Introducing the new variable ϑ instead of x, by

$$\vartheta = \int^x \frac{d\zeta}{A(t,\zeta)},$$

we get for $w(t,x) = \bar{w}(t,\vartheta)$ that $A(t,x)\,w_x = \bar{w}_\vartheta$ and equation (3.4) is transformed to

$$\bar{w}_{\vartheta\vartheta} - Q(t,\vartheta)\,\bar{w}_t = 0 \tag{3.5}$$

where $Q(t,\vartheta) = \overline{A}(t,\vartheta)\ \overline{B}(t,\vartheta)$, $A(t,x) = \overline{A}(t,\vartheta)$ and $B(t,x) = \overline{B}(t,\vartheta)$.

From the condition $u = v$ on $\mathscr{B}$, it follows that $\overline{w}(t,\vartheta) = 0$ on $\mathscr{B}$, which, by Theorem 1, implies that $\overline{w}(t,\vartheta) = 0$ and $u = v$ in $\mathscr{D}_0$. This contradicts the assumption on u and v. □

4. Application of the difference method

We shall use the difference method for the determination of the approximate solution of the parabolic differential equation

$$(\Phi_p(u_x))_x - \Phi_p(u_t) = 0 \tag{4.1}$$

with conditions (1.5) .

Let m be a positive integer and

$$\frac{l}{m} = h,$$
$$x_i = ih, \qquad i = 0, 1, 2, \ldots, m.$$

Obviously, $x_0 = 0$ and $x_m = l$.

Let us denote by $u_i(t)$ the solution of the first order system of ordinary difference-differential equations

$$\Phi_p\left(\frac{du_i}{dt}\right) = \frac{1}{h}\left[\Phi_p\left(\frac{u_{i+1}-u_i}{h}\right) - \Phi_p\left(\frac{u_i - u_{i-1}}{h}\right)\right]$$

or equivalently

$$\Phi_p\left(\frac{du_i}{dt}\right) = \frac{1}{h^{p+1}}\left[\Phi_p(u_{i+1}-u_i) - \Phi_p(u_i - u_{i-1})\right] \tag{4.2}$$

with the initial conditions

$$\begin{aligned} u_i(0) &= \gamma(x_i), \qquad i = 1, 2, \ldots, m-1,\\ u_0(t) &= \alpha(t),\\ u_m(t) &= \beta(t). \end{aligned} \tag{4.3}$$

Thus, system (4.2) involves $m-1$ equations with $m-1$ unknowns.

We intend to show that for an arbitrary T problem (4.2) (4.3) has a uniquely determined solution. We also show that $u(t,x_i)$ can be approximated by $u_i(t)$ with arbitrary accuracy, i. e., for every $\varepsilon > 0$, there exists an $h(\varepsilon) > 0$ such that $|u(t,x_i) - u_i(t)| < \varepsilon$ when $h < h(\varepsilon)$ in $0 \le t \le T$, $i = 1, 2, \ldots, m-1$.

Lemma 1. *For $p > 0$, the function $\overline{u}_i(t) = u(t,x_i)$ satisfies the system*

$$Q_i(t) = \Phi_p\left(\frac{d\overline{u}_i}{dt}\right) - \frac{1}{h^{p+1}}\left[\Phi_p(\overline{u}_{i+1}-\overline{u}_i) - \Phi_p(\overline{u}_i - \overline{u}_{i-1})\right], \quad i = 1, 2, \ldots, m-1$$

where $|Q_i(t)| \le Q_0(h)$, with $Q_0(h) \to 0$ as $h \to 0$.

PROOF. Applying the Taylor formula for $\overline{u}_{i+1}$ and $\overline{u}_{i-1}$

$$\overline{u}_{i+1}(t) = u(t, x_i + h) = u(t, x_i) + hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i + \theta_1 h),$$
$$\overline{u}_{i-1}(t) = u(t, x_i - h) = u(t, x_i) - hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i - \theta_2 h),$$

where $0 < \theta_1, \theta_2 < 1$ we have

$$\overline{u}_{i+1}(t) - \overline{u}_i(t) = hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i + \theta_1 h), \tag{4.4}$$
$$\overline{u}_i(t) - \overline{u}_{i-1}(t) = hu_x(t, x_i) - \frac{h^2}{2}u_{xx}(t, x_i - \theta_2 h). \tag{4.5}$$

