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THE HARMONIC INDEX FOR UNICYCLIC AND BICYCLIC GRAPHS WITH GIVEN MATCHING NUMBER

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Abstract. The harmonic index of a graph G is defined as the sum of the weights $\frac{2}{d(u)+d(v)}$ of all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G. In this paper, we present the minimum harmonic indices for unicyclic and bicyclic graphs with n vertices and matching number m ($2 \leq m \leq \lfloor \frac{n}{2} \rfloor$), respectively. The corresponding extremal graphs are also characterized.

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Keywords: harmonic index, unicyclic graph, bicyclic graph, matching number

1. INTRODUCTION

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Randić index $R(G)$, proposed by Randić [20] in 1975, is defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}},$$

where $d(u)$ denotes the degree of a vertex u of G. The Randić index is one of the most successful molecular descriptors in structure-property and structure-activity relationship studies. Mathematical properties of this descriptor have been studied extensively (see [9, 10, 14, 15, 19] and the references cited therein).

In this paper, we consider a closely related variant of the Randić index, named the harmonic index. For a graph G, the harmonic index $H(G)$ is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u)+d(v)}.$$

This index first appeared in [6], and it can also be viewed as a particular case of the general sum-connectivity index proposed by Zhou and Trinajstić in [32].

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Favaron, Mahéo and Saclé [7] considered the relation between the harmonic index and the eigenvalues of graphs. Zhong [28, 29], Zhong and Xu [30] determined the minimum and maximum harmonic indices for simple connected graphs, trees, unicyclic and bicyclic graphs, and characterized the corresponding extremal graphs. Wu, Tang and Deng [23] found the minimum harmonic index for graphs (triangle-free graphs, respectively) with minimum degree at least 2, and characterized the corresponding extremal graphs. Deng, Balachandran, Ayyaswamy and Venkatakrishnan [2] considered the relation between the harmonic index and the chromatic number of a graph by using the effect of removal of a minimum degree vertex on the harmonic index. Liu [17] proposed a conjecture concerning the relation between the harmonic index and the diameter of a connected graph, and showed that the conjecture is true for trees. Ilić [12], Xu [25], Zhong and Xu [31] established some relationships between the harmonic index and several other topological indices. The chemical applicability of the harmonic index was also recently investigated [8, 11]. See [3, 18, 24, 26] for more information of this index.

In this paper, we determine the minimum harmonic indices for unicyclic and bicyclic graphs with n vertices and matching number m $(2 \leq m \leq \lfloor \frac{n}{2} \rfloor)$, respectively. The corresponding extremal graphs are also characterized. The related problems have been well-studied for several other topological indices, such as the Randić index [16, 33], the modified Randić index [13] and the sum-connectivity index [4, 5, 21, 22].

2. Preliminaries

Let G be a graph. For any vertex $v \in V(G)$, we use $N_G(v)$ (or $N(v)$ if there is no ambiguity) to denote the set of neighbors of v in G. A pendent vertex is a vertex of degree 1. For two distinct vertices u and v of G, the distance $d(u, v)$ between u and v is the number of edges in a shortest path joining u and v in G. A unicyclic graph is a connected graph with n vertices and n edges, and a bicyclic graph is a connected graph with n vertices and $n + 1$ edges. We use C_n to denote the cycle on n vertices.

A matching M in a graph G is a subset of $E(G)$ such that no two edges in M share a common vertex. A matching M in G is said to be maximum, if for any other matching M' in G, $|M'| \leq |M|$. The matching number of G is the number of edges in a maximum matching of G. If M is a matching in G and the vertex $v \in V(G)$ is incident with an edge of M, then v is said to be M-saturated, and if every vertex in G is M-saturated, then M is a perfect matching.

For any vertex $v \in V(G)$, we use $G - v$ to denote the graph resulting from G by deleting the vertex v and its incident edges. We define $G - uv$ to be the graph obtained from G by deleting the edge $uv \in E(G)$, and $G + uv$ to be the graph obtained from G by adding an edge uv between two non-adjacent vertices u and v of G.

We now establish some lemmas which will be used frequently in later proofs.

Lemma 1. *Let G be a connected graph on $n \geq 4$ vertices with a pendent vertex u. Let v be the unique neighbor of u with $d(v) = s$, and let w be a neighbor of v different from u with $d(w) = t$.*

(i) *If $s = 2$ and w is adjacent to at most one pendent vertex in G, then*

$$H(G) \geq H(G-u-v) + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3}$$

with equality if and only if one neighbor of w has degree 1 and the other neighbors of w have degree 2.

(ii) *If v is adjacent to at most k pendent vertices in G, then*

$$H(G) \geq H(G-u) + \frac{2(s-k)}{s+2} + \frac{2(2k-s)}{s+1} - \frac{2(k-1)}{s}$$

with equality if and only if k neighbors of v have degree 1 and the other neighbors of v have degree 2.

Proof. (i) Let $N(w) = \{w_0 = v, w_1, \ldots, w_{t-1}\}$. Since w is adjacent to at most one pendent vertex in G, we may assume that $d(w_1) \geq 1$, and $d(w_i) \geq 2$ for each $2 \leq i \leq t-1$ (if $t \geq 3$). Note that $\frac{2}{t+x} - \frac{2}{t-1+x}$ is increasing for $x \geq 1$, we have

$$\begin{aligned} H(G) &= H(G-u-v) + \sum_{i=1}^{t-1}\left(\frac{2}{t+d(w_i)} - \frac{2}{t-1+d(w_i)}\right) + \frac{2}{t+2} + \frac{2}{3} \\ &\geq H(G-u-v) + \left(\frac{2}{t+1} - \frac{2}{t}\right) + (t-2)\left(\frac{2}{t+2} - \frac{2}{t+1}\right) + \frac{2}{t+2} + \frac{2}{3} \\ &= H(G-u-v) + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3} \end{aligned}$$

with equality if and only if $d(w_1) = 1$ and $d(w_i) = 2$ for each $2 \leq i \leq t-1$ (if $t \geq 3$). This proves (i).

