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THE SECOND EDGE-WIENER INDEX OF SOME COMPOSITE GRAPHS

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Abstract. In this paper we study the behavior of the second edge-Wiener index under the join and corona product of graphs. Results are applied for some classes of graphs such as suspensions, bottlenecks, and thorny graphs.

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1. INTRODUCTION

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modelled by means of molecular–graph–based structure–descriptors, which are also referred to as topological indices [14].

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The vertex version of the Wiener index is the first reported distance-based topological index which was introduced in 1947 by Wiener [16, 17], who used it for modeling the shape of organic molecules and for calculating several of their physico-chemical properties. The Wiener index $W(G)$ of G is defined as:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v),$$

where $d_G(u,v)$ is the shortest path distance between vertices u and v in G. Details on the Wiener index can be found in [7, 8, 12, 14, 15].

Edge versions of the Wiener index based on the distance between all pairs of edges in a simple connected graph G were introduced in 2009 [9]. Two possible distances between the edges $g = uv$ and $f = zt$ of a graph G can be considered. Each of them gives rise to a corresponding edge-Wiener index. The first distance is the one based on the distance between the corresponding vertices in the line graph of G and obviously, its related edge-Wiener index is equal to the ordinary Wiener index of the line graph of G. The second distance $d_{e|G}(g, f)$ between the edges $g = uv$ and

$f = zt$ of the graph G is defined as [9]:

$$d_{e|G}(g,f) = \begin{cases} 0 & g = f \\ \max\{d_G(u,z), d_G(u,t), d_G(v,z), d_G(v,t)\} & g \neq f \end{cases}$$

Related to this distance, the second edge-Wiener index $W_e(G)$ of G is defined as [9]:

$$W_e(G) = \sum_{\{g,f\}\subseteq E(G)} d_{e|G}(g,f).$$

We refer the reader to [6, 10] for more information on the edge-Wiener indices.

The first edge-Wiener index of some composite graphs was computed before [1, 2, 4]. In this paper, we are interested in the type of relationship that exists between the second edge-Wiener index of the join and corona product of graphs and their components. Results are applied for several classes of graphs by specializing components in these product graphs. So throughout the paper, by the edge-Wiener index of a graph G, we mean the second version, $W_e(G)$ and by distance between two edges g and f in G, we mean the second distance $d_{e|G}(g,f)$.

2. Preliminaries

Let $N_G(u)$ denote the neighborhood of a vertex u in G, i.e., the set of all vertices of G adjacent with u. The degree of u in G is the cardinality of $N_G(u)$ and is denoted by $deg_G(u)$. Let $\Delta(G)$ and $\Theta(G)$ denote the number of triangles in G and the number of subgraphs of G isomorphic to the 4-vertex complete graph K_4, respectively. It is easy to see that

$$\Delta(G) = \frac{1}{3} \sum_{uv\in E(G)} |N_G(u) \cap N_G(v)|,$$

$$\Theta(G) = \frac{1}{12} \sum_{uv\in E(G)} \sum_{z\in N_G(u)\cap N_G(v)} |N_G(u) \cap N_G(v) \cap N_G(z)|.$$

Corresponding to each triangle in G, there are 3 pairs of adjacent edges which are at distance 1 in G. So the number of such pairs of edges in G is equal to $3\Delta(G)$. Also, corresponding to each subgraph of G isomorphic to K_4, there are 3 pairs of nonadjacent edges which are at distance 1 in G. So the number of such pairs of edges in G is equal to $3\Theta(G)$. Hence, the total number of pairs of edges which are at distance 1 in G is equal to $3(\Delta(G) + \Theta(G))$.

Let x be a vertex of G and $g = uv$ be an edge of G. We define the distance $D_G(x,g)$ between the vertex x and the edge g of the graph G as [5]:

$$D_G(x,g) = \max\{d_G(x,u), d_G(x,v)\}.$$

The vertex-edge Wiener index $W_{ve}(G)$ of G is defined as the sum of such distances over all vertices $x \in V(G)$ and edges $g \in E(G)$ [5]:

$$W_{ve}(G) = \sum_{x \in V(G)} \sum_{g \in E(G)} D(x, g|G).$$

This index was studied in more details in [1, 3] under the name $Max(G)$. We refer the reader to these references for more information on the vertex-edge Wiener indices and for explicit formulas for $W_{ve}(G)$ of several classes of graphs.

