Miskolc Mathematical Notes
Vol. 15 (2014), No 1, pp. 219-225

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2014.1038

Limit cycles and invariant parabolas for an extended Kukles system

Béla Szabó and Iván Szántó

Miskolc Mathematical Notes
Vol. 15 (2014), No. 1, pp. 219–225

HU e-ISSN 1787-2413

LIMIT CYCLES AND INVARIANT PARABOLAS FOR AN EXTENDED KUKLES SYSTEM

BÉLA SZABÓ AND IVÁN SZÁNTÓ

Received 31 October, 2013

Abstract. A class of polynomial systems of odd degree with limit cycles, invariant parabolas and invariant straight lines, is examined. The limit cycles can be obtain as a bifurcation of a non hyperbolic focus at the origin as Hopf bifurcations. We will also obtain the necessary and sufficient conditions for the critical point at the interior of bounded region to be a center.

2010 *Mathematics Subject Classification:* 92D25; 34C; 58F14; 58F21

Keywords: limit cycle, center, bifurcation

1. INTRODUCTION

Let us consider a real autonomous system of ordinary differential equations on the plane with polynomial nonlinearities.

$$\begin{cases} \dot{x} = P(x,y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j \\ \dot{y} = Q(x,y) = \sum_{i+j=0}^{n} b_{ij} x^i y^j \end{cases}, \quad \text{with } a_{ij},\, b_{ij} \in \mathbb{R} \tag{1.1}$$

Suppose that the origin of (1.1) is a critical point of center-focus type. We are concerned with two closely related questions, both of which are significant elements in work on Hilbert's 16th problem. The first is the number of limit cycles (that is, isolated periodic solutions) which bifurcate from a critical point and the second is the derivation of necessary and sufficient conditions for a critical point to be a center (that is, all orbits in the neighborhood of the critical point are closed).

In order to describe Hilbert's 16th problem more precisely, let S_n be the collection of systems of form (1.1), with P and Q of degree at most n, and let $\pi(P,Q)$ be the number of limit cycles of (1.1). We let (P,Q) denote system (1.1). Define the so-called Hilbert numbers by

$$H_n = \mathrm{Sup}\{\pi(P,Q);\ (P,Q) \in S_n\}\ .$$

The problem consists of estimating H_n in terms of n and obtaining the possible relative configurations of limit cycles.
This is the second part of the 16th problem, which is contained in the famous list of

problems proposed by Hilbert at the International Congress of Mathematicians held in Paris in 1900.

Let us assume that the origin is a critical point of (1.1) and transform the system to canonical form

$$\dot{x} = \lambda x + y + p(x,y) , \qquad \dot{y} = -x + \lambda y + q(x,y) ,$$

where p, q are polynomials without linear terms. For the origin to be a center we must have $\lambda = 0$. If $\lambda = 0$ and the origin is not a center, it is said to be a *fine focus.*

The necessary conditions for a center are obtained by computing the *focal values.* These are polynomials in the coefficients arising in p and q and are defined as follows. There is a function V, analytic in a neighbourhood of the origin, such that the rate of change along orbits, $\dot{V}$, is of the form

$$\eta_2 r^2 + \eta_4 r^4 + \cdots , \quad \text{where } r^2 = x^2 + y^2 .$$

The focal values are the η_{2k}, and the origin is a center if and only if they are all zero. However, since they are polynomials, the ideal they generate has a finite basis, so there is M such that $\eta_{2\ell} = 0$, for $\ell \leq M$, implies that $\eta_{2\ell} = 0$ for all ℓ. The value of M is not known *a priori*, so it is not clear in advance how many focal values should be calculated.

The software Mathematica [4] is used to calculate the first few focal values. These are then 'reduced' in the sense that each is computed modulo the ideal generated by the previous ones: that is, the relations $\eta_2 = \eta_4 = \cdots = \eta_{2l} = 0$ are used to eliminate some of the variables in η_{2l+2}. The reduced focal value η_{2l+2}, with strictly positive factors removed, is known as the *Liapunov quantity* L_l. Common factors of the reduced focal values are removed and the computation proceeds until it can be shown that the remaining expressions cannot be zero simultaneously. The circumstances under which the calculated focal values are zero yield the necessary center conditions. The origin is a fine focus of *order* l if $L_i = 0$ for $i = 0, 1, ..., l-1$ and $L_l \neq 0$. At most l limit cycles can bifurcate out of a fine focus of order l; these are called *small amplitude* limit cycles.

Various methods used to prove the sufficiency of the possible center conditions. Of particular interest to us in this paper is the symmetry.

