The harmonic index for unicyclic and bicyclic graphs with given matching number

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THE HARMONIC INDEX FOR UNICYCLIC AND BICYCLIC
GRAPHS WITH GIVEN MATCHING NUMBER

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Abstract. The harmonic index of a graph $G$ is defined as the sum of the weights $\frac{2}{d(u)+d(v)}$ of all edges $uv$ of $G$, where $d(u)$ denotes the degree of a vertex $u$ in $G$. In this paper, we present the minimum harmonic indices for unicyclic and bicyclic graphs with $n$ vertices and matching number $m$ ($2 \leq m \leq \lfloor \frac{n}{2} \rfloor$), respectively. The corresponding extremal graphs are also characterized.

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1. INTRODUCTION

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Randić index $R(G)$, proposed by Randić [20] in 1975, is defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}},$$

where $d(u)$ denotes the degree of a vertex $u$ of $G$. The Randić index is one of the most successful molecular descriptors in structure-property and structure-activity relationship studies. Mathematical properties of this descriptor have been studied extensively (see [9, 10, 14, 15, 19] and the references cited therein).

In this paper, we consider a closely related variant of the Randić index, named the harmonic index. For a graph $G$, the harmonic index $H(G)$ is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u)+d(v)}.$$

This index first appeared in [6], and it can also be viewed as a particular case of the general sum-connectivity index proposed by Zhou and Trinajstić in [32].

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Favaron, Mahéo and Saclé [7] considered the relation between the harmonic index and the eigenvalues of graphs. Zhong [28, 29], Zhong and Xu [30] determined the minimum and maximum harmonic indices for simple connected graphs, trees, unicyclic and bicyclic graphs, and characterized the corresponding extremal graphs. Wu, Tang and Deng [23] found the minimum harmonic index for graphs (triangle-free graphs, respectively) with minimum degree at least 2, and characterized the corresponding extremal graphs. Deng, Balachandran, Ayyaswamy and Venkatakrishnan [2] considered the relation between the harmonic index and the chromatic number of a graph by using the effect of removal of a minimum degree vertex on the harmonic index. Liu [17] proposed a conjecture concerning the relation between the harmonic index and the diameter of a connected graph, and showed that the conjecture is true for trees. Ilić [12], Xu [25], Zhong and Xu [31] established some relationships between the harmonic index and several other topological indices. The chemical applicability of the harmonic index was also recently investigated [8, 11]. See [3, 18, 24, 26] for more information of this index.

In this paper, we determine the minimum harmonic indices for unicyclic and bicyclic graphs with \( n \) vertices and matching number \( m \) (\( 2 \leq m \leq \lfloor \frac{n}{2} \rfloor \)), respectively. The corresponding extremal graphs are also characterized. The related problems have been well-studied for several other topological indices, such as the Randić index [16, 33], the modified Randić index [13] and the sum-connectivity index [4, 5, 21, 22].

2. Preliminaries

Let \( G \) be a graph. For any vertex \( v \in V(G) \), we use \( N_G(v) \) (or \( N(v) \) if there is no ambiguity) to denote the set of neighbors of \( v \) in \( G \). A pendent vertex is a vertex of degree 1. For two distinct vertices \( u \) and \( v \) of \( G \), the distance \( d(u, v) \) between \( u \) and \( v \) is the number of edges in a shortest path joining \( u \) and \( v \) in \( G \). A unicyclic graph is a connected graph with \( n \) vertices and \( n \) edges, and a bicyclic graph is a connected graph with \( n \) vertices and \( n + 1 \) edges. We use \( C_n \) to denote the cycle on \( n \) vertices.

A matching \( M \) in a graph \( G \) is a subset of \( E(G) \) such that no two edges in \( M \) share a common vertex. A matching \( M \) in \( G \) is said to be maximum, if for any other matching \( M' \) in \( G \), \( |M'| \leq |M| \). The matching number of \( G \) is the number of edges in a maximum matching of \( G \). If \( M \) is a matching in \( G \) and the vertex \( v \in V(G) \) is incident with an edge of \( M \), then \( v \) is said to be \( M \)-saturated, and if every vertex in \( G \) is \( M \)-saturated, then \( M \) is a perfect matching.

For any vertex \( v \in V(G) \), we use \( G - v \) to denote the graph resulting from \( G \) by deleting the vertex \( v \) and its incident edges. We define \( G - uv \) to be the graph obtained from \( G \) by deleting the edge \( uv \in E(G) \), and \( G + uv \) to be the graph obtained from \( G \) by adding an edge \( uv \) between two non-adjacent vertices \( u \) and \( v \) of \( G \).

We now establish some lemmas which will be used frequently in later proofs.
Lemma 1. Let $G$ be a connected graph on $n \geq 4$ vertices with a pendent vertex $u$. Let $v$ be the unique neighbor of $u$ with $d(v) = s$, and let $w$ be a neighbor of $v$ different from $u$ with $d(w) = t$.

(i) If $s = 2$ and $w$ is adjacent to at most one pendent vertex in $G$, then

$$H(G) \geq H(G - u - v) + \frac{2(t - 1)}{t + 2} - \frac{2(t - 3)}{t + 1} - \frac{2}{t} + \frac{2}{3}$$

with equality if and only if one neighbor of $w$ has degree 1 and the other neighbors of $w$ have degree 2.

(ii) If $v$ is adjacent to at most $k$ pendent vertices in $G$, then

$$H(G) \geq H(G - u) + \frac{2(s - k)}{s + 2} + \frac{2(2k - s)}{s + 1} - \frac{2(k - 1)}{s}$$

with equality if and only if $k$ neighbors of $v$ have degree 1 and the other neighbors of $v$ have degree 2.

Proof: (i) Let $N(w) = \{w_0 = v, w_1, \ldots, w_{t-1}\}$. Since $w$ is adjacent to at most one pendent vertex in $G$, we may assume that $d(w_1) \geq 1$, and $d(w_i) \geq 2$ for each $2 \leq i \leq t - 1$ (if $t \geq 3$). Note that $\frac{2}{t+2} - \frac{2}{t-1+x}$ is increasing for $x \geq 1$, we have

$$H(G) = H(G - u - v) + \sum_{i=1}^{t-1} \left( \frac{2}{t + d(w_i)} - \frac{2}{t-1 + d(w_i)} \right) + \frac{2}{t+2} + \frac{2}{3}$$

$$\geq H(G - u - v) + \left( \frac{2}{t+1} - \frac{2}{t} \right) + (t-2) \left( \frac{2}{t+2} - \frac{2}{t+1} \right) + \frac{2}{t+2} + \frac{2}{3}$$

$$= H(G - u - v) + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3}$$

with equality if and only if $d(w_1) = 1$ and $d(w_i) = 2$ for each $2 \leq i \leq t - 1$ (if $t \geq 3$). This proves (i).

