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Best proximity point theorems for F-contractive non-self mappings

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BEST PROXIMITY POINT THEOREMS FOR F-CONTRACTIVE NON-SELF MAPPINGS

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Abstract. In this article, we prove the existence of a best proximity point for $F-$contractive non-self mappings and state some results in the complete metric spaces. Also we define two kinds of $F-$proximal contraction and extend some best proximity theorems and improve the recent results.

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1. INTRODUCTION

Let us assume that A, B be two nonempty subsets of a metric space (X, d) and $T : A \longrightarrow B$. Clearly $T(A) \cap A \neq \varnothing$ is a necessary condition for the existence of a fixed point of T. Now if $T(A) \cap A = \varnothing$ then to find an element $x \in A$ such that $d(x, Tx) = d(A, B)$ which called best proximity point is the idea of best proximity point theorems. In the other words, we determine an approximate solution x such that the error of equation $d(x, Tx) = 0$ is minimum. Several authors such as Prolla [2], Reich [4], Sehgal and Singh [6, 7], Vertivel, Veermani and Bhattacharyya [8] and others[1] generalized and extended the best proximity point theorems in many directions. In 2011, Sadiq Basha [5] stated the best proximity points theorems for proximal contractions. On the other hand, Wardowski [9] introduced a new type of contraction which called $F-$contraction and proved a fixed point result in complete metric spaces. In this paper, by using Wardowski's contraction, we prove the existence of a best proximity point. Moreover we define $F-$proximal contractions of the first and second kind and establish the best proximity point theorems in spire of Wardowski's contraction.

2. PRELIMINARY

Let A, B be two nonempty subsets of a metric space X. The following notations will be used throughout this paper:

$d(y,A) := \inf\{d(x,y) : x \in A\}$,

$d(A,B) := \inf\{d(x,y) : x \in A$ and $y \in B\}$,

$A_0 := \{x \in A : d(x,y) = d(A,B)$ for some $y \in B\}$,

$B_0 := \{y \in B : d(x,y) = d(A,B)$ for some $x \in A\}$.

We recall that $x \in A$ is a best proximity point of the mapping $T : A \longrightarrow B$ if $d(x,Tx) = d(A,B)$. It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 1 ([3])**.** Let (A,B) be a pair of nonempty subsets of a metric space X with $A \neq \varnothing$. Then the pair (A,B) is said to have the P-property if and only if

$$\left.\begin{array}{l} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{array}\right\} \Longrightarrow d(x_1,x_2) = d(y_1,y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is clear that, for any nonempty subset A of X, the pair (A,A) has the P-property.

Definition 2 ([5])**.** A is said to be approximatively compact with respect to B if every sequence $\{x_n\}$ of A satisfying the condition that $d(y,x_n) \longrightarrow d(y,A)$ for some y in B has a convergent subsequence.

It is easy to see that every set is approximatively compact with respect to itself.

Definition 3 ([5])**.** Given $T : A \longrightarrow B$ and an isometry $g : A \longrightarrow A$, the mapping T is said to preserve isometric distance with respect to g if

$$d(Tgx_1, Tgx_2) = d(Tx_1, Tx_2)$$

for all x_1 and x_2 in A.

Recently Wardowski [9] defined the following contraction which was called $F-$contraction.

Definition 4. Let $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing, i.e. for all $a, b \in \mathbb{R}_+$ such that $\alpha < \beta \Longrightarrow F(\alpha) < F(\beta)$;

(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

A mapping $T : X \longrightarrow X$ is said to be an $F-$contraction if there exists $\tau > 0$ such that

$$For\ all\ \ x, y \in X, (d(Tx,Ty)) > 0 \Longrightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)). \quad (2.1)$$

Now, we consider the mapping F in the different types, and we obtain the variety of contractions.

