The Cauchy-Schwarz inequality in Cayley graph and tournament structures on finite fields

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THE CAUCHY-SCHWARZ INEQUALITY IN CAYLEY GRAPH AND TOURNAMENT STRUCTURES ON FINITE FIELDS

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Abstract. The Cayley graph construction provides a natural grid structure on a finite vector space over a field of prime or prime square cardinality, where the characteristic is congruent to 3 modulo 4, in addition to the quadratic residue tournament structure on the prime subfield. Distance from the null vector in the grid graph defines a Manhattan norm. The Hermitian inner product on these spaces over finite fields behaves in some respects similarly to the real and complex case. An analogue of the Cauchy-Schwarz inequality is valid with respect to the Manhattan norm. With respect to the non-transitive order provided by the quadratic residue tournament, an analogue of the Cauchy-Schwarz inequality holds in arbitrarily large neighborhoods of the null vector, when the characteristic is an appropriate large prime.

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1. MANHATTAN NORMS AND GRID GRAPHS

We consider the finite fields $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ of prime and prime square cardinality, where $p \equiv 3 \mod 4$. The field $\mathbb{F}_{p^2}$ has a natural graph structure with the field elements as vertices, two distinct vertices $u, z$ being adjacent if $(z - u)^4 = 1$. The subfield $\mathbb{F}_p$ of $\mathbb{F}_{p^2}$ then induces a subgraph in which $x$ and $y$ are adjacent if and only if $(y - x)^2 = 1$. The graph $\mathbb{F}_{p^2}$ is isomorphic to the Cartesian square $C_p^2 = C_p \square C_p$, where $C_p$ is a $p$-cycle and within $\mathbb{F}_{p^2}$ the induced subgraph $\mathbb{F}_p$ is itself a $p$-cycle. Clearly the graph $\mathbb{F}_{p^2}$ is not planar, but can be drawn as a grid on the torus.

For any connected graph whose vertex set is a group, the distance of any vertex $z$ from the identity element of the group is called the norm of $z$, denoted $N(z)$. In general, distances and norms measured in connected subgraphs induced by subgroups can be larger than distances and norms measured with reference to the whole graph. However, with respect to the distance-preserving subgraph induced by $\mathbb{F}_p$ in $\mathbb{F}_{p^2}$, the norm of any $z \in \mathbb{F}_p$ is the same as its norm with respect to the whole graph $\mathbb{F}_{p^2}$: this is simply the length of the shortest path from 0 to $z$ in the cycle induced by $\mathbb{F}_p$.
For \( q = p \) or \( q = p^2 \), the \( n \)-dimensional vector space \( \mathbb{F}_q^n \) is also endowed with the Cartesian product graph structure \( \mathbb{F}_q \square \cdots \square \mathbb{F}_q \) isomorphic to \( C_p^n \) or \( C_{p^2}^{2n} \). The norm of a vector \( \mathbf{v} = (v_1, \ldots, v_n) \) in \( \mathbb{F}_q^n \) is then equal to the sum \( N(v_1) + \cdots + N(v_n) \) and we also write \( N(\mathbf{v}) \) for this vector norm.

The Gaussian integers \( \mathbb{Z}[i] \) also constitute a graph in which \( u \) and \( v \) are adjacent if and only if \( (z - u)^2 = 1 \).

It is easy to see that the norm in this infinite Manhattan grid satisfies the triangle and submultiplicative inequalities

\[
N(u + z) \leq N(u) + N(z)
\]

\[
N(uz) \leq N(u)N(z)
\]

To emphasize that the norms on \( \mathbb{F}_{p^2}, \mathbb{F}_{p^2}^n \) and \( \mathbb{Z}[i] \) are understood with reference to the specific grid graphs defined above, we call these norms Manhattan norms. Throughout this paper we think of \( \mathbb{F}_{p^2} \) as the ring quotient \( \mathbb{Z}[i]/(p) \).