3. Main results

Throughout this section, let G_1 and G_2 be two simple connected graphs and n_i and e_i denote the numbers of vertices and edges of G_i, respectively, where $i \in \{1, 2\}$. Our aim is to compute the edge-Wiener index of join and corona product of G_1 and G_2.

3.1. *Join*

The join $G_1 + G_2$ of graphs G_1 and G_2 is defined as the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup S,$$

where $S = \{u_1u_2 \,|u_1 \in V(G_1), u_2 \in V(G_2)\}$. All distinct vertices of $G_1 + G_2$ are either at distance 1 or 2. The vertices at distance 2 are precisely those of G_1 that are not adjacent in G_1, and those of G_2 that are not adjacent in G_2. So all distinct edges of $G_1 + G_2$ are either at distance 1 or 2. The join of two graphs is also known as their **sum**. Its definition can be extended inductively to more than two graphs in a straightforward manner. It is a commutative operation and hence both its components will appear symmetrically in any formula including distance-based invariants.

Theorem 1. *Let G_1 and G_2 be two simple connected graphs. Then*

$$\begin{aligned} W_e(G_1 + G_2) = 2\binom{n_1n_2}{2} + 2\binom{e_1}{2} + 2\binom{e_2}{2} + e_1n_2(2n_1 - 3) + n_1e_2(2n_2 - 3) \\ - e_1e_2 - 3(n_2 + 1)\Delta(G_1) - 3(n_1 + 1)\Delta(G_2) - 3\left(\Theta(G_1) + \Theta(G_2)\right). \end{aligned}$$

Proof. Let Q be the set of all pairs of edges of $G_1 + G_2$. We partition Q into six disjoint sets as follows:
$Q_1 = \{\{g, f\}|g, f \in E(G_1)\}$;
$Q_2 = \{\{g, f\}|g, f \in E(G_2)\}$;
$Q_3 = \{\{g, f\}|g \in E(G_1), f \in E(G_2)\}$;
$Q_4 = \{\{g, f\}|g \in E(G_1), f \in S\}$;
$Q_5 = \{\{g, f\}|g \in E(G_2), f \in S\}$;
$Q_6 = \{\{g, f\}|g, f \in S\}$.
The edge-Wiener index of $G_1 + G_2$ is obtained by summing the contributions of all

pairs of edges over those six sets. We proceed to evaluate their contributions in order of increasing complexity.

The case of Q_3 is the simplest. Let $\{g,f\} \in Q_3$, where $g = u_1v_1 \in E(G_1)$ and $f = u_2v_2 \in E(G_2)$. Then

$$\begin{aligned} d_{e|G_1+G_2}(g,f) =& \max\{d_{G_1+G_2}(u_1,u_2), d_{G_1+G_2}(u_1,v_2), d_{G_1+G_2}(v_1,u_2), \\ & d_{G_1+G_2}(v_1,v_2)\} \\ =& \max\{1,1,1,1\} = 1. \end{aligned}$$

There are e_1e_2 such pairs of edges in Q_3 and each of them contributes 1 to the edge-Wiener index. Hence, the total contribution of pairs from Q_3 is equal to e_1e_2.

The set Q_6 contains pairs of edges from S. Let $\{g,f\} \in Q_6$ and $g = u_1u_2$, $f = v_1v_2$, where $u_1, v_1 \in V(G_1)$, $u_2, v_2 \in V(G_2)$. Then

$$\begin{aligned} d_{e|G_1+G_2}(g,f) =& \max\{d_{G_1+G_2}(u_1,v_1), d_{G_1+G_2}(u_1,v_2), d_{G_1+G_2}(u_2,v_1), \\ & d_{G_1+G_2}(u_2,v_2)\} \\ =& \max\{d_{G_1+G_2}(u_1,v_1), 1, 1, d_{G_1+G_2}(u_2,v_2)\}. \end{aligned}$$