Example 1 ([9]). Let $F:\mathbb{R}_+ \longrightarrow \mathbb{R}$ be given by the formula $F(\alpha)=\ln\alpha$. It is clear that F satisfies the (F1)-(F3) ((F3) for any $k\in(0,1)$) conditions of Definition 4. Each mapping $T:X\longrightarrow X$ satisfying (2.1) is an $F-$contraction such that

$$d(Tx,Ty)\le e^{-\tau}d(x,y),\quad for\ all\ \ x,y\in X,\ \ Tx\neq Ty.$$

It is clear that for $x,y\in X$ such that $Tx=Ty$ the inequality $d(Tx,Ty)\le e^{-\tau}d(x,y)$ also holds, i.e. T is a Banach contraction.

3. Main Results

Now, let us state our main result.

Theorem 1. *Let A and B be non-empty, closed subsets of a complete metric space X such that A_0 is nonempty. Let $T:A\longrightarrow B$ be an $F-$contraction non-self mapping such that $T(A_0)\subseteq B_0$. Assume that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*,Tx^*)=d(A,B)$.*

Proof. Choose $x_0\in A_0$. Since $Tx_0\in T(A_0)\subseteq B_0$, there exists $x_1\in A_0$ such that $d(x_1,Tx_0)=d(A,B)$. Again, since $Tx_1\in T(A_0)\subseteq B_0$, there exists $x_2\in A_0$ such that $d(x_2,Tx_1)=d(A,B)$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1},Tx_n)=d(A,B),\quad for\ all\ n\in\mathbb{N}. \tag{3.1}$$

(A,B) satisfies the P-property, therefore from (3.1) we obtain

$$d(x_n,x_{n+1})=d(Tx_{n-1},Tx_n),\quad for\ all\ n\in\mathbb{N}. \tag{3.2}$$

We will prove that the sequence $\{x_n\}$ is convergent in A_0. If there exists $n_0\in\mathbb{N}$ such that $d(Tx_{n_0-1},Tx_{n_0})=0$, then by (3.2) we have $d(x_{n_0},x_{n_0+1})=0$ that implies $x_{n_0}=x_{n_0+1}$. Therefore

$$Tx_{n_0}=Tx_{n_0+1}\Longrightarrow d(Tx_{n_0},Tx_{n_0+1})=0 \tag{3.3}$$

From (3.2) and (3.3) we receive that

$$d(x_{n_0+2},x_{n_0+1})=d(Tx_{n_0+1},Tx_{n_0})=0\Longrightarrow x_{n_0+2}=x_{n_0+1}$$

Therefore $x_n=x_{n_0},\ for\ all\ n\ge n_0$ and $\{x_n\}$ is convergent in A_0.
Now let $d(Tx_{n-1},Tx_n)\neq 0,\ for\ all\ n\in\mathbb{N}$. T is a $F-$contraction and (3.2) holds, hence for any positive integer n we have

$$\tau+F(d(Tx_n,Tx_{n-1}))\le F(d(x_n,x_{n-1}))$$

$$\Longrightarrow F(d(x_{n+1},x_n)) \leq F(d(x_n,x_{n-1}))-\tau$$
$$\vdots$$
$$\leq F(d(x_1,x_0))-n\tau. \tag{3.4}$$

Put $\beta_n := d(x_{n+1},x_n)$.
From (3.4), we obtain $\lim_{n\to\infty} F(\beta_n) = -\infty$ that together with $(F2)$ gives

$$\lim_{n\to\infty} \beta_n = 0 \tag{3.5}$$

Also from $(F3)$ we have

$$\exists k \in (0,1) \;\; such\ that \;\; \beta_n^k F(\beta_n) = 0 \tag{3.6}$$

By (3.3), the following holds for all $n \in \mathbb{N}$:

$$F(\beta_n) - F(\beta_0) \leq -n\tau.$$

Therefore

$$\beta_n^k F(\beta_n) - \beta_n^k F(\beta_0) \leq -n\beta_n^k \tau \leq 0.$$

Letting $k \to \infty$ in the above inequality and using (3.5),(3.6), we obtain

$$\lim_{n\to\infty} n\beta_n^k = 0.$$

Hence there exists $n_1 \in \mathbb{N}$ such that $n\beta_n^k \leq 1$ for all $n \geq n_1$. Therefore for any $n \geq n_1$,