### 2. Graph Quotients and Cayley Graphs

Given a graph \( G \) (undirected, with possible loops) on vertex set \( V \) and an equivalence relation \( \equiv \) on \( V \), the quotient graph \( G/\equiv \) is defined as follows: the vertices of \( G/\equiv \) are the equivalence classes of \( \equiv \), and classes \( A, B \) are adjacent if for some \( a \in A, b \in B \), the elements \( a, b \) are adjacent in \( G \). Note that the distance of \( A \) to \( B \) in the quotient graph is at most equal to, but possibly less than the minimum of the distances \( a \) to \( b \) for all \( a \in A, b \in B \). Note also that \( G/\equiv \) can have loops even if \( G \) has not.

Given a group \( G \) with identity element \( e \) and a set \( \Gamma \) of group elements that generates \( G \), the (left) Cayley graph \( \mathcal{C}(G, \Gamma) \) of \( G \) with respect to \( \Gamma \) has vertex set \( G \), elements \( a, b \in G \) being considered adjacent if \( ab^{-1} \) or \( ba^{-1} \) belongs to \( \Gamma \). For each congruence \( \equiv \) of the group \( G \), corresponding to some normal subgroup \( H \), \( \Gamma \) yields a generating set \( \Gamma_\equiv \) of \( G/\equiv \) consisting with those classes of \( \equiv \) that intersect \( \Gamma \). The graph quotient of \( \mathcal{C}(G, \Gamma) \) by the equivalence \( \equiv \) coincides with the Cayley graph of the quotient graph \( G/\equiv \) with respect to \( \Gamma_\equiv \). For \( R \subseteq G \) inducing a connected subgraph \([R]\) in \( \mathcal{C}(G, \Gamma) \), denote by \( d_R(x, y) \) the distance function of the subgraph \([R]\). Denoting by \( xH \) the \( H \)-coset of any \( x \in G \), this relates to norms in \( \mathcal{C}(G, \Gamma) \) and \( \mathcal{C}(G, \Gamma)/\equiv \) as follows: for all \( x \in R \),

\[
d_R(x, e) \geq N(x) \geq N(xH)
\]

Both inequalities can be strict. However, we have:

**Cayley Graph Quotient Lemma.** Let a group \( G \) with identity \( e \) be generated by \( \Gamma \subseteq G \), and consider any normal subgroup \( H \) with corresponding congruence \( \equiv \). There is a set \( R \subseteq G \) having exactly one element in common with each congruence class.
class modulo $H$, and such that for every $x \in R$

$$d_R(x,e) = N(x) = N(xH)$$

Proof. We can define the unique (representative) element $r(A) \in R \cap A$ for each coset $A$ by induction on the distance $d(H,A)$ of $A$ from $H$ in $\mathcal{C}(G, \Gamma)/\equiv$. Let $r(H) = e$. Assuming $r(A)$ defined for all $A$ with $d(H,A) \leq m$, let a coset $B$ have distance $m + 1$ from $H$. Choose any coset $A$ adjacent to $B$ with $d(H,A) = m$ and elements $a \in A, b \in B$ that are adjacent in $\mathcal{C}(G, \Gamma)$. Let $r(B) = ba^{-1}r(A)$. □

We can apply the above lemma in the case where $G = \mathbb{Z}[i]$, $\Gamma = \{1,i\}$ and $H = p\mathbb{Z}[i] = \{pa + phi : a, b \in \mathbb{Z}\}$ for a prime integer $p \equiv 3 \pmod{4}$. Now $\mathcal{C}(G, \Gamma)$ and $\mathcal{C}(G, \Gamma)/\equiv$ are the Manhattan grid graphs on $\mathbb{Z}[i]$ and $\mathbb{Z}[i]/H = \mathbb{F}_p$, respectively. Referring to the set $R$ of representatives in the lemma, for any $H$-cosets $X,Y$ let $x,y$ be the unique elements in $X \cap R, Y \cap R$. As $xy \in XY$, we have $N(XY) \leq N(xy)$. By the submultiplicative inequality in $\mathbb{Z}[i]$ we have $N(xy) \leq N(x)N(y)$. Using the lemma we have $N(x)N(y) = N(X)N(Y)$. This yields a submultiplicative inequality in $\mathbb{F}_p$ and a similar reasoning on the coset $X + Y$ yields a triangle inequality:

**Triangle and Submultiplicative Inequalities in $\mathbb{F}_p$.** For all $u, z$ in $\mathbb{F}_p$

$$N(u + z) \leq N(u) + N(z)$$

$$N(uz) \leq N(u)N(z)$$

This indicates that Manhattan distance provides a well-behaved notion of neighborhood of 0 in the finite fields $\mathbb{F}_p$.