It is easy to see that, if $u_1 = v_1, u_2v_2 \in E(G_2)$ or $u_2 = v_2, u_1v_1 \in E(G_1)$ or $u_1v_1 \in E(G_1), u_2v_2 \in E(G_2)$ then $d_{e|G_1+G_2}(g,f) = 1$, otherwise $d_{e|G_1+G_2}(g,f) = 2$.
The total number of pairs of edges in Q_6 is equal to $\binom{n_1n_2}{2}$. Among them there are $n_1e_2 + n_2e_1 + 2e_1e_2$ pairs that contribute 1 to the edge-Wiener index, and all other pairs contribute 2. Hence the total contribution of pairs from Q_6 is equal to

$$2\binom{n_1n_2}{2} - n_1e_2 - n_2e_1 - 2e_1e_2.$$

Now, we compute the contribution of pairs from Q_4. Let $\{g,f\} \in Q_4$ and $g = u_1v_1 \in E(G_1)$ and $f = z_1u_2 \in S$, where $u_1, v_1, z_1 \in V(G_1)$, $u_2 \in V(G_2)$. Then

$$\begin{aligned} d_{e|G_1+G_2}(g,f) =& \max\{d_{G_1+G_2}(u_1,z_1), d_{G_1+G_2}(u_1,u_2), d_{G_1+G_2}(v_1,z_1), \\ & d_{G_1+G_2}(v_1,u_2)\} \\ =& \max\{d_{G_1+G_2}(u_1,z_1), 1, d_{G_1+G_2}(v_1,z_1), 1\}. \end{aligned}$$

It is easy to see that, if $z_1 = u_1$ or $z_1 = v_1$ or $u_1z_1, v_1z_1 \in E(G_1)$ then $d_{e|G_1+G_2}(g,f) = 1$, otherwise $d_{e|G_1+G_2}(g,f) = 2$.
The total number of pairs from Q_4 is equal to $e_1n_1n_2$. Among them there are $2e_1n_2 + 3n_2\Delta(G_1)$ pairs that contribute 1 to the edge-Wiener index, and all other pairs contribute 2. Hence the total contribution of pairs from Q_4 is equal to

$$2e_1n_1n_2 - 2e_1n_2 - 3n_2\Delta(G_1).$$

By symmetry, the total contribution of pairs from Q_5 is equal to

$$2n_1n_2e_2 - 2n_1e_2 - 3n_1\Delta(G_2).$$

It remains to compute the contributions of Q_1 and Q_2. Let $\{g, f\} \in Q_1$, where $g = u_1v_1$, $f = z_1t_1$. Then

$$\begin{aligned} d_{e|G_1+G_2}(g,f) =& \max\{d_{G_1+G_2}(u_1,z_1), d_{G_1+G_2}(u_1,t_1), d_{G_1+G_2}(v_1,z_1), \\ & d_{G_1+G_2}(v_1,t_1)\}. \end{aligned}$$

By definition of $G_1 + G_2$, the distances $d_{G_1+G_2}(u_1,z_1)$, $d_{G_1+G_2}(u_1,t_1)$, $d_{G_1+G_2}(v_1,z_1)$ and $d_{G_1+G_2}(v_1,t_1)$ are equal to 0, 1 or 2, so $d_{e|G_1+G_2}(g,f)$ is either equal to 1 or 2. The total number of pairs in Q_1 is equal to $\binom{e_1}{2}$. As mentioned before, $3\,(\Delta(G_1) + \Theta(G_1))$ pairs contribute 1 to the edge-Wiener index, and all other pairs contribute 2. Hence the total contribution of pairs from Q_1 is equal to

$$2\binom{e_1}{2} - 3\,(\Delta(G_1) + \Theta(G_1))\,.$$

Again, the total contribution of Q_2 is obtained by the symmetry as

$$2\binom{e_2}{2} - 3\,(\Delta(G_2) + \Theta(G_2))\,.$$

Now, the formula of the Theorem follows by adding the contributions of $Q_1, \ldots,$ Q_6 and simplifying the resulting expression. □

As expected, G_1 and G_2 appear symmetrically in the above formula. It is interesting to note that the formula does not depend on the connectivity of G_1 and G_2. That allows us to compute the edge-Wiener index of joins of graphs that are not themselves connected.