We rewrite problem (4.1) by

$$\begin{aligned}\Phi_p\left((\overline{u}_i)_t\right) &= \left(\Phi_p\left(u_x\left(t, x_i\right)\right)\right)_x \\ &= p\left|u_x\left(t, x_i\right)\right|^{p-1} u_{xx}\left(t, x_i\right).\end{aligned}$$

From this, together with (4.4) and (4.5), it follows that

$$\begin{aligned}Q_i(t) = p\left|u_x\left(t, x_i\right)\right|^{p-1} u_{xx}\left(t, x_i\right) - \frac{1}{h^{p+1}}\Bigg[\Phi_p\left(hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i + \theta_1 h)\right) \\ - \Phi_p\left(hu_x(t, x_i) - \frac{h^2}{2}u_{xx}(t, x_i - \theta_2 h)\right)\Bigg].\end{aligned}$$

By the Lagrange Mean Value Theorem (3.3), we can write

$$\begin{aligned}\Phi_p\left(hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i + \theta_1 h)\right) - \Phi_p\left(hu_x(t, x_i) - \frac{h^2}{2}u_{xx}(t, x_i - \theta_2 h)\right) \\ = \frac{h^2}{2}\left[u_{xx}(t, x_i + \theta_1 h) + u_{xx}(t, x_i - \theta_2 h)\right] p\left|\xi\right|^{p-1},\end{aligned}$$

where

$$\xi = hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i + \theta_3 h), \quad -\theta_2 < \theta_3 < \theta_1.$$

Then we have

$$Q_i(t) = p\left|u_x\left(t, x_i\right)\right|^{p-1} u_{xx}\left(t, x_i\right) - \frac{p}{2}\left[u_{xx}(t, x_i + \theta_1 h) + u_{xx}(t, x_i - \theta_2 h)\right]\left|\eta\right|^{p-1},$$

where

$$\eta = u_x(t, x_i) + \frac{h}{2}u_{xx}(t, x_i + \theta_3 h) = u_x(t, x_i) + O(h), \qquad h \to 0,$$

and

$$
\begin{aligned}
Q_i(t) = {} & p\,|u_x\,(t,x_i)|^{p-1}\,u_{xx}\,(t,x_i) \\
& \quad - \frac{p}{2}\,[u_{xx}(t,x_i+\theta_1 h) + u_{xx}(t,x_i-\theta_2 h)]\,|u_x(t,x_i) + O(h)|^{p-1} \\
= {} & \frac{p}{2}\,|u_x\,(t,x_i)|^{p-1}\Big[u_{xx}\,(t,x_i) - u_{xx}(t,x_i+\theta_1 h) \\
& \quad + u_{xx}\,(t,x_i) - u_{xx}(t,x_i-\theta_2 h)\Big] + O(h^{p-1}).
\end{aligned}
$$

We note that $u_x\,(t,x_i)$ is bounded, $u_{xx}\,(t,x_i)$ is uniformly continuous in $\mathscr{D}_0$ and $p > 1$, then for any $\varepsilon > 0$ there exists a function $h(\varepsilon)$ such that $|Q_i(t)| < Q_0(h)$ when $h < h(\varepsilon)$, $0 \le t \le T$. □

Now our goal is to state a maximum principle for problem (4.2), (4.3) .

Theorem 3. *Consider a solution $u_i(t)$, $i = 1,2,\ldots,m-1$, of* (4.2), (4.3)*, where the function γ is positive, increasing, and convex. Then the maxima of $u_i(t)$, $i = 1,2,\ldots,m-1$, cannot be greater than the maxima of $\alpha(t)$, $\beta(t)$ and $h(x)$; moreover, the minima of $u_i(t)$, $i = 1,2,\ldots,m-1$, cannot be less than the minima of $\alpha(t)$, $\beta(t)$, and $\gamma(x)$.*

Proof. Let us suppose that there exists a $u_k(t^*)$ where $t^* > 0$ and $k \ne 0$, $k \ne m$ such that $\max_{i,t} u_i(t) = u_k(t^*)$. Then the following two cases are possible:

(i) At least one of the inequalities $u_{k+1} - u_k < 0$ and $u_k - u_{k-1} > 0$ holds, moreover $u_k'(t_0) \ge 0$ if $t^* \ne T$ and $u_k'(t^*) > 0$ if $t^* = T$. In (4.2) we have different signs on the left side and on the right side, which is a contradiction. By the convexity of function γ we see that

$$
\Phi_p\left(u_i'(O)\right) = \frac{1}{h^{p+1}}\left[\Phi_p\left(\gamma_{i+1} - \gamma_i\right) - \Phi_p\left(\gamma_i - \gamma_{i-1}\right)\right] > 0,
$$

then the maxima of $u_i(t)$ cannot be taken at $t^* = 0$. This means that $k = 0$ or $k = m$ can be taken for any $t^* \in (0,T]$.