(ii) Let $r (1 \leq r \leq k)$ be the number of pendent neighbors of $v$ in $G$, and let $N(v) = \{v_0 = u, v_1, \ldots, v_{s-1}\}$. Without loss of generality, we may assume that $d(v_1) = 1$ for each $1 \leq i \leq r - 1$ (if $r \geq 2$), and $d(v_i) \geq 2$ for each $r \leq i \leq s - 1$ (if $s \geq r + 1$). Note that $\frac{2}{s+1} - \frac{2}{s+1+x}$ is increasing for $x \geq 1$ and $\frac{4}{s+1} - \frac{2}{s+2} - \frac{2}{s} < 0$, we have

$$H(G) = H(G - u) + (r-1) \left( \frac{2}{s+1} - \frac{2}{s} \right)$$

$$+ \sum_{i=r}^{s-1} \left( \frac{2}{s + d(v_i)} - \frac{2}{s-1 + d(v_i)} \right) + \frac{2}{s+1}$$

$$\geq H(G - u) + (r-1) \left( \frac{2}{s+1} - \frac{2}{s} \right) + (s-r) \left( \frac{2}{s+2} - \frac{2}{s+1} \right) + \frac{2}{s+1}$$

$$= H(G - u) + r \left( \frac{4}{s+1} - \frac{2}{s+2} - \frac{2}{s} \right) + \frac{2s}{s+2} + \frac{2s}{s+1} + \frac{2}{s}$$
\[ H(G - u) + k \left( \frac{4}{s+1} - \frac{2}{s+2} - \frac{2}{s} \right) + \frac{2s}{s+2} - \frac{2s}{s+1} + \frac{2}{s} \]

\[ = H(G - u) + \frac{2(s-k)}{s+2} + \frac{2(2k-s)}{s+1} - \frac{2(k-1)}{s} \]

with equalities if and only if \( r = k \) and \( d(v_i) = 2 \) for each \( k \leq i \leq s - 1 \) (if \( s \geq k + 1 \)). This completes the proof of the lemma.

**Lemma 2.**

(i) The function \( \frac{2(x-1)}{x+2} - \frac{2(x-3)}{x+1} - \frac{2}{x} \) is decreasing for \( x \geq 2 \).

(ii) For \( k \geq 1 \), the function \( \frac{2(x-k)}{x+2} + \frac{2(2k-x)}{x+1} - \frac{2(k-1)}{x} \) is decreasing for \( x \geq k + 1 \).

**Proof.** (i) Let \( f(x) = \frac{2(x-1)}{x+2} - \frac{2(x-3)}{x+1} - \frac{2}{x} \). For \( x \geq 2 \), we have

\[ f'(x) = -\frac{8}{(x+1)^2} + \frac{6}{(x+2)^2} + \frac{2}{x^2} = \frac{-8x^3 + 24x + 8}{x^2(x+1)^2(x+2)^2} \]

\[ = \frac{-8(x^2 - 4) - 8(x - 1)}{x^2(x+1)^2(x+2)^2} < 0, \]

and hence (i) holds.

(ii) Let \( g(x) = \frac{2(x-k)}{x+2} + \frac{2(2k-x)}{x+1} - \frac{2(k-1)}{x} \) and \( g_1(x) = \frac{2(k-1)}{x} + \frac{2(x-1-k)}{x+1} \). Then \( g(x) = g_1(x+1) - g_1(x) \). For \( x \geq k + 1 \geq 2 \), we have

\[ g''_1(x) = \frac{4(k-1)}{x^3} - \frac{4(k+2)}{(x+1)^3} = \frac{-12x^3 + 12(k-1)x^2 + 12(k-1)x + 4(k-1)}{x^3(x+1)^3} \]

\[ = -\frac{12x^2(x-k) - 12x(x-k) - 4(3x-k+1)}{x^3(x+1)^3} < 0, \]

and \( g'(x) = g'_1(x+1) - g'_1(x) < 0 \). So the assertion of the lemma holds.

**Lemma 3.** Let \( G \) be a connected graph, and let \( u \) be a vertex of degree 2 in \( G \) with two neighbors \( v \) and \( w \) such that \( d(v) \geq 2 \) and \( vw \notin E(G) \). Let \( G' = G - uw + vw \), then \( H(G) > H(G') \).

**Proof.** Let \( d(v) = p \geq 2 \) and let \( N(v) = \{v_0 = u, v_1, \ldots, v_{p-1}\} \). Then

\[ H(G) - H(G') = \sum_{i=1}^{p-1} \left( \frac{2}{p + d(v_i)} + \frac{2}{p + d(w)} \right) - \left( \sum_{i=1}^{p-1} \frac{2}{p + 1 + d(v_i)} + \frac{2}{p + 1 + d(w)} \right) \]

\[ = \sum_{i=1}^{p-1} \left( \frac{2}{p + d(v_i)} - \frac{2}{p + 1 + d(v_i)} \right) + \left( \frac{2}{p + d(w)} - \frac{2}{p + 1 + d(w)} \right) > 0. \]

This proves the lemma.
3. Minimum Harmonic Index for Unicyclic Graphs with Given Matching Number

Let $U_n$ be the set of unicyclic graphs with $n \geq 3$ vertices, and let $U_{n,m}$ be the set of unicyclic graphs with $n$ vertices and matching number $m$, where $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. In this section, we determine the minimum harmonic index for graphs in $U_{n,m}$, and characterize the corresponding extremal graphs.

For a unicyclic graph $G$ with the cycle $C_p$, the forest obtained from $G$ by deleting the edges in $C_p$ consists of $p$ vertex-disjoint trees, each containing a vertex of $C_p$, which is called the root of this tree in $G$. These trees are called the branches of $G$.

Chang and Tian [1] showed the following lemma.

**Lemma 4.** Let $G \in U_{2m,m}$ ($m \geq 3$), and let $T$ be a branch of $G$ with root $r$. If $u \in V(T)$ is a pendent vertex which is furthest from the root $r$ with $d(u,r) \geq 2$, then $u$ is adjacent to a vertex of degree 2.

The second lemma was proved by Yu and Tian [27].

**Lemma 5.** Let $G \in U_{n,m}$ ($n > 2m$) and $G \not\cong C_n$. Then there exists a maximum matching $M$ and a pendant vertex $u$ in $G$ such that $u$ is not $M$-saturated.

Zhong [29] proved the following result.

