CONVERGENCE AND SUBSEQUENTIAL CONVERGENCE OF
REGULARLY GENERATED SEQUENCES

SEFA ANIL SEZER AND İBRAHIM ÇANAK

Received 03 November, 2013

Abstract. In this paper we recover convergence and subsequential convergence of a sequence
of real numbers regularly generated by another sequence in some sequence spaces under certain
conditions. We also give some information about the behavior of a sequence whose generator is
given in terms of a moderately divergent sequence.

2010 Mathematics Subject Classification: 40A05; 40E05
Keywords: regularly generated sequences, slow oscillation, moderate oscillation, moderate di-
vergence, subsequential convergence, weighted means

1. INTRODUCTION

Throughout this paper, \( \mathbb{N}_0 \) will denote the set of all nonnegative integers. Let
\( u = (u_n) \) be a sequence of real numbers and any term with a negative index be zero.
Let \( p = (p_n) \) be a sequence of nonnegative numbers such that \( p_0 > 0 \) and
\[
P_n := \sum_{k=0}^{n} p_k \to \infty, \quad n \to \infty.
\]
(1.1)
The \( n \)th weighted mean of the sequence \( (u_n) \) is defined by
\[
\sigma_{n,p}(u) := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k
\]
(1.2)
for all \( n \in \mathbb{N}_0 \).

The sequence \( (u_n) \) is said to be summable by the weighted mean method determined by the sequence \( p \); in short, \( (\mathbb{N}, p) \) summable to a finite number \( s \) if
\[
\lim_{n \to \infty} \sigma_{n,p}(u) = s.
\]
The difference between \( u_n \) and its \( n \)th weighted mean \( \sigma_{n,p}(u) \), which is called the
weighted Kronecker identity, is given by
\[
u_n - \sigma_{n,p}(u) = V_{n,p} (\Delta u),
\]
(1.3)
where
\[ V_{n,p}(\Delta u) := \frac{1}{P_n} \sum_{k=0}^{n} P_{k-1} \Delta u_k \]  
(1.4)
and
\[ \Delta u_n = u_n - u_{n-1}. \]  
(1.5)

The \((\mathcal{N}, p)\) summability method is regular if and only if \(P_n \to 1\) as \(n \to \infty\). If \(p_n = 1\) for all \(n \in \mathbb{N}_0\), then \((\mathcal{N}, p)\) summability method reduces to Cesàro summability method.

A sequence \((u_n)\) is slowly oscillating \([14]\) if
\[
\lim_{\lambda \to 1^+} \limsup_{n \to \infty} \max_{n+1 \leq k \leq \lfloor \lambda n \rfloor} |u_k - u_n| = 0, 
\]  
(1.6)
where \(\lfloor \lambda n \rfloor\) denotes the integer part of \(\lambda n\).

The space of all slowly oscillating sequences is denoted by \(\mathcal{SO}\). Dik \([9]\) proved that if a sequence \((u_n)\) is slowly oscillating, then \((V_{n,1}(\Delta u))\) is bounded and slowly oscillating.

A generalization of slow oscillation is given as follows.

A sequence \((u_n)\) is moderately oscillating \([14]\) if
\[
\limsup_{n \to \infty} \max_{n+1 \leq k \leq \lfloor \lambda n \rfloor} |u_k - u_n| < \infty \]  
(1.7)
for \(\lambda > 1\). The space of all moderately oscillating sequences is denoted by \(\mathcal{MO}\).

Set
\[ t_n = n P_n / P_{n-1}, \]  
(1.8)
for \(n \in \mathbb{N}_0\). We say that \((u_n)\) is regularly generated by a sequence \(\alpha = (\alpha_n)\) in some sequence space \(\mathcal{A}\) and \(\alpha\) is called a generator of \((u_n)\) if
\[ u_n = \alpha_n + \sum_{k=1}^{n} \frac{l_k}{k} \alpha_k. \]  
(1.9)

The space of all sequences which are regularly generated by sequences in \(\mathcal{A}\) is denoted by \(U(\mathcal{A})\).