(ii) Let r $(1 \leq r \leq k)$ be the number of pendent neighbors of v in G, and let $N(v) = \{v_0 = u, v_1, \ldots, v_{s-1}\}$. Without loss of generality, we may assume that $d(v_i) = 1$ for each $1 \leq i \leq r-1$ (if $r \geq 2$), and $d(v_i) \geq 2$ for each $r \leq i \leq s-1$ (if $s \geq r+1$). Note that $\frac{2}{s+x} - \frac{2}{s-1+x}$ is increasing for $x \geq 1$ and $\frac{4}{s+1} - \frac{2}{s+2} - \frac{2}{s} < 0$, we have

$$\begin{aligned} H(G) &= H(G-u) + (r-1)\left(\frac{2}{s+1} - \frac{2}{s}\right) \\ &\quad + \sum_{i=r}^{s-1}\left(\frac{2}{s+d(v_i)} - \frac{2}{s-1+d(v_i)}\right) + \frac{2}{s+1} \\ &\geq H(G-u) + (r-1)\left(\frac{2}{s+1} - \frac{2}{s}\right) + (s-r)\left(\frac{2}{s+2} - \frac{2}{s+1}\right) + \frac{2}{s+1} \\ &= H(G-u) + r\left(\frac{4}{s+1} - \frac{2}{s+2} - \frac{2}{s}\right) + \frac{2s}{s+2} - \frac{2s}{s+1} + \frac{2}{s} \end{aligned}$$

$$\geq H(G-u)+k\left(\frac{4}{s+1}-\frac{2}{s+2}-\frac{2}{s}\right)+\frac{2s}{s+2}-\frac{2s}{s+1}+\frac{2}{s}$$
$$= H(G-u)+\frac{2(s-k)}{s+2}+\frac{2(2k-s)}{s+1}-\frac{2(k-1)}{s}$$

with equalities if and only if $r=k$ and $d(v_i)=2$ for each $k \leq i \leq s-1$ (if $s \geq k+1$). This completes the proof of the lemma. □

Lemma 2. *(i) The function $\frac{2(x-1)}{x+2}-\frac{2(x-3)}{x+1}-\frac{2}{x}$ is decreasing for $x \geq 2$.*
(ii) For $k \geq 1$, the function $\frac{2(x-k)}{x+2}+\frac{2(2k-x)}{x+1}-\frac{2(k-1)}{x}$ is decreasing for $x \geq k+1$.

Proof. (i) Let $f(x)=\frac{2(x-1)}{x+2}-\frac{2(x-3)}{x+1}-\frac{2}{x}=\frac{8}{x+1}-\frac{6}{x+2}-\frac{2}{x}$. For $x \geq 2$, we have

$$f'(x) = -\frac{8}{(x+1)^2}+\frac{6}{(x+2)^2}+\frac{2}{x^2}=\frac{-8x^3+24x+8}{x^2(x+1)^2(x+2)^2}$$
$$=\frac{-8x(x^2-4)-8(x-1)}{x^2(x+1)^2(x+2)^2}<0,$$

and hence (i) holds.

(ii) Let $g(x)=\frac{2(x-k)}{x+2}+\frac{2(2k-x)}{x+1}-\frac{2(k-1)}{x}$ and $g_1(x)=\frac{2(k-1)}{x}+\frac{2(x-1-k)}{x+1}$. Then $g(x)=g_1(x+1)-g_1(x)$. For $x \geq k+1 \geq 2$, we have

$$g_1''(x)=\frac{4(k-1)}{x^3}-\frac{4(k+2)}{(x+1)^3}=\frac{-12x^3+12(k-1)x^2+12(k-1)x+4(k-1)}{x^3(x+1)^3}$$
$$=\frac{-12x^2(x-k)-12x(x-k)-4(3x-k+1)}{x^3(x+1)^3}<0,$$

and $g'(x)=g_1'(x+1)-g_1'(x)<0$. So the assertion of the lemma holds. □

Lemma 3. *Let G be a connected graph, and let u be a vertex of degree 2 in G with two neighbors v and w such that $d(v) \geq 2$ and $vw \notin E(G)$. Let $G'=G-uw+vw$, then $H(G)>H(G')$.*

Proof. Let $d(v)=p \geq 2$ and let $N(v)=\{v_0=u, v_1, \ldots, v_{p-1}\}$. Then

$$H(G)-H(G')$$
$$=\left(\sum_{i=1}^{p-1}\frac{2}{p+d(v_i)}+\frac{2}{2+d(w)}\right)-\left(\sum_{i=1}^{p-1}\frac{2}{p+1+d(v_i)}+\frac{2}{p+1+d(w)}\right)$$
$$=\sum_{i=1}^{p-1}\left(\frac{2}{p+d(v_i)}-\frac{2}{p+1+d(v_i)}\right)+\left(\frac{2}{2+d(w)}-\frac{2}{p+1+d(w)}\right)>0.$$

This proves the lemma. □

3. MINIMUM HARMONIC INDEX FOR UNICYCLIC GRAPHS WITH GIVEN MATCHING NUMBER

Let $\mathscr{U}_n$ be the set of unicyclic graphs with $n \geq 3$ vertices, and let $\mathscr{U}_{n,m}$ be the set of unicyclic graphs with n vertices and matching number m, where $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. In this section, we determine the minimum harmonic index for graphs in $\mathscr{U}_{n,m}$, and characterize the corresponding extremal graphs.

For a unicyclic graph G with the cycle C_p, the forest obtained from G by deleting the edges in C_p consists of p vertex-disjoint trees, each containing a vertex of C_p, which is called the root of this tree in G. These trees are called the branches of G. Chang and Tian [1] showed the following lemma.

Lemma 4. *Let $G \in \mathscr{U}_{2m,m}$ $(m \geq 3)$, and let T be a branch of G with root r. If $u \in V(T)$ is a pendent vertex which is furthest from the root r with $d(u,r) \geq 2$, then u is adjacent to a vertex of degree 2.*

The second lemma was proved by Yu and Tian [27].

Lemma 5. *Let $G \in \mathscr{U}_{n,m}$ $(n > 2m)$ and $G \ncong C_n$. Then there exists a maximum matching M and a pendant vertex u in G such that u is not M-saturated.*

Zhong [29] proved the following result.

Lemma 6. *Let $G \in \mathscr{U}_n$ with $n \geq 3$. Then $H(G) \leq \frac{n}{2}$ with equality if and only if $G \cong C_n$.*

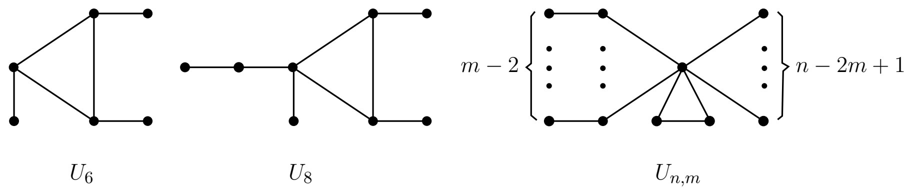

FIGURE 1. The graphs U_6, U_8 and $U_{n,m}$.

Let U_6 be the unicyclic graph on 6 vertices obtained by attaching a pendent vertex to every vertex of a triangle, and let U_8 be the unicyclic graph on 8 vertices obtained by attaching a path on two vertices to one vertex of degree 3 of U_6. For $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$, we use $U_{n,m}$ to denote the unicyclic graph on n vertices obtained by attaching $n-2m+1$ pendent vertices and $m-2$ paths on two vertices to one vertex of a triangle. See Figure 1 for an illustration.