$$\beta_n \leq \frac{1}{n^{\frac{1}{k}}}. \tag{3.7}$$

This means that series $\sum_{i=1}^{\infty} \beta_i$ is convergent.
Now let $m \geq n \geq n_1$. By the triangular inequality and (3.7), we have

$$d(x_m,x_n) \leq \beta_{m-1} + \beta_{m-2} + \ldots + \beta_n \leq \sum_{i=n}^{\infty} \beta_i.$$

Therefore $\{x_n\}$ is a Cauchy sequence in A. Since (X,d) is complete and A is a closed subset of X, there exist $x^* \in A$ such that

$$\lim_{n\to\infty} x_n = x^*$$

Since T is continuous, we have $Tx_n \longrightarrow Tx^*$. Hence continuity of the metric function d which implies that $d(x_{n+1},Tx_n) \longrightarrow d(x^*,Tx^*)$. From (3.1), $d(x^*,Tx^*) = d(A,B)$. So we show that x^* is a best proximity of T.
The uniqueness of the best proximity point follows from the condition that T is $F-$contraction. That is, suppose that

$$x_1,x_2 \in A \;\; such\ that \;\; x_1 \neq x_2 \;\; and \;\; d(x_1,Tx_1) = d(x_2,Tx_2) = d(A,B).$$

Then by the P-property of (A,B), we have $d(x_1,x_2) = d(Tx_1,Tx_2)$. Also

$$x_1 \neq x_2 \Longrightarrow d(x_1,x_2) \neq 0.$$

Therefore

$$F(d(x_1,x_2)) = F(d(Tx_1,Tx_2)) \leq F(d(x_1,x_2)) - \tau < F(d(x_1,x_2)),$$

which is a contraction. Hence the best proximity point is unique. □

The following result is a special case of Theorem 1, obtained by setting $A = B$.

Corollary 1. *Let (X,d) be a complete metric space and A be a nonempty closed subset of X. Let $T : A \longrightarrow A$ be a $F-$contractive self-map. Then T has a unique fixed point x^* in A.*

The next result is an immediate consequence of Theorem 1 by taking $F(\alpha) = \ln\alpha$.

Corollary 2 (Banach Contraction Principle)**.** *Let (X,d) be a complete metric space and A be a nonempty closed subset of X. Let $T : A \longrightarrow A$ be a contractive self-map. Then T has a unique fixed point x^* in A.*

Let F be the function as in Definition 4, we define the proximal contractions.

Definition 5. A mapping $T : A \longrightarrow B$ is said to be a $F-$proximal contraction of the first kind if there exists a $\tau > 0$ such that

$$\left.\begin{array}{r} d(u_1,Tx_1) = d(A,B) \\ d(u_2,Tx_2) = d(A,B) \\ d(u_1,u_2), d(x_1,x_2) > 0 \end{array}\right\} \Longrightarrow \tau + F(d(u_1,u_2)) \leq F(d(x_1,x_2))$$

where $u_1,u_2,x_1,x_2 \in A$.

Remark 1. If $T : A \longrightarrow B$ is a $F-$proximal contraction of the first kind and (A,B) has the P-property then T is a $F-$contractive non-self mapping.

Definition 6. A mapping $T : A \longrightarrow B$ is said to be a $F-$proximal contraction of the second kind if there exists a $\tau > 0$ such that

$$\left.\begin{array}{r} d(u_1,Tx_1) = d(A,B) \\ d(u_2,Tx_2) = d(A,B) \\ d(Tu_1,Tu_2), d(Tx_1,Tx_2) > 0 \end{array}\right\} \Longrightarrow \tau + F(d(Tu_1,Tu_2)) \leq F(d(Tx_1,Tx_2))$$

where $u_1,u_2,x_1,x_2 \in A$.