The limit cycles problem and the center problem is concentrated on specific classes of systems, those may be systems of a given degree (quadratic systems or a cubic systems, for example), or they may be of a particular form; for instance, much has been written on Liénard systems, that is, systems of the form $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$ and a Kukles systems - that is, a systems of the form

$$\dot{x} = -y , \qquad \dot{y} = x + \lambda y + g(x,y) .$$

Recall that systems of this type were studied, for the first time, by Isaak Solomonovich Kukles [2], who studied a linear center with cubic nonhomogeneous nonlinearities.

A closely related problem is the derivation of conditions under which the system is integrable and other questions of interest relate to the existence of algebraic invariant curves.

Let $n = \max(\partial P, \partial Q)$, where the symbol ∂ denotes 'degree of'. A function C is said to be *invariant* with respect to (1.1) if there is a polynomial L called cofactor, with $\partial L < m$, such that $\dot{C} = CL$. Here $\dot{C} = C_x P + C_y Q$ is the rate of change of C along orbits. It is well known that the existence of invariant curves has significant repercussions on the possible phase-portraits of the system. For example, in the case of quadratic systems ($n = 2$) the existence of an invariant ellipse or hyperbola implies that there are no limit cycles other than, possibly, the ellipse itself.

Recently in [1], consider a Kukles extended system of degree three, namely, a system of the form

$$\begin{cases} \dot{x} = P(x,y) = \lambda x + y + kxy \\ \dot{y} = Q(x,y) = -x + \lambda y + \displaystyle\sum_{i+j=2}^{3} a_{ij}x^i y^j \end{cases}, \quad \text{with } k,\ a_{ij} \in \mathbb{R}$$

and the authors obtained some conditions for the existence of centre and limit cycles.

In this paper we consider another class of extended Kukles system of degree $2n+5$, $n \geq 1$ with an invariant non–degenerate conic and two invariant straight line. For these kind of second-order differential systems, we show that for certain values of the parameters, the invariant conic and the invariant straight lines, can coexist with at least n small amplitude limit cycles which are constructed by Hopf bifurcation. We conclude with some numerical simulations of the results

2. Main results

Let us consider the extended Kukles of degree $2n+5$

$$X_\mu : \begin{cases} \dot{x} = & -yL1(x)L2(x) \\ \dot{y} = & \frac{1}{k^2}\left[2xL1^2(x)L2^2(x)\right. \\ & \left.+P1(x,y)P2(x,y)\left(k^2x - 2x + k^2Ly + \sum_{i=1}^{n} a_{2i+1}y^{2i+1}\right)\right] \end{cases} \tag{2.1}$$

where $\mu = (k, \lambda, a_3, ..., a_{2n+1}) \in \mathbb{R}^{n+2}$, $k \neq 0$ and

$$P1(x,y) = -1 + k^2x^2 - k^2y\ , \qquad L1(x) = 1 - kx\ ,$$
$$P2(x,y) = -1 + k^2x^2 + k^2y\ , \qquad L2(x) = 1 + kx.$$

Lemma 1. *For all $n \geq 1$ and for all $\mu = (k, \lambda, a_3, ..., a_{2n+1}) \in \mathbb{R}^{n+2}, k \neq 0$ the straight lines $L1(x) = 1 - kx,\ L2(x) = 1 + kx$ and the parabolas $P1(x,y) = -1 + k^2x^2 - k^2y,\ P2(x,y) = -1 + k^2x^2 + k^2y$ are invariant algebraic curves of* (2.1).

Proof. It is easy to verify that

$$\dot{L1} = L1_x\dot{x} + L1_y\dot{y} = kyL1L(x,y)\ ,$$

where the cofactor is $L(x,y) = ky(1+kx)$,

$$\dot{L2} = L2_x\dot{x} + L2_y\dot{y} = -kyL2L(x,y) ,$$

where the cofactor is $L(x,y) = -ky(1-kx)$,

$$\dot{P1} = P1_x\dot{x} + P1_y\dot{y} = P1(x,y)L(x,y) ,$$

where the cofactor is $L(x,y) = -x+k^2x^3 - Ly - 2xy + k^2xy + k^2Lx^2y + k^2Ly^2 + \frac{1}{k^2}P1(x,y)\left(\sum_{i=1}^n a_{2i+1}y^{2i+1}\right)$ and

$$\dot{P2} = P2_x\dot{x} + P2_y\dot{y} = P2(x,y)L(x,y) ,$$

where the cofactor is $L(x,y) = -x+k^2x^3 - Ly + 2xy - k^2xy + k^2Lx^2y - k^2Ly^2 - \frac{1}{k^2}P2(x,y)\left(\sum_{i=1}^n a_{2i+1}y^{2i+1}\right)$.