Theorem 1. *Let $G \in \mathscr{U}_{2m,m} \setminus \{U_6, U_8\}$, where $m \geq 2$. Then*

$$H(G) \geq \frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-2)}{3} + \frac{1}{2}$$

with equality if and only if $G \cong U_{2m,m}$.

Proof. We prove the theorem by induction on m. If $m = 2$, then either $G \cong C_4$ or $G \cong U_{4,2}$. Since $H(C_4) = 2 > \frac{9}{5} = H(U_{4,2})$, we see that the assertion of the theorem holds. So we may assume that $m \geq 3$ and the result holds for graphs in $\mathscr{U}_{2(m-1),m-1} \setminus \{U_6, U_8\}$. By Lemma 6, since C_{2m} is the unique unicyclic graph on $2m$ vertices with the maximum harmonic index, we may further assume that $G \ncong C_{2m}$. Let M be a maximum matching in G, then $|M| = m$. By Lemma 4, we need only consider the following two cases.

Case 1. There exists a pendent vertex u in G which is adjacent to a vertex v of degree 2.

Let w be the neighbor of v different from u with $d(w) = t \geq 2$, and let $G' = G - u - v$. Then $uv \in M$ and $G' \in \mathscr{U}_{2(m-1),m-1}$. Since M contains exactly one edge incident with w and there are m edges of G outside M, we have $t \leq m+1$. Note that w is adjacent to at most one pendent vertex in G.

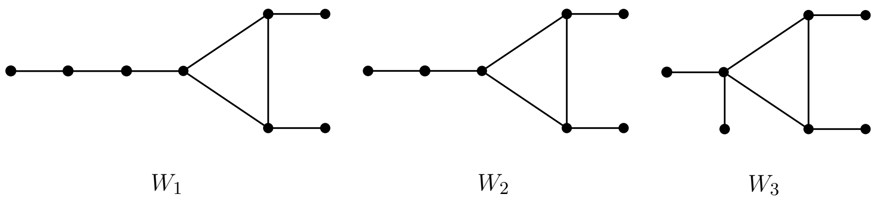

FIGURE 2. The graphs W_1, W_2 and W_3.

If $G' \cong U_6$, then we have $G \cong W_1$ (since we assume $G \ncong U_8$), see Figure 2. Since $H(W_1) = \frac{107}{30} > \frac{139}{42} = H(U_{8,4})$, we know that the result holds.

If $G' \cong U_8$, then $t \leq 5$. By Lemma 1(i) and Lemma 2(i), we have

$$\begin{aligned} H(G) &\geq H(U_8) + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3} \\ &\geq \frac{347}{105} + \frac{2\cdot(5-1)}{5+2} - \frac{2\cdot(5-3)}{5+1} - \frac{2}{5} + \frac{2}{3} = \frac{85}{21} > \frac{113}{28} = H(U_{10,5}), \end{aligned}$$

and hence the assertion of the theorem holds.

Now suppose that $G' \ncong U_6, U_8$. Then by Lemma 1(i), Lemma 2(i) and the induction hypothesis, we conclude that

$$\begin{aligned} H(G) &\geq H(G') + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3} \\ &\geq \left(\frac{2(m-1)}{(m-1)+3} + \frac{2}{(m-1)+2} + \frac{2[(m-1)-2]}{3} + \frac{1}{2}\right) \\ &\quad + \frac{2[(m+1)-1]}{(m+1)+2} - \frac{2[(m+1)-3]}{(m+1)+1} - \frac{2}{m+1} + \frac{2}{3} \end{aligned}$$

$$= \frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-2)}{3} + \frac{1}{2}$$

with equalities if and only if $G' \cong U_{2(m-1),m-1}$ and $t = m+1$, i.e., $G \cong U_{2m,m}$. This proves Case 1.

Case 2. G is a unicyclic graph with maximum degree 3 obtained by attaching $2m-p$ pendent vertices to some vertices of a cycle C_p $(m \le p \le 2m-1)$.

If $m = 3$, then G is either the unicyclic graph obtained by attaching a pendent vertex to one vertex of C_5 or the unicyclic graph obtained by attaching a pendent vertex to two adjacent vertices of C_4 (since we assume $G \ncong U_6$). Then we have $H(G) \ge \frac{79}{30} > \frac{77}{30} = H(U_{6,3})$, and the theorem holds. So we may assume that $m \ge 4$. We consider two subcases according to the value of p.

Subcase 2.1. $p = m$.

Then every vertex of C_p is attached by a pendent vertex and $H(G) = \frac{5m}{6}$. Let $f(x) = \frac{5x}{6} - \left(\frac{2x}{x+3} + \frac{2}{x+2} + \frac{2(x-2)}{3} + \frac{1}{2}\right) = \frac{x}{6} + \frac{6}{x+3} - \frac{2}{x+2} - \frac{7}{6}$. For $x \ge 4$, we have

$$f'(x) = \frac{1}{6} - \frac{6}{(x+3)^2} + \frac{2}{(x+2)^2} \ge \frac{1}{6} - \frac{6}{(4+3)^2} + \frac{2}{(x+2)^2} > 0.$$

This implies that $f(x)$ is increasing for $x \ge 4$, and thus $f(m) \ge f(4) = \frac{1}{42} > 0$, i.e., $H(G) > H(U_{2m,m})$.

Subcase 2.2. $m+1 \le p \le 2m-1$.

In this subcase, there exists at least one edge, say xy, on C_p such that $xy \in M$. Then $d(x) = d(y) = 2$; for otherwise, the pendent vertex adjacent to x or y can not be M-saturated. Let z be the neighbor of x different from y in G, and let $G'' = G - xz + yz$. Then $G'' \in \mathscr{U}_{2m,m} \setminus \{U_8\}$. By Lemma 3, we have $H(G) > H(G'')$. Comparing with the graph G, we see that the length of the unique cycle in G'' decreases by 1. Repeating this operation from G to G'', we eventually obtain the unicyclic graph described in Subcase 2.1 and the result holds. This finishes the proof of the theorem. □

Since $H(U_{6,3}) = \frac{77}{30} > \frac{5}{2} = H(U_6)$ and $H(U_{8,4}) = \frac{139}{42} > \frac{347}{105} = H(U_8)$, by Theorem 1, we immediately obtain the following two results.

Corollary 1. *Let $G \in \mathscr{U}_{6,3}$, then $H(G) \ge \frac{5}{2}$ with equality if and only if $G \cong U_6$.*

Corollary 2. *Let $G \in \mathscr{U}_{8,4}$, then $H(G) \ge \frac{347}{105}$ with equality if and only if $G \cong U_8$.*

We now prove the main result of this section.

Theorem 2. *Let $G \in \mathscr{U}_{n,m} \setminus \{U_6, U_8\}$, where $2 \le m \le \lfloor \frac{n}{2} \rfloor$. Then*

$$H(G) \ge \frac{2m}{n-m+3} + \frac{2(n-2m+1)}{n-m+2} + \frac{2(m-2)}{3} + \frac{1}{2}$$

with equality if and only if $G \cong U_{n,m}$.