The following theorem is a best proximity point theorem for non-self $F-$proximal contraction of the first kind.

Theorem 2. *Let A and B be non-empty, closed subsets of a complete metric space X such that A_0 is non-empty. Let $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions:*

(a) *T is a continuous $F-$proximal contraction of the first kind.*
(b) *g is an isometry.*
(c) $T(A_0) \subseteq B_0$.

(d) $A_0 \subseteq g(A_0)$.

Then, there exists a unique element $x \in A$ such that

$$d(gx, Tx) = d(A, B).$$

Proof. Choose $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A, B)$. If $x_0 = x_1$ then put $x_n := x_1$ for all $n \geq 2$. Also, since $Tx_1 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_2 \in A_0$ such that $d(gx_2, Tx_1) = d(A, B)$. If $x_1 = x_2$ then put $x_n := x_2$ for all $n \geq 3$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A, B), \quad for\ all\ n \in \mathbb{N}. \tag{3.8}$$

We will prove the convergence of the sequence $\{x_n\}$ in A. If there exists $n_0 \in \mathbb{N}$ such that $d(gx_{n_0}, gx_{n_0+1}) = 0$ then it is clear that the sequence $\{x_n\}$ is convergent. Hence let $d(gx_n, gx_{n+1}) \neq 0,\ for\ all\ n \in \mathbb{N}$. T is a $F-$proximal contraction of the first kind and (3.8) holds, hence for any positive integer n we have

$$\begin{aligned}
&\tau + F(d(gx_n, gx_{n+1})) \leq F(d(x_{n-1}, x_n)) \\
&\Longrightarrow F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau \\
&\vdots \\
&\qquad \leq F(d(x_0, x_1)) - n\tau.
\end{aligned} \tag{3.9}$$

Similarly as the process in the proof of Theorem 1, $\{x_n\}$ is a Cauchy sequence in A. Since X is complete metric space and A is closed subset of X, there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$.

Since, T, g and d are continuous, therefore with letting $n \longrightarrow \infty$ in (3.8), we obtain

$$d(gx, Tx) = d(A, B).$$

Now, x^* be in A such that

$$d(gx^*, Tx^*) = d(A, B).$$

We show that $x = x^*$. Suppose to the contrary, that $x \neq x^*$. Hence $d(x, x^*) \neq 0$. Since T is a $F-$proximal contraction of the first kind and g is an isometry,

$$F(d(x, x^*)) = F(d(gx, gx^*)) \leq F(d(x, x^*)) - \tau < F(d(x, x^*)),$$

which is a contraction. Therefore $x = x^*$ and this completes the proof of theorem. □

The following result is a special case of Theorem 2, if g is the identity mapping.

Corollary 3. *Let A and B be non-empty, closed subsets of a complete metric space X such that A is approximatively compact with respect to B. Further, suppose that A_0 is non-empty. Let $T : A \longrightarrow B$ satisfies the following conditions:*

(a) *T is a continuous $F-$proximal contraction of the first kind.*
(b) *$T(A_0) \subseteq B_0$.*
Then T has a unique best proximity point in A.

The following theorem is a best proximity point theorem for non-self $F-$proximal contraction of the second kind.

Theorem 3. *Let A and B be non-empty, closed subsets of a complete metric space X such that A is approximatively compact with respect to B. Further, suppose that A_0 is non-empty. Let $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions:*
(a) *T is a continuous $F-$proximal contraction of the second kind.*
(b) *g is an isometry.*
(c) *$T(A_0) \subseteq B_0$.*
(d) *$A_0 \subseteq g(A_0)$.*
(e) *T preserves isometric distance with respect to g.*
Then, there exists an element $x \in A$ such that

$$d(gx, Tx) = d(A, B).$$

Moreover, if x^* is another element of A such that $d(gx^*, Tx^*) = d(A, B)$ then $Tx = Tx^*$.