3. Squares in $\mathbb{F}_p$ and non-transitive order

For each prime $p \equiv 3 \pmod{4}$ the quadratic residue tournament on $\mathbb{F}_p$ is the directed graph with vertex set $\mathbb{F}_p$ in which there is an arrow from vertex $x$ to vertex $y$ if $y - x$ is a non-zero square in $\mathbb{F}_p$, in which case we write $x <_p y$. We write $x \leq_p y$ if $x <_p y$ or $x = y$. The relation $\leq_p$ is reflexive, anti-symmetric but not transitive, and for every $x \neq y$ exactly one of $x \leq_p y$ or $y \leq_p x$ holds. Using Dirichlet’s theorem on primes in arithmetic progressions, Kustaaheimo showed [4] that for every positive integer $k$, there is a prime $p \equiv 3 \pmod{4}$, such that $\leq_p$ is a transitive (and linear) order relation on $\{0,1,\ldots,k\} \subseteq \mathbb{F}_p$, that is, all positive integers up to $k$ are quadratic residues mod $p$. Obviously $k$ cannot exceed $(p - 1)/2$. Implications of [4] and related questions were investigated by Järnefelt, Kustaaheimo, Quist [3, 5], in particular with a view to discrete models in physics, also in subsequent application-oriented work between the 1950’s (Coish [1]) and the 1980’s (Nambu [6]). For further references see [2]. In particular [4] implies that for every positive integer $k$, there is a prime $p \equiv 3 \pmod{4}$, such that all $z \in \mathbb{F}_{p^2}$ with $N(z) \leq k$ are squares in $\mathbb{F}_{p^2}$. (Note that all elements of the prime subfield $\mathbb{F}_p$ are squares in $\mathbb{F}_{p^2}$.) To emphasise the analogy of the relation $\leq_p$ with the ordinary inequality relation $\leq$ among numbers, we say that a non-zero $z \in \mathbb{F}_{p^2}$ is positive if $z \in \mathbb{F}_p$ and $0 \leq_p z$. 

4. Inner Products Compared in Non-Transitive Order

The only non-trivial automorphism of the field $\mathbb{F}_{p^2}$ associates to each $z \in \mathbb{F}_{p^2}$ its conjugate $\overline{z}$. The inner product $\mathbf{v} \cdot \mathbf{w}$ of vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ in $\mathbb{F}_{p^2}^n$ is defined as the scalar $v_1 \overline{w_1} + \cdots + v_n \overline{w_n} \in \mathbb{F}_{p^2}$. This inner product is left and right distributive over vector addition, satisfies $\mathbf{v} \cdot \mathbf{w} = \overline{\mathbf{w}} \cdot \mathbf{v}$, $c(\mathbf{v} \cdot \mathbf{w}) = (c \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c \mathbf{w})$ for all $c \in \mathbb{F}_{p^2}$. However, while $\mathbf{v} \cdot \mathbf{v}$ belongs to the prime subfield $\mathbb{F}_p$, $\mathbf{v} \cdot \mathbf{v}$ is not necessarily positive, and can be even if $\mathbf{v} \neq \mathbf{0}$. Still, a conditional version of positive definiteness holds locally:

**Theorem 1.** For every $k \geq 1$ there is a prime $p \equiv 3 \mod 4$, such that for all $n \geq 1$ and for all vectors $\mathbf{v} \in \mathbb{F}_{p^2}^n$ of Manhattan norm $N(\mathbf{v}) \leq k$, we have $0 \leq_{\mathbb{F}_p} \mathbf{v} \cdot \mathbf{v}$ with equality if and only if $\mathbf{v} = \mathbf{0}$.