Proof. We prove Theorem 2 by induction on n. If $n = 2m$, then by Theorem 1, the assertion of the theorem holds. So we may assume that $n > 2m$ and the result holds for graphs in $\mathscr{U}_{n-1,m} \setminus \{U_6, U_8\}$. By Lemma 6, since C_n is the unique unicyclic graph on n vertices with the maximum harmonic index, we may also assume that $G \ncong C_n$. Then by Lemma 5, there exists a maximum matching M and a pendant vertex u in G such that u is not M-saturated. Let v be the unique neighbor of u with $d(v) = s \geq 2$, and let $G' = G - u$. Then $G' \in \mathscr{U}_{n-1,m}$. Since M contains exactly one edge incident with v and there are $n-m$ edges of G outside M, we have $s \leq n-m+1$. Let r be the number of pendant neighbors of v in G, where $1 \leq r \leq s-1$. Note that at least $r-1$ pendant neighbors of v are not M-saturated, and there are $n-2m$ vertices are not M-saturated in G. Then $r \leq n-2m+1$.

If $G' \cong U_6$, then $n = 7$, $m = 3$ and either $G \cong W_2$ or $G \cong W_3$ (see Figure 2). Since $H(W_2) = \frac{46}{15} > H(W_3) = \frac{284}{105} > \frac{113}{42} = H(U_{7,3})$, we see that the result holds.

If $G' \cong U_8$, then $n = 9$, $m = 4$ and $s \leq 5$. By Lemma 1(ii) (with $k = n-2m+1 = 2$) and Lemma 2(ii), we have

$$\begin{aligned} H(G) &\geq H(U_8) + \frac{2(s-2)}{s+2} + \frac{2(4-s)}{s+1} - \frac{2}{s} \\ &\geq \frac{347}{105} + \frac{2\cdot(5-2)}{5+2} + \frac{2\cdot(4-5)}{5+1} - \frac{2}{5} = \frac{24}{7} > \frac{143}{42} = H(U_{9,4}), \end{aligned}$$

and thus the assertion of the theorem holds.

Therefore we may assume that $G' \ncong U_6, U_8$. Then by Lemma 1(ii) (with $k = n-2m+1$), Lemma 2(ii) and the induction hypothesis, we conclude that

$$\begin{aligned} H(G) \geq{} & H(G') + \frac{2[s-(n-2m+1)]}{s+2} + \frac{2[2(n-2m+1)-s]}{s+1} \\ & - \frac{2[(n-2m+1)-1]}{s} \\ \geq{} & \left(\frac{2m}{(n-1)-m+3} + \frac{2[(n-1)-2m+1]}{(n-1)-m+2} + \frac{2(m-2)}{3} + \frac{1}{2}\right) \\ & + \frac{2[(n-m+1)-(n-2m+1)]}{(n-m+1)+2} + \frac{2[2(n-2m+1)-(n-m+1)]}{(n-m+1)+1} \\ & - \frac{2[(n-2m+1)-1]}{n-m+1} \\ ={} & \frac{2m}{n-m+3} + \frac{2(n-2m+1)}{n-m+2} + \frac{2(m-2)}{3} + \frac{1}{2} \end{aligned}$$

with equalities if and only if $G' \cong U_{n-1,m}$, $s = n-m+1$ and $r = n-2m+1$, i.e., $G \cong U_{n,m}$. This completes the proof of the theorem. $\square$

By applying Theorem 2, we can also obtain the minimum harmonic index for graphs in $\mathscr{U}_n$ $(n \geq 4)$. This is one of the main results in [29].

Corollary 3. *Let $G \in \mathscr{U}_n$ with $n \geq 4$. Then*

$$H(G) \geq \frac{4}{n+1} + \frac{2(n-3)}{n} + \frac{1}{2}$$

with equality if and only if $G \cong U_{n,2}$.

Proof. Let M be a maximum matching in G, then $2 \leq |M| = m \leq \lfloor \frac{n}{2} \rfloor$ (since $n \geq 4$). If $m = 2$, then by Theorem 2, we have

$$\begin{aligned} H(G) &\geq \frac{2 \cdot 2}{n-2+3} + \frac{2(n-2\cdot 2+1)}{n-2+2} + \frac{2\cdot(2-2)}{3} + \frac{1}{2} \\ &= \frac{4}{n+1} + \frac{2(n-3)}{n} + \frac{1}{2} \end{aligned}$$

with equality if and only if $G \cong U_{n,2}$. So we may assume that $m \geq 3$.

If $G \cong U_6$, then $H(G) = \frac{5}{2} > \frac{29}{14} = H(U_{6,2})$, we see that the result holds. If $G \cong U_8$, then $H(G) = \frac{347}{105} > \frac{79}{36} = H(U_{8,2})$, and the result also holds. Now suppose that $G \ncong U_6, U_8$. Then by Theorem 2 and Lemma 3, we see that $H(G) \geq H(U_{n,m}) > H(U_{n,m-1}) > \cdots > H(U_{n,2})$. So the assertion of the corollary holds. □

4. Minimum harmonic index for bicyclic graphs with given matching number

Let $\mathscr{B}_n$ be the set of bicyclic graphs with $n \geq 4$ vertices, and let $\mathscr{B}_{n,m}$ be the set of bicyclic graphs with n vertices and matching number m, where $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. In this section, we present the minimum harmonic index for graphs in $\mathscr{B}_{n,m}$, and characterize the corresponding extremal graphs.

We denote by $\tilde{\mathscr{B}}_n$ the set of bicyclic graphs with $n \geq 4$ vertices containing no pendent vertices. Let $\mathscr{B}_n^1$ be the set of bicyclic graphs on $n \geq 6$ vertices obtained by connecting two vertex-disjoint cycles by a new edge, and let $\mathscr{B}_n^2$ be the set of bicyclic graphs on $n \geq 7$ vertices obtained by connecting two vertex-disjoint cycles by a path of length at least two. Let $\mathscr{B}_n^3$ be the set of bicyclic graphs on $n \geq 5$ vertices obtained by identifying a vertex of a cycle and a vertex of the other cycle. Let $\mathscr{B}_n^4$ be the set of bicyclic graphs on $n \geq 4$ obtained from C_n by adding a new edge, and let $\mathscr{B}_n^5$ be the set of bicyclic graphs on $n \geq 5$ obtained by connecting two non-adjacent vertices by a path of length at least two. Clearly, $\tilde{\mathscr{B}}_n = \bigcup_{i=1}^{5} \mathscr{B}_n^i$.