Then straight lines $L1(x)$, $L2(x)$ and the parabolas $P1(x,y)$ and $P2(x,y)$ are invariant algebraic curves of (2.1). □

Theorem 1. *If $a_{2j+1} = 0$, $\forall j = 1,2,...,n-1$ and $a_{2n+1} \neq 0$, system (2.1) at the origin has a weak focus of order n.*

Proof. The linear part of (2.1) at the singularity $(0,0)$ is

$$DX_\mu(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix}.$$

If $\lambda = 0$, we have $\operatorname{div} X_\mu(0,0) = 0$ and $\det DX_\mu(0,0) = 1$, then the critical point $(0,0)$ is a weak focus.

As $\operatorname{div} X_\mu(0,0) = \lambda$, we have the Liapunov quantities $L_0 = \lambda$, and using Mathematica Software [4], we are able to compute the Liapunov quantities L_l, for $l \geq 1$.

If $\lambda = 0$, $L_0 = 0$ and $L_1 = \dfrac{3a_3}{8k^2}$, if $a_3 = 0$, $L_1 = 0$ and $L_2 = \dfrac{5a_5}{16k^2}$, if $a_5 = 0$, $L_2 = 0$ and $L_3 = \dfrac{35a_7}{128k^2}$ and so on. Then, if $\lambda = a_3 = a_5 = ... = a_{2n-1} = 0$, and $a_{2n+1} \neq 0$, we have $L_0 = L_1 = L_3 = ... = L_{n-1} = 0$, and for $n \geq 1$, $L_n = a_{2n+1}\dfrac{(2n+1)!!}{(2n+2)!!k^2}$. Finally, for all $l \geq n$, $L_l = a_{2l+1}R_l(k)$ where $R_l(k)$ is a polynomial on k, then system (2.1) at the origin has a weak focus of order n. □

Theorem 2. *System (2.1) has a local center at the origin if and only if $\lambda = a_3 = a_5 = ... = a_{2n+1} = 0$.*

Proof. From the linear part at the origin, it is clear that the condition $\lambda = 0$ is necessary for a center. By Theorem 1, the Liapunov quantities are given by $L_0 = \lambda$ and for $n \geq 1$, $L_n = a_{2n+1}\dfrac{(2n+1)!!}{(2n+2)!!k^2}$. As all calculated Liapunov quantities are

zero, this shows that the conditions are necessary.
If $a_{2j+1} = 0$, $j = 1,2,3,...,n$, the system (2.1) is of degree 5 and is given by

$$\begin{cases} \dot{x} = & P(x,y) = -y + k^2x^2y \\ \dot{y} = & Q(x,y) = x(1 - 2k^2x^2 + k^4x^4 + 2k^2y^2 - k^4y^2) \end{cases} \tag{2.2}$$

As the symmetries $P(-x,y) = P(x,y)$ and $Q(-x,y) = -Q(x,y)$ are satisfied, this proves the sufficiency that system (2.2) has a center at the origin. □

In order to illustrate the result stated in Theorem 1, Figure 1 shows a numerical simulation of the system (2.1) of degree five with $k = 1$, $a_3 = 0$ and $\lambda = 0$, which corresponds to the case of center. The simulation was obtained using the Pplane7 Software with MATLAB [3]

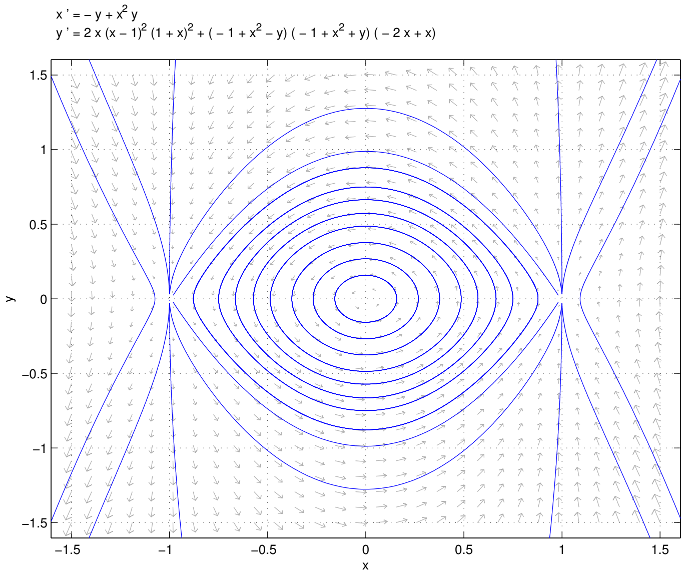

FIGURE 1. Local Center.