3.2. *Corona product*

The corona product $G_1 \circ G_2$ of graphs G_1 and G_2 is obtained by taking one copy of G_1 and n_1 copies of G_2, and joining all vertices of the i-th copy of G_2 to the i-th vertex of G_1 for $i = 1, 2, \cdots, n_1$. Unlike join, corona is a non-commutative operation, and its component graphs appear in markedly asymmetric roles. More formally, we denote the copy of G_2 related to the vertex $x \in V(G_1)$ by $G_{2,x}$ and the edge set of $G_{2,x}$ by $S_{2,x}$. By definition of $G_1 \circ G_2$, the distance between two distinct vertices $u, v \in V(G_1 \circ G_2)$ is given by:

$$d_{G_1 \circ G_2}(u,v) = \begin{cases} d_{G_1}(u,v) & u,v \in V(G_1) \\ d_{G_1}(u,x) + 1 & u \in V(G_1), v \in V(G_{2,x}) \\ 1 & uv \in S_{2,x} \\ 2 & u,v \in V(G_{2,x}), uv \notin S_{2,x} \\ d_{G_1}(x,y) + 2 & u \in V(G_{2,x}), v \in V(G_{2,y}), x \neq y \end{cases}$$

Theorem 2. *Let G_1 and G_2 be two simple connected graphs. Then*

$$W_e(G_1 \circ G_2) = W_e(G_1) + (n_2 + e_2)^2 W(G_1) + (n_2 + e_2) W_{ve}(G_1) - 6n_1\Delta(G_2)$$

$$-3n_1\Theta(G_2)+2\binom{n_1n_2}{2}+2\binom{n_1e_2}{2}+n_1e_1(n_2+e_2)$$
$$+n_1e_2(2n_1n_2-3).$$

Proof. It is obvious that the graph $G_1\circ G_2$ has $e_1+n_1e_2+n_1n_2$ edges. We partition the edge set of $G_1\circ G_2$ into three sets. The first one is the edge set of G_1, $S_1=E(G_1)$, the second one contains all edges in all copies of G_2, $S_2=\bigcup_{x\in V(G_1)}S_{2,x}$, and the third one contains all edges with one end in G_1 and the other end in some copy of G_2, $S_3=\bigcup_{x\in V(G_1)}S_{3,x}$, where $S_{3,x}=\{e|e=ux,u\in V(G_{2,x})\}$.
Now we start to compute the distances between the edges of these three sets. We consider the following six cases:
Case 1. $\{g,f\}\subseteq S_1$.
It is obvious that $d_{e|G_1\circ G_2}(g,f)=d_{e|G_1}(g,f)$, so

$$W_1=\sum_{\{g,f\}\subseteq S_1}d_{e|G_1\circ G_2}(g,f)=W_e(G_1).$$

Case 2. $\{g,f\}\subseteq S_2,\ g\in S_{2,x}$ and $f\in S_{2,y}$.
First, consider the case $x=y$ and let $g=u_{2,x}v_{2,x}$, $f=z_{2,x}t_{2,x}\in S_{2,x}$. Then

$$\begin{aligned}d_{e|G_1\circ G_2}(g,f)=&\max\{d_{G_1\circ G_2}(u_{2,x},z_{2,x}),d_{G_1\circ G_2}(u_{2,x},t_{2,x}),d_{G_1\circ G_2}(v_{2,x},z_{2,x}),\\&d_{G_1\circ G_2}(v_{2,x},t_{2,x})\}.\end{aligned}$$

Clearly, the above four distances are equal to 0, 1 or 2. So $d_{e|G_1\circ G_2}(g,f)=1$ or 2. In this case, the vertex x and its related copy, $G_{2,x}$, form a copy of K_1+G_2. So by the same reasoning as in the proof of Theorem 1, we obtain:

$$\sum_{\{g,f\}\subseteq S_{2,x}}d_{e|G_1\circ G_2}(g,f)=2\binom{e_2}{2}-3\left(\Delta(G_2)+\Theta(G_2)\right).$$

Now let $x\neq y$ and $g=u_{2,x}v_{2,x}$ and $f=u_{2,y}v_{2,y}$. Then

$$\begin{aligned}d_{e|G_1\circ G_2}(g,f)=&\max\{d_{G_1\circ G_2}(u_{2,x},u_{2,y}),d_{G_1\circ G_2}(u_{2,x},v_{2,y}),d_{G_1\circ G_2}(v_{2,x},u_{2,y}),\\&d_{G_1\circ G_2}(v_{2,x},v_{2,y})\}\\=&\max\{d_{G_1}(x,y)+2,d_{G_1}(x,y)+2,d_{G_1}(x,y)+2,d_{G_1}(x,y)+2\}\\=&d_{G_1}(x,y)+2.\end{aligned}$$