**Lemma 6.** Let $G \in U_n$ with $n \geq 3$. Then $H(G) \leq \frac{n}{2}$ with equality if and only if $G \cong C_n$.

![Figure 1](image-url)  
**Figure 1.** The graphs $U_6$, $U_8$ and $U_{n,m}$.

Let $U_6$ be the unicyclic graph on 6 vertices obtained by attaching a pendent vertex to every vertex of a triangle, and let $U_8$ be the unicyclic graph on 8 vertices obtained by attaching a path on two vertices to one vertex of degree 3 of $U_6$. For $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$, we use $U_{n,m}$ to denote the unicyclic graph on $n$ vertices obtained by attaching $n - 2m + 1$ pendent vertices and $m - 2$ paths on two vertices to one vertex of a triangle. See Figure 1 for an illustration.

**Theorem 1.** Let $G \in U_{2m,m} \setminus \{U_6, U_8\}$, where $m \geq 2$. Then

$$H(G) \geq \frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-2)}{3} + \frac{1}{2}$$

with equality if and only if $G \cong U_{2m,m}$. 

Proof. We prove the theorem by induction on \( m \). If \( m = 2 \), then either \( G \cong C_4 \) or \( G \cong U_{4,2} \). Since \( H(C_4) = 2 > \frac{9}{2} = H(U_{4,2}) \), we see that the assertion of the theorem holds. So we may assume that \( m \geq 3 \) and the result holds for graphs in \( \mathcal{U}_{2(m-1),m-1} \setminus \{U_6, U_8\} \). By Lemma 6, since \( C_{2m} \) is the unique unicyclic graph on \( 2m \) vertices with the maximum harmonic index, we may further assume that \( G \not\cong C_{2m} \). Let \( M \) be a maximum matching in \( G \), then \(|M| = m\). By Lemma 4, we need only consider the following two cases.

**Case 1.** There exists a pendent vertex \( u \) in \( G \) which is adjacent to a vertex \( v \) of degree 2.

Let \( w \) be the neighbor of \( v \) different from \( u \) with \( d(w) = t \geq 2 \), and let \( G' = G - u - v \). Then \( uv \in M \) and \( G' \in \mathcal{U}_{2(m-1),m-1} \). Since \( M \) contains exactly one edge incident with \( w \) and there are \( m \) edges of \( G \) outside \( M \), we have \( t \leq m + 1 \). Note that \( w \) is adjacent to at most one pendent vertex in \( G \).

![Figure 2. The graphs \( W_1, W_2, \) and \( W_3 \).](image)

If \( G' \cong U_8 \), then we have \( G \cong W_1 \) (since we assume \( G \not\cong U_8 \)), see Figure 2. Since \( H(W_1) = \frac{107}{30} > \frac{139}{42} = H(U_{8,4}) \), we know that the result holds.

If \( G' \cong U_8 \), then \( t \leq 5 \). By Lemma 1(i) and Lemma 2(i), we have

\[
H(G) \geq H(U_8) + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3} \geq \frac{347}{105} + \frac{2 \cdot (5-3)}{5+2} - \frac{2}{5} + \frac{2}{3} = \frac{85}{21} > \frac{113}{28} = H(U_{10,5}),
\]

and hence the assertion of the theorem holds.

Now suppose that \( G' \not\cong U_6, U_8 \). Then by Lemma 1(i), Lemma 2(i) and the induction hypothesis, we conclude that

\[
H(G) \geq H(G') + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3} \geq \left( \frac{2(m-1)}{(m-1)+3} + \frac{2}{(m-1)+2} + \frac{2[(m-1)-2]}{3} + \frac{1}{2} \right) + \frac{2[(m+1)-1]}{(m+1)+2} - \frac{2[(m+1)-3]}{(m+1)+1} - \frac{2}{m+1} + \frac{2}{3}.
\]
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\[
\frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-2)}{3} + \frac{1}{2}
\]

with equalities if and only if \( G' \cong U_{2(m-1),m-1} \) and \( t = m+1 \), i.e., \( G \cong U_{2m,m} \). This proves Case 1.

Case 2. \( G \) is a unicyclic graph with maximum degree 3 obtained by attaching 2\( m - p \) pendant vertices to some vertices of a cycle \( C_p \) (\( m \leq p \leq 2m - 1 \)).

If \( m = 3 \), then \( G \) is either the unicyclic graph obtained by attaching a pendant vertex to one vertex of \( C_5 \) or the unicyclic graph obtained by attaching a pendant vertex to two adjacent vertices of \( C_4 \) (since we assume \( G \not= U_6 \)). Then we have \( H(G) \geq \frac{70}{36} > \frac{77}{36} = H(U_{6,3}) \), and the theorem holds. So we may assume that \( m \geq 4 \).

We consider two subcases according to the value of \( p \).

Subcase 2.1. \( p = m \).

Then every vertex of \( C_p \) is attached by a pendant vertex and \( H(G) = \frac{5m}{6} \). Let
\[
f(x) = \frac{5x}{6} - \left( \frac{2x}{x+3} + \frac{2}{x+2} + \frac{2(x-2)}{3} + \frac{1}{2} \right) = \frac{5x}{6} + \frac{6}{x+3} - \frac{2}{x+2} - \frac{7}{6}.
\]
For \( x \geq 4 \), we have
\[
f'(x) = \frac{1}{6} - \frac{6}{(x+3)^2} + \frac{2}{(x+2)^2} \geq \frac{1}{6} - \frac{6}{(4+3)^2} + \frac{2}{(x+2)^2} > 0.
\]
This implies that \( f(x) \) is increasing for \( x \geq 4 \), and thus \( f(m) \geq f(4) = \frac{1}{42} > 0 \), i.e., \( H(G) > H(U_{2m,m}) \).

Subcase 2.2. \( m + 1 \leq p \leq 2m - 1 \).

In this subcase, there exists at least one edge, say \( xy \), on \( C_p \) such that \( xy \in M \). Then \( d(x) = d(y) = 2 \); for otherwise, the pendant vertex adjacent to \( x \) or \( y \) cannot be \( M \)-saturated. Let \( z \) be the neighbor of \( x \) different from \( y \) in \( G \), and let \( G'' = G - xz + yz \). Then \( G'' \in \mathcal{W}_{2m,m} \setminus \{U_8\} \). By Lemma 3, we have \( H(G) > H(G'') \). Comparing with the graph \( G \), we see that the length of the unique cycle in \( G'' \) decreases by 1. Repeating this operation from \( G \) to \( G'' \), we eventually obtain the unicyclic graph described in Subcase 2.1 and the result holds. This finishes the proof of the theorem.