If \((u_n)\) is regularly generated by a sequence \((\alpha_n)\) where \((\Delta \alpha_n) \in \mathcal{SO}\), we write \((u_n) \in U(\mathcal{SO}_\Delta)\). If \((u_n)\) is regularly generated by a sequence \((\alpha_n)\) where \((\alpha_n) \in \mathcal{SO}\), we write \((u_n) \in U(\mathcal{SO})\).

A positive sequence \((u_n)\) is O-regularly varying \([12]\) if
\[
\limsup_{n \to \infty} \frac{u[\lambda n]}{u_n} < \infty \]  
(1.10)
for \(\lambda > 1\) and it is slowly varying if
\[
\lim_{n \to \infty} \frac{u[\lambda n]}{u_n} = 1. \]  
(1.11)
It was proved by [11] that if a positive sequence \((u_n)\) is \(O\)-regularly varying, then \((\log u_n)\) is slowly varying.

A positive sequence \((u_n)\) is moderately divergent [13] if for every \(\lambda > 1\)
\[
 u_n = o(n^{\lambda-1}), \ n \to \infty
\]  
and
\[
 \sum_{n=1}^{\infty} \frac{u_n}{n^\lambda} < \infty.
\]

We denote the space of all moderately divergent sequences by \(MD\). Note that every slowly oscillating sequence of positive numbers is moderately divergent.

The convergence of a sequence \((u_n)\) implies that \((u_n)\) is bounded and \(\Delta u_n = o(1)\) as \(n \to \infty\). But it is clear that the converse of this implication is not true in general. In the case where \((u_n)\) is bounded with \(\Delta u_n = o(1)\) as \(n \to \infty\), we may not recover convergence of \((u_n)\) but we may have convergence of some subsequences of \((u_n)\). A new kind of convergence is defined as follows (See [8] for more details on subsequentially convergent sequences):

A sequence \(u = (u_n)\) is said to be subsequentially convergent if there exists a finite interval \(I(u)\) such that all accumulation points of \((u_n)\) are in \(I(u)\) and every point of \(I(u)\) is an accumulation point of \((u_n)\).

Recently, several results in terms of regularly generated sequences for different purposes have been obtained by Dik et al. [10], Çanak et al. [1], Çanak and Totur [3], Çanak et al. [2], Çanak et al. [7], Çanak and Totur [5] and many more. In this paper, we first recover convergence and subsequential convergence of a sequence which is regularly generated by another sequence in some sequence spaces under certain conditions. Secondly, we give some information about the behavior of a sequence whose generator is given in terms of a moderately divergent sequence.

2. The preliminary results

We need the following lemmas for the proof of our results.

**Lemma 1** ([8]). Let \((u_n)\) be a bounded sequence of real numbers. If \(\Delta u_n = o(1)\) as \(n \to \infty\), then \((u_n)\) converges subsequentially.

**Lemma 2.** If \((\sum_{k=1}^{n} t_k \alpha_k)\) is moderately oscillating, then \((\sum_{k=1}^{\lceil \lambda n \rceil} t_k \alpha_k)\) converges.

**Proof.** Set \(R_n := \exp(\sum_{k=1}^{n} t_k \alpha_k)\). Then we have
\[
 \frac{\frac{R_{[\lambda n]}}{R_n}}{\exp\left(\sum_{k=[n+1]}^{[\lambda n]} t_k \alpha_k\right)}.
\]  

(2.1)
Taking limsup of both sides of (2.1) as \( n \to \infty \) gives

\[
\limsup_{n \to \infty} \frac{R_{[\lambda n]}}{R_n} \leq \exp \left( \limsup_{n \to \infty} \left| \sum_{\lambda k + 1} t_k \alpha_k \right| \right).
\]

(2.2)