Now,

$$\begin{aligned}W_2&=\sum_{\{g,f\}\subseteq S_2}d_{e|G_1\circ G_2}(g,f)\\&=\sum_{x\in V(G_1)}\sum_{\{g,f\}\subseteq S_{2,x}}d_{e|G_1\circ G_2}(g,f)+\sum_{\{x,y\}\subseteq V(G_1)}\sum_{g\in S_{2,x}}\sum_{f\in S_{2,y}}d_{e|G_1\circ G_2}(g,f)\end{aligned}$$

$$= \sum_{x\in V(G_1)} \left(2\binom{e_2}{2} - 3\left(\Delta(G_2)+\Theta(G_2)\right)\right) + e_2^2 \sum_{\{x,y\}\subseteq V(G_1)} \left(d_{G_1}(x,y)+2\right)$$

$$= n_1\left(2\binom{e_2}{2} - 3\left(\Delta(G_2)+3\Theta(G_2)\right)\right) + e_2^2\left(W(G_1)+2\binom{n_1}{2}\right).$$

Case 3. $\{g,f\}\subseteq S_3,\ g\in S_{3,x}$ and $f\in S_{3,y}$.
If $x=y$ and $g=u_{2,x}x,\ f=v_{2,x}x$ where $u_{2,x},v_{2,x}\in V(G_{2,x})$, then

$$\begin{aligned} d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_{2,x},v_{2,x}), d_{G_1\circ G_2}(u_{2,x},x), d_{G_1\circ G_2}(x,v_{2,x}),\\ & d_{G_1\circ G_2}(x,x)\}\\ =& \max\{d_{G_1\circ G_2}(u_{2,x},v_{2,x}),1,1,0\}. \end{aligned}$$

If $v_{2,x}$ is adjacent to $u_{2,x}$ in $G_{2,x}$, then $d_{G_1\circ G_2}(u_{2,x},v_{2,x})=1$, so $d_{e|G_1\circ G_2}(g,f)=1$, otherwise $d(u_{2,x},v_{2,x})=2$, so $d_{e|G_1\circ G_2}(g,f)=2$. Hence for each edge $g=u_{2,x}x\in S_{3,x}$, $deg_{G_{2,x}}(u_{2,x})$ edges of $S_{3,x}$ are at distance 1 from g and all other edges are at distance 2. So

$$\begin{aligned} &\sum_{\{g,f\}\subseteq S_{3,x}} d_{e|G_1\circ G_2}(g,f)\\ &= \frac{1}{2}\sum_{u_{2,x}\in V(G_{2,x})} \left(deg_{G_{2,x}}(u_{2,x}) + 2\left(n_2-1-deg_{G_{2,x}}(u_{2,x})\right)\right)\\ &= \frac{1}{2}\left(2e_2+2n_2(n_2-1)-4e_2\right) = n_2(n_2-1)-e_2. \end{aligned}$$

If $x\neq y$ and $g=u_{2,x}x\in S_{3,x},\ f=u_{2,y}y\in S_{3,y}$, then

$$\begin{aligned} d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_{2,x},u_{2,y}), d_{G_1\circ G_2}(u_{2,x},y), d_{G_1\circ G_2}(x,u_{2,y}),\\ & d_{G_1\circ G_2}(x,y)\}\\ =& \max\{d_{G_1}(x,y)+2, d_{G_1}(x,y)+1, d_{G_1}(x,y)+1, d_{G_1}(x,y)\}\\ =& d_{G_1}(x,y)+2. \end{aligned}$$

Now,

$$\begin{aligned} W_3 &= \sum_{\{g,f\}\subseteq S_3} d_{e|G_1\circ G_2}(g,f)\\ &= \sum_{x\in V(G_1)}\sum_{\{g,f\}\subseteq S_{3,x}} d_{e|G_1\circ G_2}(g,f) + \sum_{\{x,y\}\subseteq V(G_1)}\sum_{g\in S_{3,x}}\sum_{f\in S_{3,y}} d_{e|G_1\circ G_2}(g,f)\\ &= \sum_{x\in V(G_1)} \left(n_2(n_2-1)-e_2\right) + \sum_{\{x,y\}\subseteq V(G_1)}\sum_{g\in S_{3,x}}\sum_{f\in S_{3,y}} \left(d_{G_1}(x,y)+2\right)\\ &= n_1\left(n_2(n_2-1)-e_2\right) + n_2^2\sum_{\{x,y\}\subseteq V(G_1)} \left(d_{G_1}(x,y)+2\right) \end{aligned}$$

$$= n_1\left(n_2(n_2-1)-e_2\right)+n_2^2W(G_1)+2n_2^2\binom{n_1}{2}$$
$$= 2\binom{n_1n_2}{2}-n_1e_2+n_2^2W(G_1).$$