Since \( H(U_{6,3}) = \frac{77}{36} > \frac{5}{2} = H(U_6) \) and \( H(U_{8,4}) = \frac{139}{42} > \frac{347}{105} = H(U_8) \), by Theorem 1, we immediately obtain the following two results.

Corollary 1. Let \( G \in \mathcal{W}_{6,3} \), then \( H(G) \geq \frac{5}{2} \) with equality if and only if \( G \cong U_6 \).

Corollary 2. Let \( G \in \mathcal{W}_{8,4} \), then \( H(G) \geq \frac{347}{105} \) with equality if and only if \( G \cong U_8 \).

We now prove the main result of this section.

Theorem 2. Let \( G \in \mathcal{W}_{n,m} \setminus \{U_6, U_8\} \), where \( 2 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \). Then
\[
H(G) \geq \frac{2m}{n-m+3} + \frac{2(n-2m+1)}{n-m+2} + \frac{2(m-2)}{3} + \frac{1}{2}
\]
with equality if and only if \( G \cong U_{n,m} \).
Proof. We prove Theorem 2 by induction on \( n \). If \( n = 2m \), then by Theorem 1, the assertion of the theorem holds. So we may assume that \( n > 2m \) and the result holds for graphs in \( \mathbb{U}_{n-1,m} \setminus \{ U_6, U_8 \} \). By Lemma 6, since \( C_n \) is the unique unicyclic graph on \( n \) vertices with the maximum harmonic index, we may also assume that \( G \not\cong C_n \).

Then by Lemma 5, there exists a maximum matching \( M \) and a pendant vertex \( u \) in \( G \) such that \( u \) is not \( M \)-saturated. Let \( v \) be the unique neighbor of \( u \) with \( d(v) = s \geq 2 \), and let \( G' = G - u \). Then \( G' \in \mathbb{U}_{n-1,m} \). Since \( M \) contains exactly one edge incident with \( v \) and there are \( n - m \) edges of \( G \) outside \( M \), we have \( s \leq n - m + 1 \). Let \( r \) be the number of pendant neighbors of \( v \) in \( G \), where \( 1 \leq r \leq s - 1 \). Note that at least \( r - 1 \) pendant neighbors of \( v \) are not \( M \)-saturated, and there are \( n - 2m \) vertices are not \( M \)-saturated in \( G \). Then \( r \leq n - 2m + 1 \).

If \( G' \cong U_6 \), then \( n = 7 \), \( m = 3 \) and either \( G \cong W_2 \) or \( G \cong W_3 \) (see Figure 2). Since \( H(W_2) = \frac{42}{15} > H(W_3) = \frac{184}{105} > \frac{113}{42} = H(U_{7,3}) \), we see that the result holds.

If \( G' \cong U_8 \), then \( n = 9 \), \( m = 4 \) and \( s \leq 5 \). By Lemma 1(ii) (with \( k = n - 2m + 1 = 2 \)) and Lemma 2(ii), we have

\[
H(G) \geq H(U_8) + \frac{2(s - 2)}{s + 2} + \frac{2(4 - s)}{s + 1} - \frac{2}{s} \geq \frac{347}{105} + \frac{2(5 - 2)}{5 + 2} + \frac{2(4 - 5)}{5 + 1} - \frac{2}{5} = \frac{24}{7} \geq \frac{143}{42} = H(U_{9,4}),
\]

and thus the assertion of the theorem holds.

Therefore we may assume that \( G' \not\cong U_6, U_8 \). Then by Lemma 1(ii) (with \( k = n - 2m + 1 \)), Lemma 2(ii) and the induction hypothesis, we conclude that

\[
H(G) \geq H(G') + \frac{2[s - (n - 2m + 1)]}{s + 2} + \frac{2[2(n - 2m + 1) - s]}{s + 1} - \frac{2[(n - 2m + 1) - 1]}{s} \geq \left( \frac{2m}{(n - 1) - m + 3} + \frac{2[(n - 1) - 2m + 1]}{(n - 1) - m + 2} + \frac{2(m - 2)}{3} + \frac{1}{2} \right) + \frac{2[(n - m + 1) - (n - 2m + 1)]}{(n - m + 1) + 2} + \frac{2[2(n - 2m + 1) - (n - m + 1)]}{(n - m + 1) + 1} - \frac{2[(n - 2m + 1) - 1]}{n - m + 1} = \frac{2m}{n - m + 3} + \frac{2(n - 2m + 1)}{n - m + 2} + \frac{2(m - 2)}{3} + \frac{1}{2}
\]

with equalities if and only if \( G' \cong U_{n-1,m}, s = n - m + 1 \) and \( r = n - 2m + 1 \), i.e., \( G \cong U_{n,m} \). This completes the proof of the theorem. \( \square \)

By applying Theorem 2, we can also obtain the minimum harmonic index for graphs in \( \mathbb{U}_n \) \((n \geq 4)\). This is one of the main results in [29].
Corollary 3. Let $G \in \mathcal{U}_n$ with $n \geq 4$. Then
\[
H(G) \geq \frac{4}{n+1} + \frac{2(n-3)}{n} + \frac{1}{2}
\]
with equality if and only if $G \cong U_{n,2}$.

Proof. Let $M$ be a maximum matching in $G$, then $2 \leq |M| = m \leq \left\lfloor \frac{n}{2} \right\rfloor$ (since $n \geq 4$). If $m = 2$, then by Theorem 2, we have
\[
H(G) \geq \frac{2 \cdot 2}{n-2+3} + \frac{2(n-2 \cdot 2 + 1)}{n-2+2} + \frac{2 \cdot (2-2)}{3} + \frac{1}{2}
\]
with equality if and only if $G \cong U_{n,2}$. So we may assume that $m \geq 3$.

If $G \cong U_6$, then $H(G) = \frac{3}{2} > \frac{29}{14} = H(U_{6,2})$, we see that the result holds. If $G \cong U_8$, then $H(G) = \frac{347}{105} > \frac{79}{36} = H(U_{8,2})$, and the result also holds. Now suppose that $G \not\cong U_6, U_8$. Then by Theorem 2 and Lemma 3, we see that $H(G) \geq H(U_{n,m}) > H(U_{n,m-1}) > \cdots > H(U_{n,2})$. So the assertion of the corollary holds. \[ \square \]

4. Minimum harmonic index for bicyclic graphs with given matching number

Let $\mathcal{B}_n$ be the set of bicyclic graphs with $n \geq 4$ vertices, and let $\mathcal{B}_{n,m}$ be the set of bicyclic graphs with $n$ vertices and matching number $m$, where $2 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor$. In this section, we present the minimum harmonic index for graphs in $\mathcal{B}_{n,m}$, and characterize the corresponding extremal graphs.