Proof. Choose $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A, B)$. If $Tx_0 = Tx_1$ then put $x_n := x_1$ for all $n \geq 2$. Otherwise again since $Tx_1 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_2 \in A_0$ such that $d(gx_2, Tx_1) = d(A, B)$. If $Tx_1 = Tx_2$) then put $x_n := x_2$ for all $n \geq 3$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A, B), \quad for\ all\ n \in \mathbb{N}. \tag{3.10}$$

We will prove the convergence of the sequence $\{Tx_n\}$ in B. If there exists $n_0 \in \mathbb{N}$ such that $d(Tgx_{n_0}, Tgx_{n_0+1}) = 0$ then it is clear that the sequence $\{Tx_n\}$ is convergent. Hence let $d(Tgx_n, Tgx_{n+1}) \neq 0,\ for\ all\ n \in \mathbb{N}$. T is a $F-$proximal contraction of the second kind, T preserves isometric distance with respect to g and (3.10) holds, hence for any positive integer n we have

$$\begin{aligned}
&\tau + F(d(Tgx_n, Tgx_{n+1})) \leq F(d(Tx_{n-1}, Tx_n)) \\
&\Longrightarrow F(d(Tx_n, Tx_{n+1})) \leq F(d(Tx_{n-1}, Tx_n)) - \tau \\
&\vdots \\
&\qquad \leq F(d(Tx_0, Tx_1)) - n\tau.
\end{aligned} \tag{3.11}$$

Similarly as the process in the proof of Theorem 1, we receive that $\{Tx_n\}$ is a Cauchy sequence in B. Since X is complete metric space and B is closed subset of X, there exists $y \in B$ such that $\lim_{n\to\infty} Tx_n = y$.
By the triangular inequality, we have,

$$\begin{aligned} d(y,A) \le d(y,gx_n) &\le d(y,Tx_{n-1}) + d(Tx_{n-1},gx_n) \\ &= d(y,Tx_{n-1}) + d(A,B) \\ &\le (y,Tx_{n-1}) + d(y,A) \end{aligned}$$

Letting $k \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(y,gx_n) = d(y,A).$$

Since A is approximatively compact with respect to B, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that converging to some $z \in A$. Therefore

$$d(z,y) = \lim_{k\to\infty} d(gx_{n_k}, Tx_{n_k-1}) = d(A,B).$$

This implies that $z \in A_0$. Since $A_0 \subseteq g(A_0)$, there exists $x \in A_0$ such that $z = gx$. As $\lim_{n\to\infty} g(x_{n_k}) = g(x)$ and g is an isometry, we have

$$\lim_{n\to\infty} x_{n_k} = x.$$

T is continuous and $\{Tx_n\}$ is convergent to y, Therefore

$$\lim_{n\to\infty} Tx_{n_k} = Tx = y.$$

Thus, it follows that

$$d(gx,Tx) = \lim_{n\to\infty} d(gx_{n_k}, Tx_{n_k}) = d(A,B).$$

Now let x^* be another element in A such that

$$d(gx^*,Tx^*) = d(A,B).$$

We will show that $Tx = Tx^*$. Suppose to the contrary, that $Tx \ne Tx^*$. Hence $d(Tx,Tx^*) \ne 0$. Since T preserves isometric distance with respect to g and T is a $F-$proximal contraction of the second kind,

$$F(d(Tx,Tx^*)) = F(d(Tgx,Tgx^*)) \le F(d(Tx,Tx^*)) - \tau < F(d(Tx,Tx^*)),$$

which is a contraction. Therefore $Tx = Tx^*$. □

The next result is an immediate consequence of the Theorem 3, if g is the identity mapping.

Corollary 4. *Let A and B be non-empty, closed subsets of a complete metric space X such that A is approximatively compact with respect to B. Further, suppose that A_0 is non-empty. Let $T : A \longrightarrow B$ satisfies the following conditions:*
(a) *T is a continuous $F-$proximal contraction of the second kind.*
(b) $T(A_0) \subseteq B_0$.
Then, T has a best proximity point in A. Moreover, if x^* is another best proximity point of T then $Tx = Tx^*$.

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