Case 4. $g \in S_1, f \in S_2$.
Let $g = u_1v_1 \in S_1,\ f = u_{2,x}v_{2,x} \in S_{2,x}$, for some $x \in V(G_1)$. Then

$$\begin{aligned} d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_1,u_{2,x}), d_{G_1\circ G_2}(u_1,v_{2,x}), d_{G_1\circ G_2}(v_1,u_{2,x}),\\ & d_{G_1\circ G_2}(v_1,v_{2,x})\}\\ =& \max\{d_{G_1}(u_1,x)+1, d_{G_1}(u_1,x)+1, d_{G_1}(v_1,x)+1,\\ & d_{G_1}(v_1,x)+1\}\\ =& \max\{d_{G_1}(u_1,x), d_{G_1}(v_1,x)\}+1\\ =& D_{G_1}(x,g)+1. \end{aligned}$$

Now,

$$\begin{aligned} W_4 =& \sum_{x\in V(G_1)}\sum_{f\in S_{2,x}}\sum_{g\in S_1} d_{e|G_1\circ G_2}(g,f) = \sum_{x\in V(G_1)}\sum_{f\in S_{2,x}}\sum_{g\in S_1}\left(D_{G_1}(x,g)+1\right)\\ =& e_2W_{ve}(G_1)+n_1e_1e_2 = e_2\left(W_{ve}(G_1)+n_1e_1\right). \end{aligned}$$

Case 5. $g \in S_1, f \in S_3$
Let $g = u_1v_1 \in S_1,\ f = u_{2,x}x \in S_{3,x}$, where $u_1, v_1, x \in V(G_1)$, $u_{2,x} \in V(G_{2,x})$. Then

$$\begin{aligned} d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_1,u_{2,x}), d_{G_1\circ G_2}(u_1,x), d_{G_1\circ G_2}(v_1,u_{2,x}),\\ & d_{G_1\circ G_2}(v_1,x)\}\\ =& \max\{d_{G_1}(u_1,x)+1, d_{G_1}(u_1,x), d_{G_1}(v_1,x)+1, d_{G_1}(v_1,x)\}\\ =& \max\{d_{G_1}(u_1,x), d_{G_1}(v_1,x)\}+1\\ =& D_{G_1}(x,g)+1. \end{aligned}$$

Now,

$$\begin{aligned} W_5 =& \sum_{x\in V(G_1)}\sum_{f\in S_{3,x}}\sum_{g\in S_1} d_{e|G_1\circ G_2}(g,f) = \sum_{x\in V(G_1)}\sum_{f\in S_{3,x}}\sum_{g\in S_1}\left(D_{G_1}(x,g)+1\right)\\ =& n_2W_{ve}(G_1)+n_1e_1n_2 = n_2\left(W_{ve}(G_1)+n_1e_1\right). \end{aligned}$$

Case 6. $g \in S_2, f \in S_3$
Let $g = u_{2,x}v_{2,x} \in S_{2,x},\ f = z_{2,x}x \in S_{3,x}$, where $x \in V(G_1)$ and $u_{2,x}, v_{2,x}, z_{2,x} \in V(G_{2,x})$. Then

$$\begin{aligned} d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_{2,x},z_{2,x}), d_{G_1\circ G_2}(u_{2,x},x), d_{G_1\circ G_2}(v_{2,x},z_{2,x}),\\ & d_{G_1\circ G_2}(v_{2,x},x)\} \end{aligned}$$

$$=\max\{d_{G_1\circ G_2}(u_{2,x},z_{2,x}),1,d_{G_1\circ G_2}(v_{2,x},z_{2,x}),1\}.$$