We denote by $\mathcal{B}_n$ the set of bicyclic graphs with $n \geq 4$ vertices containing no pendant vertices. Let $\mathcal{B}_{n}^1$ be the set of bicyclic graphs on $n \geq 6$ vertices obtained by connecting two vertex-disjoint cycles by a new edge, and let $\mathcal{B}_{n}^2$ be the set of bicyclic graphs on $n \geq 7$ vertices obtained by connecting two vertex-disjoint cycles by a path of length at least two. Let $\mathcal{B}_{n}^3$ be the set of bicyclic graphs on $n \geq 5$ vertices obtained by identifying a vertex of a cycle and a vertex of the other cycle. Let $\mathcal{B}_{n}^4$ be the set of bicyclic graphs on $n \geq 4$ obtained from $C_n$ by adding a new edge, and let $\mathcal{B}_{n}^5$ be the set of bicyclic graphs on $n \geq 5$ obtained by connecting two non-adjacent vertices by a path of length at least two. Clearly, $\mathcal{B}_n = \bigcup_{i=1}^{5} \mathcal{B}_n^i$.

For $i = 4, 5$, we use $B_i$ to denote the unique bicyclic graph on $i$ vertices in $\mathcal{B}_n^i$. Let $B_{n,a,b}$ be the bicyclic graph on $n$ vertices obtained by attaching $a-3$ and $b-3$ pendant vertices to the two vertices of degree 3 of $B_4$, respectively, where $a \geq b \geq 3$ and $a + b = n + 2$. Let $B_{n,a,b}'$ be the bicyclic graph on $n$ vertices obtained by attaching $a-3$ and $b-3$ pendant vertices to the two vertices of degree 3 of $B_5$, respectively, where $a \geq b \geq 3$ and $a + b = n + 1$. Then $B_4 \cong B_{4,3,3}$ and $B_5 \cong B_{5,3,3}'$. See Figure 3 and Figure 4 for an illustration. We first determine the minimum harmonic index for graphs in $\mathcal{B}_n$ with matching number 2.
\textbf{Theorem 3.} Let $G \in \mathcal{B}_{n,2}$ with $n \geq 4$. Then
\[ H(G) \geq \frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5} \]
with equality if and only if $G \cong \overline{B}_{n,n-1,3}$.

\textit{Proof.} Since $B_4$ is the unique bicyclic graph on 4 vertices in $\mathcal{B}_{4,2}$, we see that the result holds for $n = 4$. If $n = 5$, then $G \in \{ F_i | 1 \leq i \leq 3 \} \cup B_5 \cup B_{5,4,3}$, where $F_i$ $(1 \leq i \leq 3)$ are shown in Figure 5. It is easy to calculate that $H(F_1) = \frac{73}{30} > H(B_5) = \frac{12}{5} > H(F_2) = \frac{7}{3} > H(F_3) = \frac{23}{10} > \frac{226}{105} = H(B_{5,4,3})$, and hence the assertion of the theorem holds. So we may assume that $n \geq 6$. We consider three cases according to the structure of $G$.

\begin{figure}[h]
\centering
\includegraphics{figure5.png}
\caption{The graphs $F_1$, $F_2$ and $F_3$.}
\end{figure}
Case 1. $G \cong B_{n,a,b}$, where $a \geq b \geq 3$ and $a + b = n + 2$.
Let $f(x) = \frac{4}{x+1} - \frac{8}{x}$. For $x \geq 3$, we have
\[
f''(x) = \frac{8}{(x+1)^3} - \frac{16}{x^3} = \frac{-8(x^3 + 6x^2 + 6x + 2)}{x^3(x+1)^3} < 0.
\]
This implies that $f(x + 1) - f(x)$ is decreasing for $x \geq 3$. Suppose $a \geq b \geq 4$. Then
\[
H(B_{n,a+1,b-1}) - H(B_{n,a,b}) = \left(\frac{4}{a+1} + 2\frac{(a+1) - 3}{(a+1) + 1} + \frac{4}{(b-1) + 2} + 2\frac{(b-1) - 3}{(b-1) + 1}
\right)
\left(\frac{2}{(a+1) + (b-1)}\right)^2 - \left(\frac{4}{a+2} + \frac{2(a-3)}{a+1} + \frac{4}{b+2} + \frac{2(b-3)}{b+1} + \frac{2}{a+b}\right)
\]
\[
= \left(\frac{4}{a+3} - \frac{12}{a+2} + \frac{8}{a+1}\right) - \left(\frac{4}{b+2} - \frac{12}{b+1} + \frac{8}{b}\right)
= [f(a + 2) - f(a + 1)] - [f(b + 1) - f(b)] < 0.
\]
i.e., $H(B_{n,a,b}) > H(B_{n,a+1,b-1})$ for $a \geq b \geq 4$. So we conclude that $H(B_{n,a,b}) \geq H(B_{n,n-1,3})$ with equality if and only if $a = n - 1$ and $b = 3$.

Case 2. $G$ is the bicyclic graph obtained by attaching $n - 4$ pendent vertices to one vertex of degree 2 of $B_4$.

Then
\[
H(G) - H(B_{n,n-1,3}) = \left(\frac{4}{n+1} + 2\frac{(n-4)}{n-1} + \frac{4}{5} + \frac{1}{3}\right) - \left(\frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5}\right)
\]
\[
= \frac{8}{n} - \frac{2}{n+2} - \frac{6}{n-1} + \frac{1}{3} = \left(\frac{2}{n} - \frac{2}{n+2}\right) - \frac{6}{n(n-1)} + \frac{1}{3} 
\]
\[
\geq \left(\frac{2}{n} - \frac{2}{n+2}\right) - \frac{6}{6 \cdot (6-1)} + \frac{1}{3} > 0.
\]
So Case 2 holds.

Case 3. $G \cong B_{n,a,b}'$, where $a \geq b \geq 3$ and $a + b = n + 1$.
Let $x$ be one vertex of degree 2, and let $y, z$ be the two vertices of degree at least 3 in $G$, see Figure 4. Let $G' = G - xz + yz$, then $G' \cong B_{n,a+1,b}$. By Lemma 3, we have $H(G) > H(G')$. Hence by the argument in Case 1, we deduce that $H(G) > H(B_{n,n-1,3})$. This completes the proof of the theorem. □

The following lemma was proved by Zhu, Liu and Wang [33], which will be used in the following argument.

