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# Functions preserving rank-$k$ matrices of order $n$ over fields

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# FUNCTIONS PRESERVING RANK-$k$ MATRICES OF ORDER $n$ OVER FIELDS

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ABSTRACT. Let $\mathbb{F}$ be an arbitrary field and $n$ is an integer with $n \geq 2$. For a fixed positive integer $k$ satisfying $k < n$, we determine the general form of all functions preserving rank-$k$ matrices of order $n$. This article generalizes the recent results of J. Kalinowski [1, 2].

*Mathematics Subject Classification:* 15A03

*Keywords:* field, rank-$k$ matrix, function

## 1. INTRODUCTION

Suppose $\mathbb{F}$ is an arbitrary field and $\mathbb{R}$ is the field of the real numbers. Let $n$ be an integer with $n \geq 2$. For a function $f : \mathbb{F} \to \mathbb{F}$ and a matrix $A = [a_{ij}]$ over $\mathbb{F}$, denote the matrix $[f(a_{ij})]$ by $A^f$. We say that a function $f : \mathbb{F} \to \mathbb{F}$ preserves ranks of matrices if $\operatorname{rank} A^f = \operatorname{rank} A$ for all matrices (of any order) over $\mathbb{F}$, and preserves rank-$k$ matrices of order $n$ if $\operatorname{rank} A^f = \operatorname{rank} A$ for every rank-$k$ matrix of order $n$.

Kalinowski [1] investigated that a monotonic and continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ preserves ranks of matrices if and only if it is linear, i. e., $f(x) = cx$ for every $x \in \mathbb{R}$ and some non-zero $c \in \mathbb{R}$. Furthermore, in [2], Kalinowski generalized the result in [1] by removing any restrictions on the function $f$.

Inspired by [1, 2], in this article we prove the following two theorems which generalize the result in [2].

**Theorem 1.** *Let $k$ be a fixed integer satisfying $2 \leq k < n$. Then $f : \mathbb{F} \to \mathbb{F}$ is a function preserving rank-$k$ matrices of order $n$ if and only if there exist a non-zero scalar $c$ and an injective field endomorphism $\delta$ of $\mathbb{F}$ such that $f = c\delta$.*

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**Theorem 2.** *$f : \mathbb{F} \to \mathbb{F}$ is a function preserving rank-1 matrices of order $n$ if and only if either $f$ is a non-zero constant function or $f = c\kappa$, where $c$ is a non-zero scalar and $\kappa : \mathbb{F} \to \mathbb{F}$ is a multiplicative function with $\kappa(0) = 0$ and $\kappa(1) = 1$.*

As pointed out by Marková [3], these results obtained in [1, 2] play a important role in the theory of $g$-calculus (see [4] for the concept of $g$-calculus and the relevant topics). Therefore, Theorems 1 and 2 will be helpful for studying extensively $g$-calculus.

We end this section by introducing the notation which will be used in the next section. Denote by $\oplus$ the usual direct sum of matrices. For a positive integer $k$, let $I_k$ be the $k \times k$ identity matrix over $\mathbb{F}$

## 2. Proofs of Theorems 1 and 2

The Proof of Theorem 1. The "if" part is obvious. The proof of the "only if" part is divided into the following four steps.

*Step* 1: *$f(0) = 0$ and $f(d) \neq 0$ for every non-zero scalar $d$.* For any non-zero scalar $d$, it follows from $\operatorname{rank}(dI_k \oplus 0) = k$ and the definition of $f$ that

$$\operatorname{rank}\begin{bmatrix} f(d) & f(0) & \cdots & \cdots & \cdots & f(0) \\ f(0) & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & f(d) & \ddots & & \vdots \\ \vdots & & \ddots & f(0) & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & f(0) \\ f(0) & \cdots & \cdots & \cdots & f(0) & f(0) \end{bmatrix} = k,$$

where the number of occurrences of $f(d)$ is equal to $k$. This, together with the inequality $2 \leq k < n$, completes the present step.

*Step* 2: *$f(1)f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{F}$.* For any $x, y \in \mathbb{F}$, since

$$\operatorname{rank}\left(\begin{bmatrix} 1 & x \\ y & xy \end{bmatrix} \oplus I_{k-1} \oplus 0\right) = k,$$

it follows from Step 1 (i. e., $f(0) = 0$) and the definition of $f$ that

$$\operatorname{rank}\left(\begin{bmatrix} f(1) & f(x) \\ f(y) & f(xy) \end{bmatrix} \oplus f(1)I_{k-1} \oplus 0\right) = k,$$

and hence

$$\det\left(\begin{bmatrix} f(1) & f(x) \\ f(y) & f(xy) \end{bmatrix} \oplus f(1)I_{k-1}\right) = 0.$$

By direct computation, one shows that $f(1)^{k-1}(f(1)f(xy) - f(x)f(y)) = 0$. This, together with Step 1 (i. e., $f(1) \neq 0$), gives $f(1)f(xy) = f(x)f(y)$.