By definition of $G_1 \circ G_2$, the distances $d_{G_1\circ G_2}(u_{2,x},z_{2,x})$ and $d_{G_1\circ G_2}(v_{2,x},z_{2,x})$ are equal to 0, 1 or 2. It is easy to see that, if $z_{2,x} = u_{2,x}$ or $z_{2,x} = v_{2,x}$ or $u_{2,x}z_{2,x}, v_{2,x}z_{2,x} \in S_{2,x}$, then $d_{e|G_1\circ G_2}(g, f) = 1$, otherwise $d_{e|G_1\circ G_2}(g, f) = 2$. So, for each edge $g = u_{2,x}v_{2,x} \in S_{2,x}$, the edges $u_{2,x}x, v_{2,x}x, z_{2,x}x \in S_{3,x}$, where $z_{2,x} \in N_{G_{2,x}}(u_{2,x}) \cap N_{G_{2,x}}(v_{2,x})$ are at distance 1 from g and all other edges are at distance 2. So

$$\begin{aligned}
&\sum_{g\in S_{2,x}}\sum_{f\in S_{3,x}} d_{e|G_1\circ G_2}(g,f)\\
&= \sum_{u_{2,x}v_{2,x}\in S_{2,x}} \Big(2+|N_{G_{2,x}}(u_{2,x})\cap N_{G_{2,x}}(v_{2,x})|\\
&\qquad\qquad + 2\left(n_2-2-|N_{G_{2,x}}(u_{2,x})\cap N_{G_{2,x}}(v_{2,x})|\right)\Big)\\
&= \sum_{u_{2,x}v_{2,x}\in S_{2,x}} \left(2n_2-2-|N_{G_{2,x}}(u_{2,x})\cap N_{G_{2,x}}(v_{2,x})|\right) = 2e_2(n_2-1)-3\Delta(G_2).
\end{aligned}$$

If $g = u_{2,x}v_{2,x} \in S_{2,x}$, $f = u_{2,y}y \in S_{3,y}$, where $x, y \in V(G_1)$ and $x \neq y$, then

$$\begin{aligned}
d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_{2,x},u_{2,y}), d_{G_1\circ G_2}(u_{2,x},y), d_{G_1\circ G_2}(v_{2,x},u_{2,y}),\\
& d_{G_1\circ G_2}(v_{2,x},y)\}\\
=& \max\{d_{G_1}(x,y)+2, d_{G_1}(x,y)+1, d_{G_1}(x,y)+2, d_{G_1}(x,y)+1\}\\
=& d_{G_1}(x,y)+2.
\end{aligned}$$

Now,

$$\begin{aligned}
W_6 &= \sum_{x,y\in V(G_1)}\sum_{g\in S_{2,x}}\sum_{f\in S_{3,y}} d_{e|G_1\circ G_2}(g,f)\\
&= \sum_{x\in V(G_1)}\sum_{g\in S_{2,x}}\sum_{f\in S_{3,x}} d_{e|G_1\circ G_2}(g,f)\\
&+ \sum_{x\in V(G_1)}\sum_{y\in V(G_1)\setminus\{x\}}\sum_{g\in S_{2,x}}\sum_{f\in S_{3,y}} d_{e|G_1\circ G_2}(g,f)\\
&= \sum_{x\in V(G_1)} (2e_2(n_2-1)-3\Delta(G_2))\\
&+ \sum_{x\in V(G_1)}\sum_{y\in V(G_1)\setminus\{x\}}\sum_{g\in S_{2,x}}\sum_{f\in S_{3,y}} (d_{G_1}(x,y)+2)\\
&= n_1\left(2e_2(n_2-1)-3\Delta(G_2)\right) + n_2e_2 \sum_{x\in V(G_1)}\sum_{y\in V(G_1)\setminus\{x\}} (d_{G_1}(x,y)+2)\\
&= n_1\left(2e_2(n_2-1)-3\Delta(G_2)\right) + 2n_2e_2\left(W(G_1)+n_1(n_1-1)\right).
\end{aligned}$$

Now the formula for the edge-Wiener index of $G_1 \circ G_2$ follows by adding all six contributions and simplifying the resulting expression. $\square$

Again, it is interesting to note that the formula of Theorem 2 does not include any invariants of G_2 that depend on its connectivity. It is, hence, possible to apply Theorem 2 to corona products $G_1 \circ G_2$ with disconnected G_2.