**Lemma 7.** Let $G \in \mathcal{B}_{n,m}$ ($n > 2m \geq 6$) and $G$ contains at least one pendent vertex. Then there exists a maximum matching $M$ and a pendent vertex $u$ in $G$ such that $u$ is not $M$-saturated.
Let \( B_8 \) be the bicyclic graph on 8 vertices obtained by attaching a pendent vertex to every vertex of \( B_4 \). For \( 3 \leq m \leq \lfloor \frac{8}{3} \rfloor \), we use \( B_{n,m} \) to denote the bicyclic graph on \( n \) vertices obtained by attaching \( n - 2m + 1 \) pendent vertices and \( m - 3 \) paths on two vertices to the vertex of degree 4 of \( F_2 \), see Figure 6.

**Lemma 8.** Let \( G \in \mathcal{B}_{2m,m} \setminus \{B_8\} \) \((m \geq 3)\) and no pendent vertex has neighbor of degree 2. Then

\[
H(G) \geq \frac{2(m + 1)}{m + 4} + \frac{2}{m + 3} + \frac{2(m - 3)}{3} + 1
\]

with equality if and only if \( G \cong B_{6,3} \).

**Proof.** Let \( M \) be a maximum matching in \( G \), then \(|M| = m\) and every vertex in \( G \) is adjacent to at most one pendent vertex. Since \( G \in \mathcal{B}_{2m,m} \setminus \{B_8\} \) and no pendent vertex has neighbor of degree 2, we see that \( G \) can be obtained by attaching some pendent vertices to a bicyclic graph \( \tilde{G} \in \mathcal{B}_k \) \((m \leq k \leq 2m)\). We consider two cases according to \( G \) contains vertices of degree 2 or not.

**Case 1.** There is no vertex of degree 2 in \( G \).

Then either \( k = m \) or \( k = m + 1 \). If \( k = m \), then \( G \) can be obtained by attaching a pendent vertex to every vertex of a bicyclic graph \( \tilde{G} \in \mathcal{B}_m \). If \( k = m + 1 \), then \( G \) can be obtained by attaching a pendent vertex to every vertex of degree 2 of a bicyclic graph \( \tilde{G} \in \mathcal{B}_m \setminus \{B_4\} \).

**Figure 7.** The graphs \( Q_1 \) and \( Q_2 \).

If \( m = 3 \), then \( \tilde{G} \cong B_4 \) and \( G \cong Q_1 \) (see Figure 7). Since \( H(Q_1) = \frac{8}{3} = \frac{52}{21} = \frac{2(3+1)}{3+4} + \frac{2}{3+3} + \frac{2(3-3)}{3} + 1 \), we know that the lemma holds.
If \( m = 4 \), since we assume \( G \not\cong B_8 \), we have \( \tilde{G} \cong F_1 \) and \( G \cong Q_2 \) (see Figure 7). So the assertion of the lemma holds because
\[
H(Q_2) = \frac{7}{2} \geq \frac{269}{84} = \frac{2(4+1)}{4+4} + \frac{2}{4+3} + \frac{2(4-3)}{3} + 1.
\]

Now assume that \( m \geq 5 \). Then
\[
H(G) = \begin{cases}
\frac{5m}{6} - \frac{59}{420}, & \text{if } \tilde{G} \in \mathcal{B}_m^1 \cup \mathcal{B}_m^4, \\
\frac{5m}{6} - \frac{16}{105}, & \text{if } \tilde{G} \in \mathcal{B}_m^2 \cup \mathcal{B}_m^5, \\
\frac{5m}{6} - \frac{6}{5}, & \text{if } \tilde{G} \in \mathcal{B}_m^3, \\
\frac{5m}{6} + \frac{4}{5}, & \text{if } \tilde{G} \in \mathcal{B}_{m+1}^1 \cup \mathcal{B}_{m+1}^4.
\end{cases}
\]

Let \( f(x) = \left( \frac{5x}{6} - \frac{1}{6} \right) - \left( \frac{2(x+1)}{x+4} + \frac{2}{x+3} + \frac{2(x-3)}{3} + 1 \right) = \frac{x}{6} + \frac{6}{x+4} - \frac{2}{x+3} - \frac{7}{6}. \) For \( x \geq 5 \), we have
\[
f'(x) = \frac{1}{6} - \frac{6}{(x+4)^2} + \frac{2}{(x+3)^2} \geq \frac{1}{6} - \frac{6}{(5+4)^2} + \frac{2}{(x+3)^2} > 0.
\]

This implies that \( f(x) \) is increasing for \( x \geq 5 \), and thus \( f(m) \geq f(5) = \frac{1}{12} > 0 \), i.e.,
\[
H(G) > \frac{2(m+1)}{m+4} + \frac{2}{m+3} + \frac{2(m-3)}{3} + 1.
\]

**Case 2.** There exists a vertex, say \( u \), of degree 2 in \( G \).

Let \( v \) and \( w \) be the two neighbors of \( u \) in \( G \) such that \( d(v) = s \geq 2 \) and \( d(w) = t \geq 2 \). By the symmetry between \( v \) and \( w \), we may assume that \( uv \in M \).

Suppose that no vertex of degree 2 is contained in the cycles of \( G \). Since no pendant vertex has neighbor of degree 2 in \( G \), we conclude that \( \tilde{G} \in \mathcal{B}_s^2 \) and \( u \) lies on the path connecting two vertex-disjoint cycles of \( G \). Hence \( vw \notin E(\tilde{G}) \). Let \( G' = G - uv + vw \); then \( G' \in \mathcal{B}_{2m,m} \setminus \{B_8\} \). By Lemma 3, we have \( H(G) > H(G') \). Comparing with the graph \( G \), we see that the number of vertices of degree 2 in \( G' \) decreases by 1. Repeating this operation from \( G \) to \( G' \), we finally obtain a bicyclic graph described in Case 1, and hence the result holds.

So we may choose \( u \) such that \( u \) lies on some cycle of \( G \). Let \( N(u) = \{w_0 = u, w_1, \ldots, w_{t-1}\} \), and let \( G'' = G - uw \). Then \( G'' \) is a unicyclic graph on \( 2m \) vertices with a perfect matching \( M \), i.e., \( G'' \in \mathcal{W}_{2m,m} \). Note that \( 2 \leq s, t \) and \( w \) is adjacent to at most one pendant vertex. Since \( \frac{2}{s+2} - \frac{2}{s+1} \) is increasing for \( s \geq 2, \frac{2}{t+2} - \frac{2}{t-1+2} \) is increasing for \( x \geq 1 \) and by Lemma 2(i), we have
\[
H(G) = H(G'') + \sum_{i=1}^{t-1} \left( \frac{2}{t + d(w_i)} - \frac{2}{t - 1 + d(w_i)} \right) + \frac{2}{t+2} + \left( \frac{2}{s+2} - \frac{2}{s+1} \right)
\]
\[
\geq H(G'') + \left( \frac{2}{t+1} - \frac{2}{t} \right) + (t-2) \left( \frac{2}{t+2} - \frac{2}{t+1} \right) + \frac{2}{t+2}
\]
\[
+ \left( \frac{2}{2+2} - \frac{2}{2+1} \right)
\]
\[
= H(G'') + \left( \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} \right) - \frac{1}{6}
\]
\[ H(G'') \geq H(G'') + \left( \frac{2 \cdot (5 - 1)}{5 + 2} - \frac{2 \cdot (5 - 3)}{5 + 1} - \frac{2}{5} \right) - \frac{1}{6} \]

\[ = H(G'') - \frac{19}{210} \quad (*) \]

with equalities if and only if \( s = 2, t = 5 \), one neighbor of \( w \) has degree 1 and the other neighbors of \( w \) have degree 2.