Since \( \sum_{k=1}^n t_k \alpha_k \) is moderately oscillating, we have

\[
\limsup_{n \to \infty} \frac{R_{[\lambda n]}}{R_n}
\]

is finite for \( \lambda > 1 \). This says that \( (R_n) \) is O-regularly varying. Since \( (R_n) \) is O-regularly varying, \( (\log R_n) \) is slowly varying. It follows that

\[
\sum_{n=1}^{\infty} \frac{1}{np} \left| \sum_{k=1}^{n} t_k \alpha_k \right| < \infty
\]

for \( p > 1 \). This implies that

\[
\sum_{n=1}^{\infty} \frac{t_n}{n} \alpha_n < \infty.
\]

(2.5)

**Lemma 3.** If \( \sum_{k=1}^n t_k \alpha_k \) converges, then \( \sigma_{n,p}(\alpha) = o(1) \), \( n \to \infty \).

**Proof.** Set \( \gamma_n := \sum_{k=1}^n t_k \alpha_k \). Then we obtain

\[
\alpha_n = \frac{P_{n-1}}{p_n} \Delta \gamma_n
\]

(2.6)

and

\[
\sigma_{n,p}(\alpha) = V_{n,p}(\Delta \gamma)
\]

(2.7)

for \( n \in \mathbb{N}_0 \). Since \( (\gamma_n) \) converges, it follows by the weighted Kronecker identity

\[
\gamma_n - \sigma_{n,p}(\gamma) = V_{n,p}(\Delta \gamma)
\]

(2.8)

that

\[
V_{n,p}(\Delta \gamma) = o(1), \quad n \to \infty.
\]

This completes the proof. \( \Box \)

**Lemma 4 ([16]).** Let \( (p_n) \) satisfy the condition

\[
1 \leq \frac{p_n}{n} \to 1, \quad n \to \infty.
\]

(2.9)

If \( (u_n) \) is slowly oscillating, then \( (V_{n,p}(\Delta u)) \) is slowly oscillating and bounded.

**Lemma 5 ([15]).** Let \( (u_n) \) be Cesàro summable to \( s \). If \( (u_n) \) is slowly oscillating, then \( (u_n) \) converges to \( s \).
3. The Main Results

**Theorem 1.** Suppose that

\[
\left( \sum_{k=1}^{n} t_k \alpha_k \right) \in \mathcal{M} \Theta,
\]

\[1 \leq \frac{P_n}{n} \rightarrow 1, \quad n \rightarrow \infty, \quad (3.1)\]

\[t_n = O(1), n \rightarrow \infty. \quad (3.2)\]

If \((u_n) \in U(\Theta \Delta)\), then \((u_n)\) converges subsequentially.

**Proof.** Since \((u_n) \in U(\Theta \Delta)\), \((u_n)\) can be written as

\[u_n = \alpha_n + \sum_{k=1}^{n} \frac{t_k}{k} \alpha_k, \quad (3.4)\]

where \((\Delta \alpha_n) \in \Theta \Theta\). Moderate oscillation of \((\sum_{k=1}^{n} t_k \alpha_k)\) implies convergence of \((\gamma_n) = (\sum_{k=1}^{n} \frac{t_k}{k} \alpha_k)\) by Lemma 2 and \(\sigma_{n,p}(\alpha) = o(1)\) as \(n \rightarrow \infty\) by Lemma 3. Hence, \((u_n)\) is \((N,p)\) summable to the limit of \((\gamma_n)\). By the condition (3.3), it follows that

\[\frac{\alpha_n}{n} \rightarrow 0, n \rightarrow \infty \quad (3.5)\]

by Lemma 3. Since \((\Delta \alpha_n) \in \Theta \Theta\), we have that

\[\Delta \alpha_n \rightarrow 0, n \rightarrow \infty \quad (3.6)\]

by Lemma 5. Taking the backward difference of (3.4), we have

\[\Delta u_n = \Delta \alpha_n + \alpha_n \frac{P_n}{P_{n-1}} \quad (3.7)\]

for \(n \in \mathbb{N}_0\).