For $i = 4, 5$, we use B_i to denote the unique bicyclic graph on i vertices in $\mathscr{B}_n^i$. Let $B_{n,a,b}$ be the bicyclic graph on n vertices obtained by attaching $a-3$ and $b-3$ pendent vertices to the two vertices of degree 3 of B_4, respectively, where $a \geq b \geq 3$ and $a+b = n+2$. Let $B'_{n,a,b}$ be the bicyclic graph on n vertices obtained by attaching $a-3$ and $b-3$ pendent vertices to the two vertices of degree 3 of B_5, respectively, where $a \geq b \geq 3$ and $a+b = n+1$. Then $B_4 \cong B_{4,3,3}$ and $B_5 \cong B'_{5,3,3}$. See Figure 3 and Figure 4 for an illustration. We first determine the minimum harmonic index for graphs in $\mathscr{B}_n$ with matching number 2.

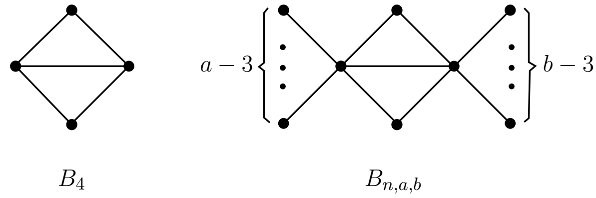

FIGURE 3. The graphs B_4 and $B_{n,a,b}$.

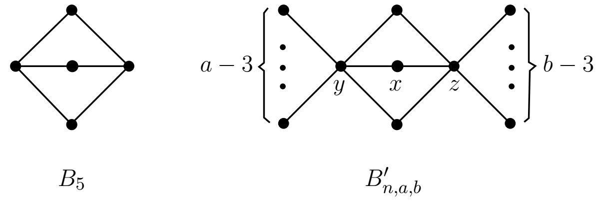

FIGURE 4. The graphs B_5 and $B'_{n,a,b}$.

Theorem 3. *Let $G \in \mathscr{B}_{n,2}$ with $n \geq 4$. Then*

$$H(G) \geq \frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5}$$

with equality if and only if $G \cong B_{n,n-1,3}$.

Proof. Since B_4 is the unique bicyclic graph on 4 vertices in $\mathscr{B}_{4,2}$, we see that the result holds for $n = 4$. If $n = 5$, then $G \in \{F_i | 1 \leq i \leq 3\} \cup B_5 \cup B_{5,4,3}$, where F_i $(1 \leq i \leq 3)$ are shown in Figure 5. It is easy to calculate that $H(F_1) = \frac{73}{30} > H(B_5) = \frac{12}{5} > H(F_2) = \frac{7}{3} > H(F_3) = \frac{23}{10} > \frac{226}{105} = H(B_{5,4,3})$, and hence the assertion of the theorem holds. So we may assume that $n \geq 6$. We consider three cases according to the structure of G.

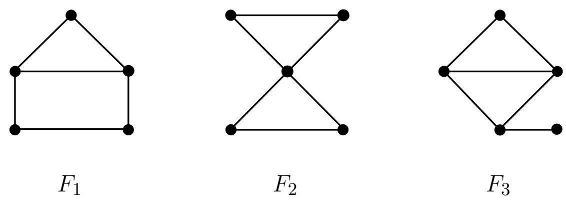

FIGURE 5. The graphs F_1, F_2 and F_3.

Case 1. $G \cong B_{n,a,b}$, where $a \geq b \geq 3$ and $a+b = n+2$.

Let $f(x) = \frac{4}{x+1} - \frac{8}{x}$. For $x \geq 3$, we have

$$f''(x) = \frac{8}{(x+1)^3} - \frac{16}{x^3} = \frac{-8(x^3+6x^2+6x+2)}{x^3(x+1)^3} < 0.$$

This implies that $f(x+1) - f(x)$ is decreasing for $x \geq 3$. Suppose $a \geq b \geq 4$. Then

$$\begin{aligned}
&H(B_{n,a+1,b-1}) - H(B_{n,a,b}) \\
&= \left(\frac{4}{(a+1)+2} + \frac{2[(a+1)-3]}{(a+1)+1} + \frac{4}{(b-1)+2} + \frac{2[(b-1)-3]}{(b-1)+1}\right. \\
&\quad \left. + \frac{2}{(a+1)+(b-1)}\right) - \left(\frac{4}{a+2} + \frac{2(a-3)}{a+1} + \frac{4}{b+2} + \frac{2(b-3)}{b+1} + \frac{2}{a+b}\right) \\
&= \left(\frac{4}{a+3} - \frac{12}{a+2} + \frac{8}{a+1}\right) - \left(\frac{4}{b+2} - \frac{12}{b+1} + \frac{8}{b}\right) \\
&= [f(a+2) - f(a+1)] - [f(b+1) - f(b)] < 0,
\end{aligned}$$

i.e., $H(B_{n,a,b}) > H(B_{n,a+1,b-1})$ for $a \geq b \geq 4$. So we conclude that $H(B_{n,a,b}) \geq H(B_{n,n-1,3})$ with equality if and only if $a = n-1$ and $b = 3$.

Case 2. G is the bicyclic graph obtained by attaching $n-4$ pendent vertices to one vertex of degree 2 of B_4.

Then

$$\begin{aligned}
&H(G) - H(B_{n,n-1,3}) \\
&= \left(\frac{4}{n+1} + \frac{2(n-4)}{n-1} + \frac{4}{5} + \frac{1}{3}\right) - \left(\frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5}\right) \\
&= \frac{8}{n} - \frac{2}{n+2} - \frac{6}{n-1} + \frac{1}{3} = \left(\frac{2}{n} - \frac{2}{n+2}\right) - \frac{6}{n(n-1)} + \frac{1}{3} \\
&\geq \left(\frac{2}{n} - \frac{2}{n+2}\right) - \frac{6}{6\cdot(6-1)} + \frac{1}{3} > 0.
\end{aligned}$$

So Case 2 holds.

Case 3. $G \cong B'_{n,a,b}$, where $a \geq b \geq 3$ and $a+b = n+1$.

Let x be one vertex of degree 2, and let y, z be the two vertices of degree at least 3 in G, see Figure 4. Let $G' = G - xz + yz$, then $G' \cong B_{n,a+1,b}$. By Lemma 3, we have $H(G) > H(G')$. Hence by the argument in Case 1, we deduce that $H(G) > H(B_{n,n-1,3})$. This completes the proof of the theorem. □

The following lemma was proved by Zhu, Liu and Wang [33], which will be used in the following argument.

Lemma 7. *Let $G \in \mathscr{B}_{n,m}$ ($n > 2m \geq 6$) and G contains at least one pendent vertex. Then there exists a maximum matching M and a pendent vertex u in G such that u is not M-saturated.*

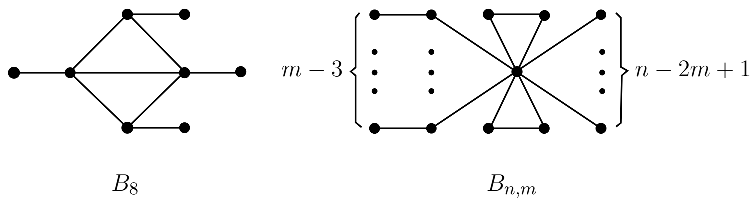

FIGURE 6. The graphs B_8 and $B_{n,m}$.