![Figure 8. The graphs \( R_1 \) and \( R_2 \).](image)

If \( G'' \cong U_6 \), then either \( G'' \cong R_1 \) or \( G'' \cong R_2 \) (see Figure 8). Since \( H(R_1) = \frac{14}{5} > H(R_2) = \frac{533}{210} \), the assertion of the lemma holds. If \( G'' \cong U_8 \), then by (\( * \)), we have

\[ H(G) \geq H(U_8) - \frac{19}{210} = \frac{347}{105} \quad \text{or} \quad \frac{19}{210} = \frac{45}{14} \]

and the result holds. So suppose that \( G'' \not\cong U_6, U_8 \). It follows from Lemma 2(i) that

\[ \frac{2[(m + 2) - 1]}{(m + 2) + 2} - \frac{2[(m + 2) - 3]}{(m + 2) + 1} - \frac{2}{m + 2} \leq \frac{2 \cdot [(3 + 2) - 1]}{(3 + 2) + 2} - \frac{2 \cdot [(3 + 2) - 3]}{(3 + 2) + 1} - \frac{2}{3 + 2} = \frac{8}{105} \]

since \( m \geq 3 \). Then by (\( * \)) and Theorem 1, we have

\[ H(G) \geq H(G'') - \frac{19}{210} \]

\[ \geq \left( \frac{2m}{m + 3} + \frac{2}{m + 2} + \frac{2(m - 2)}{3} + \frac{1}{2} \right) - \frac{19}{210} \]

\[ = \left( \frac{2m}{m + 3} + \frac{2}{m + 2} + \frac{2(m - 3)}{3} + 1 \right) + \frac{8}{105} \]

\[ \geq \left( \frac{2m}{m + 3} + \frac{2}{m + 2} + \frac{2(m - 3)}{3} + 1 \right) \]
\[ \frac{2(m+1)}{m+4} + \frac{2}{m+3} + \frac{2(m-3)}{3} + 1 \]

with equalities if and only if \( s = 2, t = 5, G'' \cong U_{2m,m} \) and \( m = 3 \), i.e., \( G \cong B_{6,3} \).

This finishes the proof of the lemma.

\[ \square \]

**Theorem 4.** Let \( G \in \mathcal{B}_{2m,m} \setminus \{B_8\} \), where \( m \geq 3 \). Then

\[
H(G) \geq \frac{2(m+1)}{m+4} + \frac{2}{m+3} + \frac{2(m-3)}{3} + 1
\]

with equality if and only if \( G \cong B_{2m,m} \).

**Proof.** We prove Theorem 4 by induction on \( m \). If \( m = 3 \), then by Lemma 7, we may assume that there exists a pendent vertex in \( G \) whose neighbor is a vertex of degree 2. Hence \( G \) is the bicyclic graph obtained from \( B_4 \) by attaching a path on two vertices to either one vertex of degree 3 or one vertex of degree 2. Then we have \( H(G) \geq \frac{289}{105} > \frac{52}{27} = H(B_{6,3}) \), and the assertion of the theorem holds. So we may assume that \( m \geq 4 \) and the result holds for graphs in \( \mathcal{B}_{2(m-1),m-1} \setminus \{B_8\} \). Let \( M \) be a maximum matching in \( G \), then \( |M| = m \). If no pendent vertex has neighbor of degree 2 in \( G \), then by Lemma 7, we see that the result holds.

Now suppose that there exists a pendent vertex \( u \) in \( G \) whose neighbor \( v \) is a vertex of degree 2. Let \( w \) be the neighbor of \( v \) different from \( u \) with \( d(w) = t \geq 2 \). Hence \( G' = G - u - v \). Then \( uv \in M \) and \( G' \in \mathcal{B}_{2(m-1),m-1} \). Since \( M \) contains exactly one edge incident with \( w \) and there are \( m+1 \) edges of \( G \) outside \( M \), we have \( t \leq m+2 \). Note that \( w \) is adjacent to at most one pendent vertex in \( G \).

If \( G' \cong B_8 \), then \( t \leq 5 \). By Lemma 1(i) and Lemma 2(i), we have

\[
H(G) \geq H(B_8) + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} = \frac{2}{t+3} + \frac{2}{3} = \frac{551}{140} > \frac{47}{12} = H(U_{10,5}),
\]

and hence the assertion of the theorem holds.

So we may further assume that \( G' \not\cong B_8 \). Then by Lemma 1(i), Lemma 2(i) and the induction hypothesis, we conclude that

\[
H(G) \geq H(G') + \left( \frac{2[(m-1)+1]}{(m-1)+4} + \frac{2[(m-1)+3]}{m+2} + \frac{2}{m+2} \right)
\]

and hence the assertion of the theorem holds.
with equalities if and only if \( G' \cong B_{2(m-1),m-1} \) and \( t = m + 2 \), i.e., \( G \cong B_{2m,m} \). So Theorem 4 holds.

Since \( H(B_{8,4}) = \frac{269}{34} > \frac{447}{140} = H(B_8) \), by Theorem 4, we immediately obtain the following result.

**Corollary 4.** Let \( G \in \mathcal{B}_{8,4} \), then \( H(G) \geq \frac{447}{140} \) with equality if and only if \( G \cong B_8 \).

We now present the minimum harmonic index for graphs in \( \mathcal{B}_{n,m} \setminus \{B_8\} \), where \( 3 \leq m \leq \lfloor \frac{n}{2} \rfloor \).

**Theorem 5.** Let \( G \in \mathcal{B}_{n,m} \setminus \{B_8\} \), where \( 3 \leq m \leq \lfloor \frac{n}{2} \rfloor \). Then

\[
H(G) \geq \frac{2(m+1)}{n-m+4} + \frac{2(n-2m+1)}{n-m+3} + \frac{2(m-3)}{3} + 1
\]

with equality if and only if \( G \cong B_{n,m} \).