It follows by (3.3), (3.5) and (3.6) that

\[\Delta u_n = o(1), n \rightarrow \infty. \quad (3.8)\]

To complete the proof, it suffices to prove that \((u_n)\) is bounded. Applying Lemma 4 to \((v_n) = (\sum_{k=1}^{n} \alpha_k t_k)\), and taking \((v_n) \in \Theta \Theta\) into account, we obtain \((V_{n,p}(\alpha t))\) is bounded and slowly oscillating, where \(\alpha t = (\alpha_n t_n)\).

From the weighted Kronecker identity

\[S_n(\alpha) - \sigma_{n,p}(S(\alpha)) = V_{n,p}(\alpha) \quad (3.9)\]

where \(S(\alpha) = (S_n(\alpha)) = (\sum_{k=0}^{n} \alpha_k)\), we have

\[\alpha_n - \frac{P_n}{P_{n-1}} V_{n,p}(\alpha) = \Delta V_{n,p}(\alpha). \quad (3.10)\]
Replacing $\alpha_n$ by $\alpha_n t_n$ in (3.10) and then dividing by $t_n$, we have

$$\alpha_n = \frac{V_{n,p}(\alpha t)}{n} + \frac{P_{n-1}}{np_n} \Delta V_{n,p}(\alpha t).$$

(3.11)

It follows from (3.11) that $(\alpha_n)$ is bounded. Hence, $(u_n)$ is bounded. By Lemma 1, $(u_n)$ is subsequentially convergent. □

**Theorem 2.** Suppose that

$$\sum_{k=1}^{n} t_k \alpha_k \in M \Theta,$$

(3.12)

$$1 \leq \frac{P_n}{n} \to 1, n \to \infty,$$

(3.13)

$$1 < \liminf_{n \to \infty} \frac{P_{[\lambda n]}}{P_n} \leq \limsup_{n \to \infty} \frac{P_{[\lambda n]}}{P_n} < \infty, \quad \text{for} \quad \lambda > 1,$$

(3.14)

$$1 < \liminf_{n \to \infty} \frac{P_{[\lambda n]}}{P_n} \leq \limsup_{n \to \infty} \frac{P_{[\lambda n]}}{P_n} < \infty, \quad \text{for} \quad 0 < \lambda < 1,$$

(3.15)

$$t_n = O(1), n \to \infty.$$  

(3.16)

If $(u_n) \in U(\$ \Theta)$, then $(u_n)$ converges.

**Proof.** Assume that $(u_n) \in U(\$ \Theta)$. Then, $(u_n)$ can be written as

$$u_n = \alpha_n + \sum_{k=1}^{n} \frac{t_k}{k} \alpha_k$$

(3.17)

where $(\alpha_n) \in \$ \Theta$. From (3.17), we have

$$V_{n,p}(\Delta u) = V_{n,p}(\Delta \alpha) + \sigma_{n,p}(\alpha).$$

(3.18)

Moderate oscillation of $\sum_{k=1}^{n} t_k \alpha_k$ implies convergence of $(\gamma_n) = \sum_{k=1}^{n} \frac{t_k}{k} \alpha_k$ by Lemma 2 and $\sigma_{n,p}(\alpha) = o(1)$ as $n \to \infty$ by Lemma 3. Therefore, $(u_n)$ is $(N,p)$ summable to the limit of $(\gamma_n)$.

Since $(\alpha_n)$ is slowly oscillating, $(V_{n,p}(\Delta \alpha))$ is bounded and slowly oscillating by Lemma 4.

It follows from (3.18) that $(V_{n,p}(\Delta u)) \in \$ \Theta$ and bounded. Since $(u_n)$ is $(N,p)$ summable, $(u_n)$ converges to $\lim_{n \to \infty} \sigma_{n,p}(u)$ by Theorem 6 in [4]. □

**Theorem 3.** Suppose that $(u_n)$ is regularly generated by $(\alpha_n)$ and

$$\frac{p_{n}}{P_{n-1}} - \frac{p_{n+1}}{P_n} = O \left( \frac{1}{n^2} \right), n \to \infty.$$  

(3.19)

If

$$\sum_{k=1}^{n} \alpha_k = m_n$$

(3.20)
for some \((m_n) \in M\mathcal{D}\) and some \(\gamma \in (0, 1)\), then

i) \((u_n)\) is \((\overline{N}, p)\) summable.

ii) \(u_n = \Delta(n^\gamma m_n) + \beta_n\), where \(\beta_n = o(1), n \to \infty\).

iii) \(u_n = o(n), n \to \infty\) and \(\sum_{n=1}^{\infty} \frac{u_n}{n^\gamma} < \infty\).