4. Examples and corollaries

Now, we can obtain explicit formulas for the edge-Wiener indices of some classes of graphs by specializing components in joins and coronas. We start by computing the edge Wiener-index of a suspension of a graph G. For a given graph G, we call the graph $K_1 + G$ the **suspension** of G, where K_1 denotes the single vertex graph.

Corollary 1. *Let G be a simple graph of order n and size e. Then*

$$W_e(K_1+G) = W_e(K_1 \circ G) = 2\binom{n+e}{2} - 3e - 6\Delta(G) - 3\Theta(G).$$

Now the formulas for the **wheel graph** $W_n = K_1 + C_n$ and for the **fan graph** $K_1 + P_n$ follow at once. Both graphs allow alternative representations as $K_1 \circ C_n$ and $K_1 \circ P_n$, respectively.

Corollary 2.

$$W_e(K_1+C_n) = \begin{cases} 15 & n=3 \\ 4n^2-5n & n \geq 4 \end{cases};$$

$$W_e(K_1+P_n) = 4n^2-9n+5, \quad n \geq 2.$$

Our next example is the **windmill graph**. The windmill graph $D_n^{(m)}$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common. The case $n=3$ therefore corresponds to the **Dutch windmill graph**. One can easily see that the windmill graph is the suspension of m copies of K_{n-1}. Also, it is easy to see that, $\Delta(K_n) = \binom{n}{3}$ and $\Theta(K_n) = \binom{n}{4}$, for $n \geq 1$. So using Corollary 1 , we can get the formula for the edge-Wiener index of the windmill graph $D_n^{(m)}$.

Corollary 3. *For $n \geq 2$ and $m \geq 1$,*

$$W_e(D_n^{(m)}) = \frac{1}{4}m\binom{n}{2}\left(4m\binom{n}{2} - n^2 + n - 2\right).$$

In particular, the edge-Wiener index of the Dutch windmill graph $D_3^{(m)}$ is given by:

Corollary 4. *For $m \geq 1$,*

$$W_e(D_3^{(m)}) = 3m(3m-2).$$

Now, we turn our attention toward coronas. Coronas sometimes appear in chemical literature as plerographs of the usual hydrogen-suppressed molecular graphs known as kenographs; see [13] for definitions and more information. The **t-thorny graph** of a given graph G is obtained as $G \circ \overline{K}_t$, where $\overline{K}_t$ denotes the empty graph on t vertices [11]. For the t-thorny graph of a graph G, we obtain the following formula.

Corollary 5. *Let G be a simple connected graph of order n and size e. Then*

$$W_e(G \circ \overline{K}_t) = W_e(G) + t^2 W(G) + t W_{ve}(G) + nt(nt + e - 1).$$

Now, we present two formulas for the edge-Wiener indices of the **t-thorny cycle** $C_n \circ \overline{K}_t$ and the **t-thorny path** $P_n \circ \overline{K}_t$. We use known results for the edge-Wiener indices of paths and cycles [9] and our results on the vertex-edge Wiener indices of these graphs obtained in [1].

Corollary 6.

$$W_e(C_n \circ \overline{K}_t) = \begin{cases} \frac{n(t+1)}{8}[n^2(t+1) + 4n(2t+1) - (t+13)] & n \ is \ odd \\ \frac{n(t+1)}{8}[n^2(t+1) + 4n(2t+1) - 8] & n \ is \ even \end{cases};$$

$$W_e(P_n \circ \overline{K}_t) = \binom{n+1}{3}(t+1)^2 + (nt + n - 1)(nt - 1).$$

Interesting classes of graphs can also be obtained by specializing the first component in the corona product. For example, for a graph G, the graph $K_2 \circ G$ is called its **bottleneck graph**. For the bottleneck graph of a graph G, we obtain the following formula.

Corollary 7. *Let G be a simple graph of order n and size e. Then*

$$W_e(K_2 \circ G) = 5(n+e)^2 + 2(n - 2e) - 12\Delta(G) - 6\Theta(G).$$

Using Corollary 7, the formulas for the **bottleneck graph of a cycle** $K_2 \circ C_n$ and the **bottleneck graph of a path** $K_2 \circ P_n$ are easily obtained.

Corollary 8.

$$W_e(K_2 \circ C_n) = \begin{cases} 162 & n = 3 \\ 20n^2 - 2n & n \geq 4 \end{cases};$$

$$W_e(K_2 \circ P_n) = 20n^2 - 22n + 9.$$

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