**Proof.** We prove the theorem by induction on \( n \). If \( n = 2m \), then by Theorem 4, the assertion of the theorem holds. So we may assume that \( n > 2m \) and the result holds for graphs in \( \mathcal{B}_{n-1,m} \setminus \{B_8\} \). If there is no pendent vertex in \( G \), then \( G \cong B_{n,m} \). It is easy to check that

\[
H(G) = \begin{cases} 
\frac{m + 13}{30}, & \text{if } G \in \mathcal{B}_{2m+1}^1 \cup \mathcal{B}_{2m+1}^4, \\
\frac{m + 7}{11}, & \text{if } G \in \mathcal{B}_{2m+1}^2 \cup \mathcal{B}_{2m+1}^3, \\
\frac{m + 3}{5}, & \text{if } G \in \mathcal{B}_{2m+1}^3.
\end{cases}
\]

This implies that

\[
H(G) - H(B_{2m+1,m}) \geq \left( \frac{m + 1}{3} \right) - \left( \frac{2(m+1)}{(2m+1)-m+4} + \frac{2[(2m+1)-2m+1]}{(2m+1)-m+3} + \frac{2(m-3)}{3} + 1 \right)
\]

\[
= \frac{m}{3} + \frac{8}{m+5} - \frac{4}{m+4} - \frac{2}{3} = \frac{m-2}{3} + \frac{4(m+3)}{(m+4)(m+5)} > 0,
\]

i.e., \( H(G) > H(B_{2m+1,m}) \).

So we may assume that \( G \) contains at least one pendent vertex. Then by Lemma 7, there exists a maximum matching \( M \) and a pendent vertex \( u \) in \( G \) such that \( u \) is not \( M \)-saturated. Let \( v \) be the unique neighbor of \( u \) with \( d(v) = s \geq 2 \), and let \( G' = G - u \). Then \( G' \in \mathcal{B}_{n-1,m} \). Since \( M \) contains exactly one edge incident with \( v \) and there are \( n+1-m \) edges of \( G \) outside \( M \), we have \( s \leq n-m+2 \). Let \( r \) be the number of pendant neighbors of \( v \) in \( G \), where \( 1 \leq r \leq s-1 \). Note that at least \( r-1 \) pendant neighbors of \( v \) are not \( M \)-saturated, and there are \( n-2m \) vertices are not \( M \)-saturated in \( G \). Then \( r \leq n-2m+1 \).
If $G' \cong B_8$, then $n = 9$, $m = 4$ and $s \leq 5$. By Lemma 1(ii) (with $k = n - 2m + 1 = 2$) and Lemma 2(ii), we deduce that

$$H(G) \geq H(B_8) + \frac{2(s - 2)}{s + 2} + \frac{2(4 - s)}{s + 1} - \frac{2}{s}$$

$$\geq \frac{447}{140} + \frac{2 \cdot (5 - 2)}{5 + 2} + \frac{2 \cdot (4 - 5)}{5 + 1} - \frac{2}{5} = \frac{1393}{420} > \frac{59}{18} = H(B_{9,4}),$$

and hence the assertion of the theorem holds.

Therefore we may assume that $G' \not\cong B_8$. Then by Lemma 1(ii) (with $k = n - 2m + 1$), Lemma 2(ii) and the induction hypothesis, we have

$$H(G) \geq H(G') + \frac{2[s - (n - 2m + 1)]}{s + 2} + \frac{2(2n - 2m + 1) - s}{s + 1}$$

$$\geq \frac{2(m + 1)}{(n - 1) - m + 4} + \frac{2[(n - 1) - 2m + 1]}{(n - 1) - m + 3} + \frac{2(m - 3)}{3} + 1$$

$$+ \frac{2[(n - m + 2) - (n - 2m + 1)]}{(n - m + 2) + 2} + \frac{2[2(n - 2m + 1) - (n - m + 2)]}{(n - m + 2) + 1}$$

$$- \frac{2[(n - 2m + 1) - 1]}{n - m + 2}$$

$$= \frac{2(m + 1)}{n - m + 4} + \frac{2(n - 2m + 1)}{n - m + 3} + \frac{2(m - 3)}{3} + 1$$

with equalities if and only if $G' \cong B_{n-1,m}$, $s = n - m + 2$ and $r = n - 2m + 1$, i.e., $G \cong B_{n,m}$. This completes the proof of the theorem. \(\square\)

We can also determine the minimum harmonic index for graphs in $B_n$ (see also in [31]) by using Theorem 3 and Theorem 5.

**Corollary 5.** Let $G \in B_n$ with $n \geq 4$. Then

$$H(G) \geq \frac{2}{n + 2} + \frac{4}{n + 1} + \frac{2(n - 4)}{n} + \frac{4}{5}$$

with equality if and only if $G \cong B_{n,n-1,3}$.

**Proof.** Let $M$ be a maximum matching in $G$, then $2 \leq |M| = m \leq \left\lfloor \frac{n}{2} \right\rfloor$ (since $n \geq 4$). If $m = 2$, then the result follows immediately from Theorem 3.

If $m = 3$, then by Theorem 5, we have

$$H(G) \geq \frac{2 \cdot (3 + 1)}{n - 3 + 4} + \frac{2(n - 2 \cdot 3 + 1)}{n - 3 + 3} + \frac{2 \cdot (3 - 3)}{3} + 1$$

$$= \frac{8}{n + 1} + \frac{2(n - 5)}{n} + 1$$
with equality if and only if \( G \cong B_{n,3} \). Note that in this case \( n \geq 6 \). Since
\[
H(B_{n,3}) - H(B_{n,n-1,3}) = \left( \frac{8}{n+1} + \frac{2(n-5)}{n} + 1 \right) - \left( \frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5} \right)
\]
\[
= \left( \frac{4}{n+1} - \frac{2}{n+2} - \frac{2}{n} \right) + \frac{1}{5} = \frac{-4}{n(n+1)(n+2)} + \frac{1}{5}
\]
\[
\geq \frac{79}{420} > 0,
\]
we know that the assertion of the corollary holds.

So we may assume that \( m \geq 4 \). If \( G \cong B_8 \), then \( H(G) = \frac{447}{140} > \frac{22}{7} = H(B_{8,7,3}) \), we see that Corollary 5 holds. Now suppose that \( G \not\cong B_8 \). Then by Theorem 5 and Lemma 3, we see that \( H(G) \geq H(B_{n,m}) > H(B_{n,m-1}) \) \( \cdots > H(B_{n,3}) > H(B_{n,n-1,3}) \). This finishes the proof of the corollary. \( \square \)

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