Let B_8 be the bicyclic graph on 8 vertices obtained by attaching a pendent vertex to every vertex of B_4. For $3 \le m \le \lfloor \frac{n}{2} \rfloor$, we use $B_{n,m}$ to denote the bicyclic graph on n vertices obtained by attaching $n-2m+1$ pendent vertices and $m-3$ paths on two vertices to the vertex of degree 4 of F_2, see Figure 6.

Lemma 8. *Let $G \in \mathscr{B}_{2m,m} \setminus \{B_8\}$ $(m \ge 3)$ and no pendent vertex has neighbor of degree 2. Then*

$$H(G) \ge \frac{2(m+1)}{m+4} + \frac{2}{m+3} + \frac{2(m-3)}{3} + 1$$

with equality if and only if $G \cong B_{6,3}$.

Proof. Let M be a maximum matching in G, then $|M| = m$ and every vertex in G is adjacent to at most one pendent vertex. Since $G \in \mathscr{B}_{2m,m} \setminus \{B_8\}$ and no pendent vertex has neighbor of degree 2, we see that G can be obtained by attaching some pendent vertices to a bicyclic graph $\tilde{G} \in \tilde{\mathscr{B}}_k$ $(m \le k \le 2m)$. We consider two cases according to G contains vertices of degree 2 or not.

Case 1. There is no vertex of degree 2 in G.

Then either $k = m$ or $k = m+1$. If $k = m$, then G can be obtained by attaching a pendent vertex to every vertex of a bicyclic graph $\tilde{G} \in \tilde{\mathscr{B}}_m$. If $k = m+1$, then G can be obtained by attaching a pendent vertex to every vertex of degree 2 of a bicyclic graph $\tilde{G} \in \mathscr{B}^1_{m+1} \cup \mathscr{B}^4_{m+1}$.

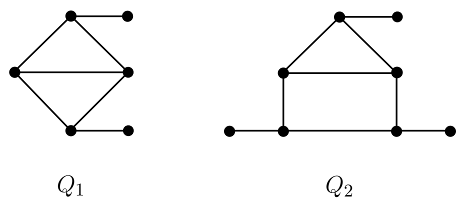

FIGURE 7. The graphs Q_1 and Q_2.

If $m = 3$, then $\tilde{G} \cong B_4$ and $G \cong Q_1$ (see Figure 7). Since $H(Q_1) = \frac{8}{3} > \frac{52}{21} = \frac{2\cdot(3+1)}{3+4} + \frac{2}{3+3} + \frac{2\cdot(3-3)}{3} + 1$, we know that the lemma holds.

If $m=4$, since we assume $G \ncong B_8$, we have $\tilde{G} \cong F_1$ and $G \cong Q_2$ (see Figure 7). So the assertion of the lemma holds because $H(Q_2)=\frac{7}{2}>\frac{269}{84}=\frac{2\cdot(4+1)}{4+4}+\frac{2}{4+3}+\frac{2\cdot(4-3)}{3}+1$.

Now assume that $m \geq 5$. Then

$$H(G)=\begin{cases} \frac{5m}{6}-\frac{59}{420}, & \text{if } \tilde{G} \in \mathscr{B}_m^1 \cup \mathscr{B}_m^4, \\ \frac{5m}{6}-\frac{16}{105}, & \text{if } \tilde{G} \in \mathscr{B}_m^2 \cup \mathscr{B}_m^5, \\ \frac{5m}{6}-\frac{1}{6}, & \text{if } \tilde{G} \in \mathscr{B}_m^3, \\ \frac{5m}{6}+\frac{1}{6}, & \text{if } \tilde{G} \in \mathscr{B}_{m+1}^1 \cup \mathscr{B}_{m+1}^4. \end{cases}$$

Let $f(x)=\left(\frac{5x}{6}-\frac{1}{6}\right)-\left(\frac{2(x+1)}{x+4}+\frac{2}{x+3}+\frac{2(x-3)}{3}+1\right)=\frac{x}{6}+\frac{6}{x+4}-\frac{2}{x+3}-\frac{7}{6}$. For $x \geq 5$, we have

$$f'(x)=\frac{1}{6}-\frac{6}{(x+4)^2}+\frac{2}{(x+3)^2} \geq \frac{1}{6}-\frac{6}{(5+4)^2}+\frac{2}{(x+3)^2}>0.$$

This implies that $f(x)$ is increasing for $x \geq 5$, and thus $f(m) \geq f(5)=\frac{1}{12}>0$, i.e., $H(G)>\frac{2(m+1)}{m+4}+\frac{2}{m+3}+\frac{2(m-3)}{3}+1$.

Case 2. There exists a vertex, say u, of degree 2 in G.

Let v and w be the two neighbors of u in G such that $d(v)=s \geq 2$ and $d(w)=t \geq 2$. By the symmetry between v and w, we may assume that $uv \in M$.

Suppose that no vertex of degree 2 is contained in the cycles of G. Since no pendent vertex has neighbor of degree 2 in G, we conclude that $\tilde{G} \in \mathscr{B}_k^2$ and u lies on the path connecting two vertex-disjoint cycles of G. Hence $vw \notin E(G)$. Let $G'=G-uw+vw$, then $G' \in \mathscr{B}_{2m,m} \setminus \{B_8\}$. By Lemma 3, we have $H(G)>H(G')$. Comparing with the graph G, we see that the number of vertices of degree 2 in G' decreases by 1. Repeating this operation from G to G', we finally obtain a bicyclic graph described in Case 1, and hence the result holds.

So we may choose u such that u lies on some cycle of G. Let $N(w)=\{w_0=u, w_1, \ldots, w_{t-1}\}$, and let $G''=G-uw$. Then G'' is a unicyclic graph on $2m$ vertices with a perfect matching M, i.e., $G'' \in \mathscr{U}_{2m,m}$. Note that $2 \leq s,t \leq 5$ and w is adjacent to at most one pendent vertex. Since $\frac{2}{s+2}-\frac{2}{s+1}$ is increasing for $s \geq 2$, $\frac{2}{t+x}-\frac{2}{t-1+x}$ is increasing for $x \geq 1$ and by Lemma 2(i), we have

$$\begin{aligned} H(G) &= H(G'')+\sum_{i=1}^{t-1}\left(\frac{2}{t+d(w_i)}-\frac{2}{t-1+d(w_i)}\right)+\frac{2}{t+2}+\left(\frac{2}{s+2}-\frac{2}{s+1}\right) \\ &\geq H(G'')+\left(\frac{2}{t+1}-\frac{2}{t}\right)+(t-2)\left(\frac{2}{t+2}-\frac{2}{t+1}\right)+\frac{2}{t+2} \\ &\quad+\left(\frac{2}{2+2}-\frac{2}{2+1}\right) \\ &= H(G'')+\left(\frac{2(t-1)}{t+2}-\frac{2(t-3)}{t+1}-\frac{2}{t}\right)-\frac{1}{6} \end{aligned}$$

$$\geq H(G'') + \left(\frac{2\cdot(5-1)}{5+2} - \frac{2\cdot(5-3)}{5+1} - \frac{2}{5}\right) - \frac{1}{6}$$

$$= H(G'') - \frac{19}{210} \qquad (*)$$

with equalities if and only if $s = 2$, $t = 5$, one neighbor of w has degree 1 and the other neighbors of w have degree 2.