**Proof.** By Kustaanheimo’s result in [4] there is a prime integer $p \equiv 3 \mod 4$ such that $0, 1, \ldots, 2k^3$ are all quadratic residues mod $p$. For $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{F}_{p^2}^n$, let $v_j = a_j + b_j i$, where $i^2 = -1$. If $N(\mathbf{v}) \leq k$ then for all $j$, $N(a_j) \leq k$ and $N(b_j) \leq k$, $v_j \overline{v_j} = a_j^2 + b_j^2$ belongs to the set of squares $\{0, \ldots, 2k^2\}$. Since $v_j$ can be non-zero for at most $k$ indices $1 \leq j \leq n$ only, the sum of the corresponding terms $a_j^2 + b_j^2$ belongs to the set of squares $\{0, 1, \ldots, 2k^2\}$. \(\square\)

Note that for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{F}_{p^2}^n$,

$$(\mathbf{v} \cdot \mathbf{w})(\mathbf{w} \cdot \mathbf{v}) = (\mathbf{v} \cdot \mathbf{w})(\overline{\mathbf{v}} \cdot \overline{\mathbf{w}}) \in \mathbb{F}_p$$

and

$$(\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) \in \mathbb{F}_p.$$

If $\mathbf{v}$ and $\mathbf{w}$ are proportional, i.e. if there exists a scalar $c \in \mathbb{F}_{p^2}$ such that $\mathbf{v} = c \mathbf{w}$ or $\mathbf{w} = c \mathbf{v}$, then the above two products are equal. Generally, they are related in the quadratic residue tournament of $\mathbb{F}_p$ as follows.

**Theorem 2 (Cauchy-Schwarz Inequality).** For every $k \geq 1$ there is a prime $p \equiv 3 \mod 4$, such that for all $n \geq 1$ and for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{F}_{p^2}^n$ of Manhattan norm at most $k$,

$$(\mathbf{v} \cdot \mathbf{w})(\mathbf{w} \cdot \mathbf{v}) \leq_{\mathbb{F}_p} (\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}).$$

**Proof.** For $n = 1$ the inequality holds trivially as the two sides are equal. Assume $n \geq 2$, $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{w} = (w_1, \ldots, w_n)$. For all $1 \leq i \leq n$, $N(v_i) \leq k$, $N(w_i) \leq k$. By Kustaanheimo’s result [4] there is a prime $p \equiv 3 \mod 4$ such that all positive integers up to $4k^6$ are quadratic residues modulo $p$. For each of the $\binom{n}{2}$ pairs $\{i, j\} \subseteq \{1, \ldots, n\}$, $i \neq j$, by the triangle and submultiplicative inequalities in $\mathbb{F}_{p^2}$

$$N((v_i w_j - v_j w_i)(\overline{v_i w_j} - \overline{v_j w_i})) \leq (k^2 + k^2)^2 = 4k^4$$

Thus the element

$$(v_i w_j \overline{v_i w_j} + v_j w_i \overline{v_j w_i}) - (v_i w_j \overline{v_j w_i} + v_j w_i \overline{v_i w_i}) = (v_i w_j - v_j w_i)(\overline{v_i w_j} - \overline{v_j w_i})$$
is a square of Manhattan norm at most $4k^4$ in $\mathbb{F}_p$, and it is non-zero for at most \( \binom{k}{2} \leq k^2 \) pairs \( \{i, j\} \). Summing over all pairs \( \{i, j\} \), all but at most \( \binom{k}{2} \leq k^2 \) terms vanish in the sum
\[
\sum [(v_i w_j \overline{v}_i \overline{w}_j + v_j w_i \overline{v}_j \overline{w}_i) - (v_i w_j \overline{v}_j \overline{w}_i + v_j w_i \overline{v}_i \overline{w}_j)]
\]
which therefore has Manhattan norm at most $4k^6$ and it must also be a square in $\mathbb{F}_p$.

But this sum is equal to the difference of products
\[
\sum_{i=1}^{n} v_i \overline{v}_i \sum_{j=1}^{n} w_j \overline{w}_j - \sum_{i=1}^{n} v_i \overline{v}_i \sum_{j=1}^{n} \overline{v}_j w_j = (v \cdot v)(w \cdot w) - (v \cdot w)(w \cdot v)
\]
which is consequently a square in $\mathbb{F}_p$.

\[\square\]

**Remark.** From the proof it is clear that, in analogy with the classical Cauchy-Schwarz inequality, for vectors $v, w$ of norm not exceeding $k$ in $\mathbb{F}_p^n$, where $p$ is related to $k$ as stipulated above, the Cauchy-Schwarz inequality with respect to $\leq_p$ holds with equality if and only if $v_i w_j - v_j w_i = 0$ for all $i, j$, i.e. if and only if $v, w$ are proportional.