*Step* 3: *$f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{F}$.* For any $x, y \in \mathbb{F}$, in view of the relation

$$\operatorname{rank}\left(\begin{bmatrix} 0 & 1 & x \\ 1 & 0 & y \\ 1 & 1 & x+y \end{bmatrix} \oplus I_{k-2} \oplus 0\right) = k,$$

it follows from Step 1 (i. e., $f(0) = 0$) and the definition of $f$ that

$$\operatorname{rank}\left(\begin{bmatrix} 0 & f(1) & f(x) \\ f(1) & 0 & f(y) \\ f(1) & f(1) & f(x+y) \end{bmatrix} \oplus f(1)I_{k-2} \oplus 0\right) = k.$$

Furthermore,

$$\det\left(\begin{bmatrix} 0 & f(1) & f(x) \\ f(1) & 0 & f(y) \\ f(1) & f(1) & f(x+y) \end{bmatrix} \oplus f(1)I_{k-2}\right) = 0.$$

Thus, $f(1)^k\,(f(y) + f(x) - f(x+y)) = 0$. This, together with Step 1 (i. e., $f(1) \neq 0$), implies that $f(x+y) = f(x) + f(y)$.

*Step* 4: *there exist a non-zero scalar $c$ and an injective field endomorphism $\delta$ of $\mathbb{F}$ such that $f = c\delta$.* If we denote $c = f(1)$ and $\delta = c^{-1}f$, then $f = c\delta$ and $c$ is a non-zero scalar. Furthermore, it is easy to verify from Steps 1–3 that $\delta$ is an injective field endomorphism of $\mathbb{F}$.

The proof is complete. □

Proof of Theorem 2. *The "if" part.* If $f$ is a non-zero constant function, then, clearly, $f$ preserves rank-1 matrices of order $n$.

Now we prove the case $f = c\kappa$, where $c$ is a non-zero scalar and $\kappa : \mathbb{F} \to \mathbb{F}$ is a multiplicative function with $\kappa(0) = 0$ and $\kappa(1) = 1$. For an arbitrary rank-1 matrix $A$, it can be written as $A = [a_i b_j]$, where $a_i, b_i \in \mathbb{F}$, $i = 1, \cdots, n$, and $a_p b_q \neq 0$ for some $p, q$. Hence $A^f = [c\kappa(a_i b_j)] = c[\kappa(a_i b_j)]$. Since $\kappa$ is multiplicative, it can be concluded that $A^f = c[\kappa(a_i)\kappa(b_j)]$, which implies $\operatorname{rank} A^f \leq 1$. On the other hand, for any non-zero $d \in \mathbb{F}$, it follows from $dd^{-1} = 1$ and the multiplicativity of $\kappa$ that $\kappa(d)\kappa(d^{-1}) = f(1)$. Using $\kappa(1) = 1$, we have $\kappa(d) \neq 0$. Therefore, $\kappa(a_p)\kappa(b_q) = \kappa(a_p b_q) \neq 0$ since $a_p b_q \neq 0$. In summary, $\operatorname{rank} A^f = 1$, i. e., $f$ is a function preserving rank-1 matrices of order $n$.

*The "only if" part.* For any non-zero scalar $d$, it follows from $\operatorname{rank}(d \oplus 0) = 1$ and the definition of $f$ that

$$\operatorname{rank}\begin{bmatrix} f(d) & f(0) & \cdots & f(0) \\ f(0) & f(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & f(0) \\ f(0) & \cdots & f(0) & f(0) \end{bmatrix} = 1. \tag{2.1}$$

*Case* 1. Suppose that $f(0) \neq 0$. Then, by (2.1), we have $f(d) = f(0)$. Since $d$ is an arbitrary non-zero scalar, we can claim that $f$ is a non-zero constant function.

*Case* 2. Suppose that $f(0) = 0$. Then, by (2.1), we have $f(d) \neq 0$ for any non-zero scalar $d$. For any $x, y \in \mathbb{F}$, since

$$\operatorname{rank}\left(\begin{bmatrix} 1 & x \\ y & xy \end{bmatrix} \oplus 0\right) = 1,$$

it follows from $f(0) = 0$ and the definition of $f$ that

$$\operatorname{rank}\left(\begin{bmatrix} f(1) & f(x) \\ f(y) & f(xy) \end{bmatrix} \oplus 0\right) = 1.$$

Thus,

$$\det\begin{bmatrix} f(1) & f(x) \\ f(y) & f(xy) \end{bmatrix} = 0,$$

i. e., $f(1)f(xy) = f(x)f(y)$. If we put $c = f(1)$ and $\kappa = c^{-1}f$, then $f = c\kappa$ and $c$ is a non-zero scalar. Furthermore, it is easily verified that $\kappa$ is multiplicative function from $\mathbb{F}$ to itself such that with $\kappa(0) = 0$ and $\kappa(1) = 1$. This completes the proof. $\square$

## References


[1] Kalinowski, J.: *On rank equivalence and preserving rank operators*, Novi Sad J. Math., **32** (2002), 133–139.

[2] Kalinowski, J.: *On functions preserving rank of matrices*, Math. Notes, Miskolc, **4** (2003), 35–37.

[3] Marková, A.: *Some remarks on the pseudo-linear algebra*, Tatra Mountains Math. Publ., **6** (1995), 123–130.

[4] Pap, E.: *g-calculus*, Zb. Prir.-Mat. Fak., Univ. Novom Sadu, Ser. Mat., **23** (1993), 145–156.


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