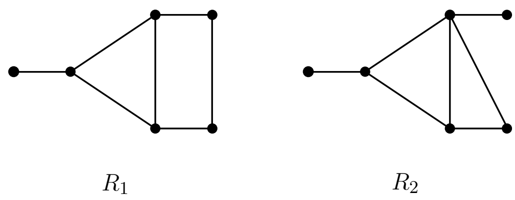

FIGURE 8. The graphs R_1 and R_2.

If $G'' \cong U_6$, then either $G'' \cong R_1$ or $G'' \cong R_2$ (see Figure 8). Since $H(R_1) = \frac{14}{5} > H(R_2) = \frac{533}{210} > \frac{52}{21} = \frac{2\cdot(3+1)}{3+4} + \frac{2}{3+3} + \frac{2\cdot(3-3)}{3} + 1$, the assertion of the lemma holds. If $G'' \cong U_8$, then by $(*)$, we have

$$H(G) \geq H(U_8) - \frac{19}{210} = \frac{347}{105} - \frac{19}{210} = \frac{45}{14}$$
$$> \frac{269}{84} = \frac{2\cdot(4+1)}{4+4} + \frac{2}{4+3} + \frac{2\cdot(4-3)}{3} + 1,$$

and the result holds. So suppose that $G'' \ncong U_6, U_8$. It follows from Lemma 2(i) that

$$\frac{2[(m+2)-1]}{(m+2)+2} - \frac{2[(m+2)-3]}{(m+2)+1} - \frac{2}{m+2}$$
$$\leq \frac{2\cdot[(3+2)-1]}{(3+2)+2} - \frac{2\cdot[(3+2)-3]}{(3+2)+1} - \frac{2}{3+2} = \frac{8}{105}$$

since $m \geq 3$. Then by $(*)$ and Theorem 1, we have

$$H(G) \geq H(G'') - \frac{19}{210}$$
$$\geq \left(\frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-2)}{3} + \frac{1}{2}\right) - \frac{19}{210}$$
$$= \left(\frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-3)}{3} + 1\right) + \frac{8}{105}$$
$$\geq \left(\frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-3)}{3} + 1\right)$$

$$+\left(\frac{2[(m+2)-1]}{(m+2)+2}-\frac{2[(m+2)-3]}{(m+2)+1}-\frac{2}{m+2}\right)$$
$$=\frac{2(m+1)}{m+4}+\frac{2}{m+3}+\frac{2(m-3)}{3}+1$$

with equalities if and only if $s=2$, $t=5$, $G''\cong U_{2m,m}$ and $m=3$, i.e., $G\cong B_{6,3}$. This finishes the proof of the lemma. □

Theorem 4. *Let $G\in\mathscr{B}_{2m,m}\setminus\{B_8\}$, where $m\geq 3$. Then*

$$H(G)\geq\frac{2(m+1)}{m+4}+\frac{2}{m+3}+\frac{2(m-3)}{3}+1$$

with equality if and only if $G\cong B_{2m,m}$.

Proof. We prove Theorem 4 by induction on m. If $m=3$, then by Lemma 7, we may assume that there exists a pendent vertex in G whose neighbor is a vertex of degree 2. Hence G is the bicyclic graph obtained from B_4 by attaching a path on two vertices to either one vertex of degree 3 or one vertex of degree 2. Then we have $H(G)\geq\frac{289}{105}>\frac{52}{21}=H(B_{6,3})$, and the assertion of the theorem holds. So we may assume that $m\geq 4$ and the result holds for graphs in $\mathscr{B}_{2(m-1),m-1}\setminus\{B_8\}$. Let M be a maximum matching in G, then $|M|=m$. If no pendent vertex has neighbor of degree 2 in G, then by Lemma 7, we see that the result holds.

Now suppose that there exists a pendent vertex u in G whose neighbor v is a vertex of degree 2. Let w be the neighbor of v different from u with $d(w)=t\geq 2$, and let $G'=G-u-v$. Then $uv\in M$ and $G'\in\mathscr{B}_{2(m-1),m-1}$. Since M contains exactly one edge incident with w and there are $m+1$ edges of G outside M, we have $t\leq m+2$. Note that w is adjacent to at most one pendent vertex in G.

If $G'\cong B_8$, then $t\leq 5$. By Lemma 1(i) and Lemma 2(i), we have

$$H(G)\geq H(B_8)+\frac{2(t-1)}{t+2}-\frac{2(t-3)}{t+1}-\frac{2}{t}+\frac{2}{3}$$
$$\geq\frac{447}{140}+\frac{2\cdot(5-1)}{5+2}-\frac{2\cdot(5-3)}{5+1}-\frac{2}{5}+\frac{2}{3}=\frac{551}{140}>\frac{47}{12}=H(U_{10,5}),$$

and hence the assertion of the theorem holds.

So we may further assume that $G'\ncong B_8$. Then by Lemma 1(i), Lemma 2(i) and the induction hypothesis, we conclude that

$$H(G)\geq H(G')+\frac{2(t-1)}{t+2}-\frac{2(t-3)}{t+1}-\frac{2}{t}+\frac{2}{3}$$
$$\geq\left(\frac{2[(m-1)+1]}{(m-1)+4}+\frac{2}{(m-1)+3}+\frac{2[(m-1)-3]}{3}+1\right)$$
$$+\frac{2[(m+2)-1]}{(m+2)+2}-\frac{2[(m+2)-3]}{(m+2)+1}-\frac{2}{m+2}+\frac{2}{3}$$

$$= \frac{2(m+1)}{m+4} + \frac{2}{m+3} + \frac{2(m-3)}{3} + 1$$

with equalities if and only if $G' \cong B_{2(m-1),m-1}$ and $t = m+2$, i.e., $G \cong B_{2m,m}$. So Theorem 4 holds. □

Since $H(B_{8,4}) = \frac{269}{84} > \frac{447}{140} = H(B_8)$, by Theorem 4, we immediately obtain the following result.