We note that the inequality established above is conditional, it holds only in a specified Manhattan neighborhood of the null vector. Every non-zero element of $\mathbb{F}_p$ can be written as a sum of two squares, in particular there are $a, b \in \mathbb{F}_p$ such that $a^2 + b^2 = -1$. For $z = a + bi$ we have $z \overline{z} = -1$. As soon as $n \geq 2$, in $\mathbb{F}_p^n$ let
\[
v = (a, b, 0, \ldots, 0) \quad \text{and} \quad w = (b \overline{z}, -a \overline{z}, 0, \ldots, 0)
\]
The inequality $(v \cdot w)(w \cdot v) \leq_p (v \cdot v)(w \cdot w)$ fails because the left-hand side is 0 and the right-hand side is $-1$. In fact if $n \geq 3$, the inequality can be invalidated with vectors $v, w$ in $\mathbb{F}_p^n$ as follows. Taking again $a, b \in \mathbb{F}_p$ with $a^2 + b^2 = -1$, let
\[
v = (1, a, b, 0, \ldots, 0) \quad \text{and} \quad w = (1, 0, 0, 0, \ldots, 0)
\]
However, the Cauchy-Schwarz inequality holds unconditionally in the 2-dimensional case for vectors with components in $\mathbb{F}_p$:

**Special case of $\mathbb{F}_p^2$.** Let $p$ be a prime congruent 3 modulo 4. For all vectors $v, w$ in $\mathbb{F}_p^2$
\[
(v \cdot w)(w \cdot v) \leq_p (v \cdot v)(w \cdot w).
\]

**Proof.** Now the conjugation appearing in the inner products is the identity. Written in components,
\[
(v \cdot v)(w \cdot w) - (v \cdot w)(w \cdot v) = (v_1^2 + v_2^2)(w_1^2 + w_2^2) - (v_1 w_1 + v_2 w_2)^2 = \]
\[
v_1^2 w_2^2 + v_2^2 w_1^2 - 2v_1 w_1 v_2 w_2 = (v_1 w_2 - v_2 w_1)^2.
\]

\[\square\]
5. MANHATTAN NORM OF INNER PRODUCT

The Manhattan norm can be seen to be submultiplicative not only on the ring $\mathbb{Z}[i]$ and its quotient field $\mathbb{F}_{p^2}$, but on all vector spaces $\mathbb{F}_{p^2}^n$, with respect to the inner product:

**Cauchy-Schwarz Inequality for Manhattan Norm on $\mathbb{F}_{p^2}^n$.** Consider any prime $p \equiv 3 \mod 4$ and let $n \geq 1$. For all $v, w \in \mathbb{F}_{p^2}^n$

$$N(v \cdot w) \leq N(v)N(w).$$

**Proof.** Let $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{F}_{p^2}^n$. Then $v \cdot w = \sum v_j w_j$. Clearly $N(z) = N(\overline{z})$ for any $z \in \mathbb{F}_{p^2}$. By the triangle and submultiplicative inequalities in $\mathbb{F}_{p^2}$ we have

$$N(v \cdot w) = N\left( \sum v_j w_j \right) \leq \sum N(v_j w_j) \leq \sum N(v_j)N(w_j) \leq \sum N(v_j)\sum N(w_j) = N(v)N(w)$$

□

**Remark.** The inequality $N(v \cdot w) \leq N(v)N(w)$ is easily interpreted and continues to hold for $v, w$ in the module $(\mathbb{Z}[i]/m\mathbb{Z}[i])^n$ for any positive integer $m$. As soon as $m$ is composite, or a prime not congruent to 3 modulo 4, the ring $\mathbb{Z}[i]/m\mathbb{Z}[i]$ fails to be an integral domain.

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