Theorem 3. *In the parameters space $\mathbb{R}^{n+2}$, there exists an open set $\mathcal{N} \neq \phi$, such that for all $\mu = (k, \lambda, a_3, ..., a_{2n+1}) \in \mathcal{N}$, $k \neq 0$, system (2.1) has at least n small-amplitude limit cycle bounded by the invariant conic.*

Proof. If $\lambda = 0$ and $a_{2j-1} = 0$, $\forall j = 2,3,...,n$ and $a_{2n+1} \neq 0$, by Theorem 1, system (2.1) has at the singularity $(0,0)$ a repelling or attracting weak focus of order n.

Let us consider the case $a_{2n+1} > 0$ and n odd number, that is an attracting weak focus of order n. Taking $a_{2n-1} < 0$, so that $L_n > 0$ and $L_{n-1} < 0$, the origin of (2.1)

is reversed and a hyperbolic repelling small amplitude limit cycle is created (Hopf bifurcation), the limit cycle created persist under new small perturbation. Furthermore the singularity at the origin is an attracting weak focus of order $n-1$.
If we adjust the parameter a_{2n-3}, so that $a_{2n-3} > 0$ small enough and $L_n > 0$, $L_{n-1} < 0$ and $L_{n-2} > 0$, the stability at the origin is reversed and new hyperbolic attracting small amplitude limit cycle is created, that persists under new small perturbation. Furthermore the singularity at the origin is an repelling weak focus of order $n-2$. By following the same process until all Liapunov quantities are non-zero, such that in each step the stability of the singularity is inverted, n hyperbolic small amplitude limit cycles are created and we obtain $n+1$ inequalities which define an a open set $\mathcal{N}$ in the parameter space, given by $\mathcal{N} = \{(k, \lambda, a_3, ..., a_{2n+1}) \in \mathbb{R}^{n+2} \mid k \neq 0, L_n > 0, L_{n-1} < 0, L_{n-2} > 0, ..., L_2 < 0, L_1 > 0, L_0 < 0\}$. □

In order to illustrate the result stated in Theorem 3, Figure 2 shows a numerical simulation of the system (2.1) with $k = 1$, $a_3 = -1$ and $\lambda = 0.1$, which corresponds to the case of attracting limit cycle. The simulation was obtained using the Pplane7 Software with MATLAB [3].

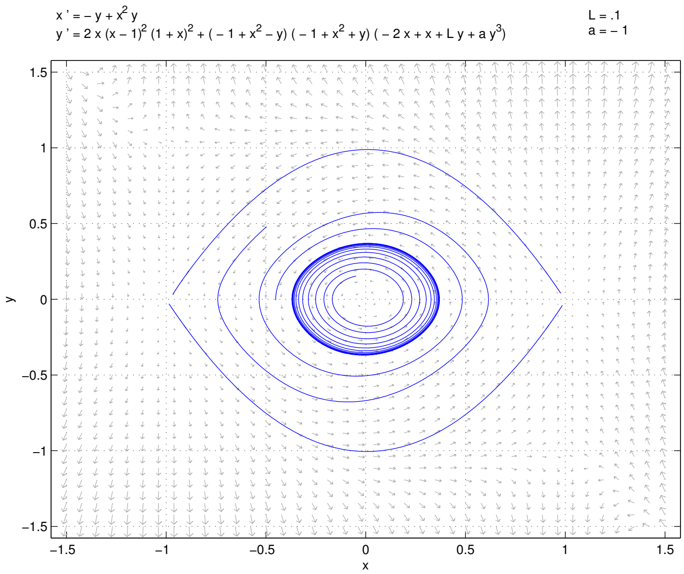

FIGURE 2. Attracting limit cycle ($n = 1$).

REFERENCES

[1] J. M. Hill, N. G. Lloyd, and J. M. Pearson, “Centers and limit cycles for an extended kukles system,” *Electronic Journal of Differential Equations*, vol. 2007, no. 119, pp. 1–23, 2007.

[2] I. S. Kukles, "Sur quelques cas de distinction entre un foyer et un centre," *Dokl. Akad. Nauk. SSSR*, vol. 42, no. 42, pp. 208–211, 1944.
[3] MatWorks, *MATLAB: The Language of technical computing Using MATLAB(version 7.0)*. Natwick,MA: MatWorks, 2004.
[4] Wolfram Research Mathematica, *A System for Doing Mathematics by Computer, Champaign, IL*, ser. Wolfram Research, Inc. Addison-Wesley, 1988.

Authors' addresses

Béla Szabó
Műszaki Tudományi Doktori Iskola, Szent István Egyetem, Páter K. u. 1., H-2100, Gödöllő, Hungary
E-mail address: `Szabo.Bela@gek.szie.hu`

Iván Szántó
Departamento de Matemática, Universidad Técnica Federico Santa María, Departamento de Matemática, Casilla 110-V, Valparaíso, Chile
E-mail address: `ivan.szanto@usm.cl`