Corollary 4. *Let $G \in \mathscr{B}_{8,4}$, then $H(G) \geq \frac{447}{140}$ with equality if and only if $G \cong B_8$.*

We now present the minimum harmonic index for graphs in $\mathscr{B}_{n,m} \setminus \{B_8\}$, where $3 \leq m \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 5. *Let $G \in \mathscr{B}_{n,m} \setminus \{B_8\}$, where $3 \leq m \leq \lfloor \frac{n}{2} \rfloor$. Then*

$$H(G) \geq \frac{2(m+1)}{n-m+4} + \frac{2(n-2m+1)}{n-m+3} + \frac{2(m-3)}{3} + 1$$

with equality if and only if $G \cong B_{n,m}$.

Proof. We prove the theorem by induction on n. If $n = 2m$, then by Theorem 4, the assertion of the theorem holds. So we may assume that $n > 2m$ and the result holds for graphs in $\mathscr{B}_{n-1,m} \setminus \{B_8\}$. If there is no pendent vertex in G, then $G \in \tilde{\mathscr{B}}_n$ and $n = 2m+1$. It is easy to check that

$$H(G) = \begin{cases} m + \frac{13}{30}, & \text{if } G \in \mathscr{B}^1_{2m+1} \cup \mathscr{B}^4_{2m+1}, \\ m + \frac{2}{5}, & \text{if } G \in \mathscr{B}^2_{2m+1} \cup \mathscr{B}^5_{2m+1}, \\ m + \frac{1}{3}, & \text{if } G \in \mathscr{B}^3_{2m+1}. \end{cases}$$

This implies that

$$\begin{aligned} &H(G) - H(B_{2m+1,m}) \\ &\geq \left(m + \frac{1}{3}\right) - \left(\frac{2(m+1)}{(2m+1)-m+4} + \frac{2[(2m+1)-2m+1]}{(2m+1)-m+3} + \frac{2(m-3)}{3} + 1\right) \\ &= \frac{m}{3} + \frac{8}{m+5} - \frac{4}{m+4} - \frac{2}{3} = \frac{m-2}{3} + \frac{4(m+3)}{(m+4)(m+5)} > 0, \end{aligned}$$

i.e., $H(G) > H(B_{2m+1,m})$.

So we may assume that G contains at least one pendent vertex. Then by Lemma 7, there exists a maximum matching M and a pendent vertex u in G such that u is not M-saturated. Let v be the unique neighbor of u with $d(v) = s \geq 2$, and let $G' = G - u$. Then $G' \in \mathscr{B}_{n-1,m}$. Since M contains exactly one edge incident with v and there are $n+1-m$ edges of G outside M, we have $s \leq n-m+2$. Let r be the number of pendant neighbors of v in G, where $1 \leq r \leq s-1$. Note that at least $r-1$ pendant neighbors of v are not M-saturated, and there are $n-2m$ vertices are not M-saturated in G. Then $r \leq n-2m+1$.

If $G' \cong B_8$, then $n = 9$, $m = 4$ and $s \leq 5$. By Lemma 1(ii) (with $k = n-2m+1 = 2$) and Lemma 2(ii), we deduce that

$$\begin{aligned} H(G) &\geq H(B_8) + \frac{2(s-2)}{s+2} + \frac{2(4-s)}{s+1} - \frac{2}{s} \\ &\geq \frac{447}{140} + \frac{2\cdot(5-2)}{5+2} + \frac{2\cdot(4-5)}{5+1} - \frac{2}{5} = \frac{1393}{420} > \frac{59}{18} = H(B_{9,4}), \end{aligned}$$

and hence the assertion of the theorem holds.

Therefore we may assume that $G' \ncong B_8$. Then by Lemma 1(ii) (with $k = n - 2m + 1$), Lemma 2(ii) and the induction hypothesis, we have

$$\begin{aligned} H(G) &\geq H(G') + \frac{2[s-(n-2m+1)]}{s+2} + \frac{2[2(n-2m+1)-s]}{s+1} \\ &\quad - \frac{2[(n-2m+1)-1]}{s} \\ &\geq \left(\frac{2(m+1)}{(n-1)-m+4} + \frac{2[(n-1)-2m+1]}{(n-1)-m+3} + \frac{2(m-3)}{3} + 1\right) \\ &\quad + \frac{2[(n-m+2)-(n-2m+1)]}{(n-m+2)+2} + \frac{2[2(n-2m+1)-(n-m+2)]}{(n-m+2)+1} \\ &\quad - \frac{2[(n-2m+1)-1]}{n-m+2} \\ &= \frac{2(m+1)}{n-m+4} + \frac{2(n-2m+1)}{n-m+3} + \frac{2(m-3)}{3} + 1 \end{aligned}$$

with equalities if and only if $G' \cong B_{n-1,m}$, $s = n-m+2$ and $r = n-2m+1$, i.e., $G \cong B_{n,m}$. This completes the proof of the theorem. □

We can also determine the minimum harmonic index for graphs in $\mathscr{B}_n$ (see also in [31]) by using Theorem 3 and Theorem 5.

Corollary 5. *Let $G \in \mathscr{B}_n$ with $n \geq 4$. Then*

$$H(G) \geq \frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5}$$

with equality if and only if $G \cong B_{n,n-1,3}$.

Proof. Let M be a maximum matching in G, then $2 \leq |M| = m \leq \lfloor \frac{n}{2} \rfloor$ (since $n \geq 4$). If $m = 2$, then the result follows immediately from Theorem 3.

If $m = 3$, then by Theorem 5, we have

$$\begin{aligned} H(G) &\geq \frac{2\cdot(3+1)}{n-3+4} + \frac{2(n-2\cdot 3+1)}{n-3+3} + \frac{2\cdot(3-3)}{3} + 1 \\ &= \frac{8}{n+1} + \frac{2(n-5)}{n} + 1 \end{aligned}$$

with equality if and only if $G \cong B_{n,3}$. Note that in this case $n \geq 6$. Since

$$\begin{aligned}
&H(B_{n,3}) - H(B_{n,n-1,3}) \\
&= \left(\frac{8}{n+1} + \frac{2(n-5)}{n} + 1\right) - \left(\frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5}\right) \\
&= \left(\frac{4}{n+1} - \frac{2}{n+2} - \frac{2}{n}\right) + \frac{1}{5} = \frac{-4}{n(n+1)(n+2)} + \frac{1}{5} \\
&\geq \frac{-4}{6\cdot(6+1)\cdot(6+2)} + \frac{1}{5} = \frac{79}{420} > 0,
\end{aligned}$$

we know that the assertion of the corollary holds.

So we may assume that $m \geq 4$. If $G \cong B_8$, then $H(G) = \frac{447}{140} > \frac{22}{9} = H(B_{8,7,3})$, we see that Corollary 5 holds. Now suppose that $G \ncong B_8$. Then by Theorem 5 and Lemma 3, we see that $H(G) \geq H(B_{n,m}) > H(B_{n,m-1}) > \cdots > H(B_{n,3}) > H(B_{n,n-1,3})$. This finishes the proof of the corollary. $\square$

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