(ii) At least one of equalities $u_{k+1} - u_k = 0$ and $u_k - u_{k-1} = 0$ holds. We assume that $u_{k+1} - u_k = 0$. Then stepping from index k to $k+1$, either we obtain a contradiction or we get

$$
u_k = u_{k+1} = \cdots = u_{m-1} = u_m = \beta\left(t^*\right),
$$

which we had to prove.

The proof concerning the minima is similar except that $t^* = 0$ is also allowed as $u_i'(0) > 0$. □

Now we consider the existence and uniqueness of a solution of problem (4.2), (4.3).

Theorem 4. *Let us suppose that function γ is continuous, increasing, convex and*

$$\min_{x,h}\Phi_{\frac{1}{p}}\left\{\frac{1}{h^{p+1}}\left[\Phi_p\left(\gamma(x+h)-\gamma(x)\right)-\Phi_p\left(\gamma(x)-\gamma(x-h)\right)\right]\right\}>L,$$

where L is independent of h, moreover, $u_i(0)=\gamma_i$ with $\gamma_i=\gamma(ih)$ for $i=1,2,\dots,m-1$, and $u_0(t)\equiv\alpha(t)$, $u_m(t)\equiv\beta(t)$, where $\alpha<\beta$, $\alpha'>0$, $\beta'>0$ for all $t\geq 0$. Then problem (4.2), (4.3) *has a uniquely determined solution for $0\leq t\leq T$, where T is positive.*

Proof. By the conditions on γ, we get that

$$\frac{du_i}{dt}=\Phi_{\frac{1}{p}}\left\{\frac{1}{h^{p+1}}\left[\Phi_p\left(u_{i+1}-u_i\right)-\Phi_p\left(u_i-u_{i-1}\right)\right]\right\} \tag{4.6}$$

are continuous and satisfy the Lipschitz condition for any $t<\tau$, with a small $\tau>0$. This implies that the solution exists and is unique for $t<\tau$. The conditions on γ also gives that

$$\frac{du_i}{dt}(0)=\Phi_{\frac{1}{p}}\left\{\frac{1}{h^{p+1}}\left[\Phi_p\left(\gamma_{i+1}-\gamma_i\right)-\Phi_p\left(\gamma_i-\gamma_{i-1}\right)\right]\right\}>L,$$

then

$$u_i'(t)>0 \text{ for } t<\tau.$$

Since $u_i(0)=\gamma(x_i)>0$, $i=1,2,\dots,m-1$, then either $u_i'(t)$ remains positive for $t>0$ or there is a smallest value τ_h for which $u_k'(\tau_h)=0$ for some $k=i$.

Taking the derivative of (4.6) we obtain

$$|u_k'|^{p-1}u_k''=\frac{1}{h^{p+1}}\left\{|\Delta u_{k+1}|^{p-1}\left(u_{k+1}'-u_k'\right)-|\Delta u_k|^{p-1}\left(u_k'-u_{k-1}'\right)\right\}, \tag{4.7}$$

where $\Delta u_k=u_k-u_{k-1}$. At $t=\tau_h$ we have that $\Delta u_k=\Delta u_{k+1}$. For small $\varepsilon>0$, in interval $(\tau_h-\varepsilon,\tau_h)$ we obtain that $|\Delta u_k|>0$, $|\Delta u_{k+1}|>0$, $u_k'=o(1)$, $u_{k+1}'-u_k'>0$, $u_k'-u_{k-1}'<0$. From these it follows that in (4.7) the right side is positive while the left side is negative as $u_k''<0$ for $t\in(\tau_h-\varepsilon,\tau_h)$.

In the case $u_{k+1}'(\tau_h)=0$, passing from k to $k+1$ and carrying on, we obtain

$$u_{m-1}'=u_m'=\beta'(\tau_h)=0,$$

which is a contradiction. From the argument above, it follows that such a finite τ_h does not exist.

Consequently, we have that

$$\Delta u_1<\Delta u_2<\cdots<\Delta u_{m-1}<\Delta u_m.$$

□

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