**Proof.**

i) By Abel’s partial summation formula, we have

\[
\sum_{k=1}^{n} \frac{p_k}{P_{k-1}} \alpha_k = \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} (S_k(\alpha) - S_{k-1}(\alpha))
\]

\[
= \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} S_k(\alpha) - \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} S_{k-1}(\alpha)
\]

\[
= \sum_{k=1}^{n} \left( \frac{p_k}{P_{k-1}} S_k(\alpha) - \frac{p_{k+1}}{P_k} S_k(\alpha) \right) + \frac{P_n}{P_{n-1}} S_n(\alpha) - \frac{P_1}{P_0} S_0
\]

\[
= \frac{P_n}{P_{n-1}} S_n(\alpha) + \sum_{k=1}^{n-1} \left( \frac{p_k}{P_{k-1}} - \frac{p_{k+1}}{P_k} \right) S_k(\alpha)
\]

(3.21)

Since \(S_n(\alpha) = n^\gamma m_n\) for some \((m_n) \in M\mathcal{D}\), we have

\[
\frac{P_n}{P_{n-1}} S_n(\alpha) = O\left( \frac{m_n}{n^{1-\gamma}} \right), \quad n \to \infty.
\]

(3.22)

By moderate divergence of \((m_n)\), we have

\[
\frac{P_n}{P_{n-1}} S_n(\alpha) = o(1), \quad n \to \infty.
\]

(3.23)

The second term on the right of (3.21) converges by (3.19). It follows from the representation

\[
u_n = \alpha_n + \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} \alpha_k.
\]

(3.24)

that \((u_n)\) is \((\overline{N}, p)\) summable.

ii) Note that the sequence \((\beta_n)\) defined by \(\beta_n = \frac{t_n \alpha_n}{n}\) for \(n \in \mathbb{N}_0\) converges to zero. From the representation and the condition (3.20) it follows that

\[
u_n = \Delta(n^\gamma m_n) + \beta_n
\]

(3.25)

where \(\beta_n = \frac{t_n}{n} \alpha_n\).

iii) By ii), we have

\[
u_n = n^\gamma m_n - (n-1)^\gamma m_{n-1} + \beta_n.
\]

(3.26)
Dividing (3.26) by $n$, we have
\[
\frac{u_n}{n} = \frac{m_n}{n^{1-\gamma}} - \frac{m_{n-1}}{(n-1)^{1-\gamma}} + \frac{\beta_n}{n}.
\] (3.27)
Since $(m_n) \in M\mathcal{D}$ and $\beta_n = o(1)$, we have
\[
\frac{u_n}{n} = o(1), n \to \infty.
\] (3.28)
By (3.26), we obtain
\[
\sum_{k=2}^{n} \frac{u_k}{k^2} = \sum_{k=2}^{n} \frac{m_k}{k^{2-\gamma}} - \sum_{k=2}^{n} \frac{m_{k-1}}{(k-1)^{2-\gamma}} + \sum_{k=2}^{n} \frac{\beta_k}{k^2}.
\] (3.29)
Taking the limit of both sides of (3.29) as $n \to \infty$, we obtain $\sum_{n=1}^{\infty} \frac{u_n}{n^2} < \infty$. □

References

Authors’ addresses

Sefa Anıl Sezer
İstanbul Medeniyet University, Department of Mathematics, 34720 Istanbul, Turkey
Current address: Ege University, Department of Mathematics, 35100 İzmir, Turkey
E-mail address: sefaanil.sezer@medeniyet.edu.tr

İbrahim Çanak
Ege University, Department of Mathematics, 35100 İzmir, Turkey
E-mail address: ibrahimcanak@yahoo.com